Robust Linear Quadratic Control of Systems with Integral Quadratic Constraints *

Huaizhong Li   Minyue Fu

Department of Electrical and Computer Engineering
The University of Newcastle
NSW 2308, Australia

Lihua Xie

School of Electrical and Electronics Engineering
Nanyang Technological University
Singapore

Abstract

In this paper, we consider the robust linear quadratic (LQ) control problem for a class of uncertain linear systems which are subject to a general type of integral quadratic constraints (IQCs). Both analysis and synthesis problems are considered in this paper. For the analysis problem, we determine if the system satisfies a desired linear quadratic performance index for all admissible uncertainties subject to the IQCs while for the synthesis problem, we look for a dynamic output regulator to guarantee certain level of performance index. We show that the two addressed problems can be effectively solved using linear matrix inequalities (LMIs). Some discussions on the optimization of the guaranteed performance index are also included.

Keywords: Robust control, LQ control, Output feedback, Linear matrix inequalities.

*This work has been supported by the Australian Research Council and Nanyang Technological University Applied Research Fund.

1 Introduction

Recently, the linear matrix inequality (LMI) approach has attracted a lot of attention for control theory, especially in H_∞ analysis and synthesis, see [5, 6], for example. Two advantageous features are offered by the LMI approach when used in H_∞ analysis and synthesis: i) It is computationally efficient due to the recent progress in convex optimization (see, e.g., [7] and [8]); and ii) It is simple to treat the so-called singular case (see [6], for example).

Linear quadratic (LQ) control is well-known in modern control theory. However, this technique only suits for systems without uncertainty. Practically, only an approximate model of a physical system is available for control design. Hence, a useful control design should cater for the potential uncertainty of the model. Usually, a robust controller is required to guarantee not only the robust stability of the system but also certain level of performance. In [2], an adaptive control scheme is used to achieve guaranteed quadratic performance. Robust LQ control techniques have been developed in [1] and [3] to cope with continuous-time systems with norm bounded uncertainty. A worst-case H_2 performance analysis and synthesis for systems which are subject to single block non-casual disturbances with bounded $\mathcal{L}_2$-induced operator norm are given in [4].

The LQ control problems reported in [1] and [3] can be summarized as follows: Given a linear system involving some norm-bounded uncertainty in the state-space matrices, the robust LQ analysis problem is to find a tight upper bound for the quadratic performance index. Similarly, the robust LQ synthesis is to design an feedback controller such that the uncertain system is stabilized and the upper bound for the specific LQ performance index is minimized. A typical solution to the robust LQ analysis/synthesis problem is to convert it into a "scaled" H_∞ analysis/synthesis one which replaces the norm-bounded uncertainty by some scaling parameters (see, [3] and [4] for example). By searching the scaling parameters iteratively, the latter problem is solved using standard ARE approach. However, this approach has three disadvantages: i) Solving AREs are not numerically efficient; ii) The scaling parameters enter the AREs nonlinearly for the "scaled" H_∞ analysis and synthesis, which makes it much more complicated to solve the AREs; and iii) The norm-bounded uncertainty assumption is not general enough and has certain limitation in describing physical systems.

In this paper, we consider the robust LQ analysis and synthesis problem for a class of uncertain linear systems where the uncertainty is described by some integral quadratic constraints (IQCs). The robust LQ analysis problem is to determine if the L_2 norm of the error output is less than a quadratic function of the initial state for all possible uncertainty satisfying the given IQCs. In parallel, the robust LQ synthesis problem is to design a feedback controller such that for all admissible uncertainties satisfying IQCs, the upper bound for the L_2 norm mentioned above is minimized. The IQCs used in our paper are very general in nature.

We apply the so-called $\mathcal{S}$-procedure [9] to the IQCs and show that the resulting problem can be solved using linear matrix inequalities (LMIs). Namely, the robust LQ analysis problem can be solved by using single LMI which is jointly linear in two sets of variables:

a positive-definite matrix for LQ performance and the scaling parameters. The robust LQ synthesis problem can be solved by using two LMIs. The first LMI is jointly linear in a positive-definite matrix for state-feedback and the inverses of the scaling parameters. The second LMI is jointly linear in a positive-definite matrix for observer design and the scaling parameters. Since one LMI is linear in the scaling parameters and the other in their inverses, the two LMIs are not jointly linear in the scaling parameters. However, the state feedback case can be dealt with by using a single LMI which is fully linear.

The rest of the paper is organized as follows: Section 2 studies the robust LQ analysis problem; section 3, the synthesis problem; and the concluding remarks are given in section 4.

2 Robust LQ Analysis

Consider the following linear system:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{p} H_{1i}\xi_i(t), \quad x(0) = x_0 \tag{1}$$

$$z(t) = Cx(t) + \sum_{i=1}^{p} H_{2i}\xi_i(t) \tag{2}$$

where $x(t) \in \mathcal{R}^n$ is the state, $z(t) \in \mathcal{R}^r$ the error output, and $\xi_i(t) \in \mathcal{R}^{k_i}$ the uncertain variables satisfying the following IQCs:

$$\int_0^T ||\xi_i(t)||^2 dt \leq \int_0^T ||E_{1i}x(t) + E_{2i}\xi(t)||^2 dt, \quad \text{as } T \to \infty, \ i = 1, \cdots, p \tag{3}$$

with

$$\xi(t) = [\xi_1^T(t) \cdots \xi_p^T(t)]^T.$$

Also, $A, C, H_{1i}, H_{2i}, E_{1i}$ and E_{2i} are constant matrices of appropriate dimensions.

We define the quadratic performance index $\mathcal{J}$ as following:

$$\begin{aligned} \mathcal{J} &= \int_0^\infty \|z(t)\|^2 dt \\ &= \int_0^\infty \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}^T R \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} dt \end{aligned} \tag{4}$$

where

$$\begin{aligned} R &= [C \;\; H_2]^T [C \;\; H_2] \\ &= \begin{bmatrix} C^T C & C^T H_2 \\ H_2^T C & H_2^T H_2 \end{bmatrix} \end{aligned} \tag{5}$$

$$H_2 = [H_{21} \;\; H_{22} \;\; \ldots \;\; H_{2p}]. \tag{6}$$

Then the robust LQ analysis problem is as follows: *Given the system (1)-(2) with uncertainties described by the IQCs (3), determine if the system is asymptotically stable and find a positive function, $G(x_0)$, such that*

$$\mathcal{J} < G(x_0) \tag{7}$$

for all admissible uncertainties satisfying (3). Furthermore, minimize the cost function $G(x_0)$.

To simplify mathematical description, we introduce some short-hand notation:

$$H_1 = [H_{11} \cdots H_{1p}]; \quad H_2 = [H_{21} \cdots H_{2p}] \tag{8}$$
$$E_1^T = [E_{11}^T \cdots E_{1p}^T]; \quad E_2^T = [E_{21}^T \cdots E_{2p}^T] \tag{9}$$
$$\tau = (\tau_1, \cdots, \tau_p) \tag{10}$$
$$J = \mathrm{diag}\{\tau_1 I_{k_1}, \cdots, \tau_p I_{k_p}\} \tag{11}$$

where $\tau_1, \cdots, \tau_p$ are scalars and k_i are the numbers of columns of H_i.

Applying the well-known $\mathcal{S}$-procedure[9, 10], we have the following fundamental result:

Lemma 1. *Given the system (1)-(2), if there exist a symmetric positive definite matrix $P \in \mathcal{R}^{n\times n}$ and scaling parameters $\tau_1, \cdots, \tau_p > 0$ such that*

$$2x^T P(Ax + \sum_{i=1}^{p} H_{1i}\xi_i) + \sum_{i=1}^{p} \tau_i(||E_{1i}x + E_{2i}\xi||^2 - ||\xi_i||^2) + \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}^T R \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} < 0,$$
$$\forall x \in \mathcal{R}^n, \xi_i \in \mathcal{R}^{k_i}, i = 1, \cdots, p \tag{12}$$

then the system is asymptotically stable and condition (7) holds with $G(x_0) = x_0^T P x_0$.

Proof. Let $V(x) = x^T P x$ be a Lyapunov function candidate of the system (1)-(2). Then, the asymptotic stability of the system is clearly implied by (12). Integrating the left hand side of the inequality in (12) along any trajectory of the system (1)-(2), we obtain:

$$x^T(T)Px(T) - x^T(0)Px(0) + \sum_{i=1}^{p} \tau_i \left\{ \int_0^T ||E_{1i}x(t) + E_{2i}\xi(t)||^2 dt - \int_0^T ||\xi_i(t)||^2 dt \right\}$$
$$+ \int_0^T \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}^T R \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} < 0$$

Since the system is asymptotically stable, $x(T) \to 0$ as $T \to \infty$. Hence, by considering (3), the above inequality leads to

$$\mathcal{J} \leq x_0^T P x_0.$$

Q.E.D.

Using this fundamental lemma, we obtain the following theorem which establishes several equivalent conditions to (12):

Theorem 2. *Given the uncertain system (1)-(2), the following conditions, all guaranteeing the solution to the linear quadratic analysis problem, are equivalent:*

(i) *There exist* $P = P^T > 0$ *and* $J > 0$ *in (11) such that (12) holds;*

(ii) *There exist* $P = P^T > 0$ *and* $J > 0$ *in (11) solving the following LMI:*

$$\mathcal{L}_1 = \begin{bmatrix} A^TP + PA + E_1^TJE_1 + C^TC & PH_1 + E_1^TJE_2 + C^TH_2 \\ H_1^TP + E_2^TJE_1 + H_2^TC & -J + E_2^TJE_2 + H_2^TH_2 \end{bmatrix} < 0 \tag{13}$$

(iii) *There exist* $P = P^T > 0$ *and* $J > 0$ *in (11) solving the following LMI:*

$$\mathcal{L}_2 = \begin{bmatrix} A^TP + PA & PH_1 & C^T & E_1^TJ \\ H_1^TP & -J & H_2^T & E_2^TJ \\ C & H_2 & -I & 0 \\ JE_1 & JE_2 & 0 & -J \end{bmatrix} < 0 \tag{14}$$

(iv) *There exists* $J > 0$ *in (11) such that the following auxiliary system is asymptotically stable and the* H_∞*-norm of the transfer function from* $\hat{w}(\cdot)$ *to* $\hat{z}(\cdot)$ *is less than 1:*

$$\dot{\hat{x}}(t) = A\hat{x}(t) + H_1J^{-1/2}\hat{w}(t) \tag{15}$$

$$\hat{z}(t) = \begin{bmatrix} C \\ J^{1/2}E_1 \end{bmatrix}\hat{x}(t) + \begin{bmatrix} H_2J^{-1/2} \\ J^{1/2}E_2J^{-1/2} \end{bmatrix}\hat{w}(t) \tag{16}$$

Proof. <u>**(i)** $\Longleftrightarrow$ **(ii)**</u>:

Rewrite the inequality (12) as follows:

$$2x^TP(Ax + H_1\xi) + (x^TE_1^T + \xi^TE_2^T)J\begin{bmatrix} E_1x \\ E_2\xi \end{bmatrix} - \xi^TJ\xi + \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}^T R \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} < 0,$$
$$\forall x \in \mathcal{R}^n, \xi_i \in \mathcal{R}^{k_i}, i = 1, \cdots, p \tag{17}$$

which is equivalent to

$$[x^T \quad \xi^T]\mathcal{L}_1\begin{bmatrix} x \\ \xi \end{bmatrix} < 0, \quad \forall x \in \mathcal{R}^n, \xi_i \in \mathcal{R}^{k_i}, i = 1, \cdots, p \tag{18}$$

i.e., (13) holds.

<u>**(ii)** $\Longleftrightarrow$ **(iv)**</u>:

Denote

$$\hat{B} = H_1J^{-1/2} \tag{19}$$

$$\hat{C}^T = [C^T \quad E_1^TJ^{1/2}] \tag{20}$$

$$\hat{D}^T = [J^{-1/2}H_2^T \quad J^{-1/2}E_2^TJ^{1/2}] \tag{21}$$

and

$$\hat{\mathcal{L}}_1 = \begin{bmatrix} A^T P + PA + \hat{C}^T\hat{C} & P\hat{B} + \hat{C}^T\hat{D} \\ \hat{B}^T P + \hat{D}^T\hat{C} & -I + \hat{D}^T\hat{D} \end{bmatrix} \tag{22}$$

the auxiliary system (15)-(16) can be rewritten as follows:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \hat{B}\hat{w}(t) \tag{23}$$

$$\hat{z}(t) = \hat{C}\hat{x}(t) + \hat{D}\hat{w}(t) \tag{24}$$

Further, we note the matrix $\mathcal{L}_1$ in (13) can be alternatively expressed as follows:

$$\mathcal{L}_1 = \mathrm{diag}\{I_n, J^{1/2}\}\hat{\mathcal{L}}_1\mathrm{diag}\{I_n, J^{1/2}\} \tag{25}$$

That is, $\mathcal{L}_1 < 0$ if and only if $\hat{\mathcal{L}}_1 < 0$. Therefore, by the well known bounded real lemma [11], A is asymptotically stable and $||\hat{D} + \hat{C}(sI - A)^{-1}\hat{B}||_\infty < 1$ if and only if $\hat{\mathcal{L}}_1 < 0$ for some $P = P^T > 0$. Hence, (ii) is equivalent to (iv).

(ii) $\Longleftrightarrow$(iii):

Note that $\hat{\mathcal{L}}_1 < 0$ if and only if the following holds:

$$\hat{\mathcal{L}}_2 = \begin{bmatrix} A^T P + PA & P\hat{B} & \hat{C}^T \\ \hat{B}^T P & -I & \hat{D}^T \\ \hat{C} & \hat{D} & -I \end{bmatrix} < 0 \tag{26}$$

which is derived from the well-known Schur complements that

$$\begin{bmatrix} X_1 & X_2^T \\ X_2 & -I \end{bmatrix} < 0 \quad <=> \quad X_1 < 0 \;\text{ and }\; X_1 + X_2^T X_2 < 0 \tag{27}$$

The equivalence between $\hat{\mathcal{L}}_2 < 0$ and $\mathcal{L}_2 < 0$ can be established by similar manipulations used on $\hat{\mathcal{L}}_1$ and $\mathcal{L}_1$. The details are hence omitted. Q.E.D.

Remark 1. H_2 performance analysis for systems with single block of uncertainty has been tackled in [4] where a Lagrange multipliers and Riccati equation approach is used. Our result in Theorem 2 is concerned with systems with multi-block of uncertainty and is obtained by applying the S-procedure and LMI approach. It can be observed from (14) that P and J are jointly linear and hence the convex optimization can be applied to obtain a tighter bound for the performance index. Generally, we are dealing with the following standard eigenvalue problem (EVP):

$$\begin{array}{ll} \text{minimize} & x_0^T P x_0 \\ \text{subject to} & P > 0, \quad J > 0, \quad \text{and LMI (14)} \end{array}$$

When there is a feasible solution to constraints $P > 0$, $J > 0$ and LMI (14), the EVP can be solved using standard optimization algorithms such as ellipsoid algorithm or the more efficient interior-point algorithm; see [8, 7] and the references therein for details about EVP optimization. To facilitate the optimization, we consider two situations:

(a) Assuming that x_0 is a random variable with $\mathcal{E} x_0 x_0^T = Q$, replace the performance index of (4) by its expectation. Then, Lemma 1 implies that $\mathcal{E}\mathcal{J} < tr(QP)$;

(b) Let x_0 be from the set

$$S = \{x_0 = Bv, \ \|v\| \leq 1\}$$

Then, it follows from Lemma 1 that $\mathcal{J} < \lambda_{max}(B^T P B)$.

A convex optimization can be set up to minimize the upper bound by using Theorem 2.

3 Robust LQ Synthesis

Consider the following uncertain system generalized from (1)-(2):

$$\dot{x}(t) = Ax(t) + Bu(t) + \sum_{i=1}^{p} H_{1i}\xi_i(t), \quad x(0) = x_0 \tag{28}$$

$$z(t) = C_1 x(t) + D_1 u(t) + \sum_{i=1}^{p} H_{2i}\xi_i(t) \tag{29}$$

$$y(t) = C_2 x(t) + D_2 u(t) + \sum_{i=1}^{p} H_{3i}\xi_i(t) \tag{30}$$

where $x(t) \in \mathcal{R}^n$ is the state, $u(t) \in \mathcal{R}$ the control input, $z(t) \in \mathcal{R}^r$ the controlled output, $y(t) \in \mathcal{R}^{r_y}$ is the measured output, and $\xi_i(t) \in \mathcal{R}^{k_i}$ the uncertain variables satisfy the following IQCs:

$$\int_0^T ||\xi_i(t)||^2 dt \leq \int_0^T ||E_{1i}x(t) + E_{2i}\xi(t) + E_{3i}u(t)||^2 dt, \quad \text{as } T \to \infty, \ i = 1, \cdots, p \tag{31}$$

Also, $A, B, C_1, C_2, D_1, D_2, H_{1i}, H_{2i}, H_{3i}, E_{1i}, E_{2i}$ and E_{3i} are constant matrices with appropriate dimensions.

We assume the following:

(A1) (A, B, C_2) is stabilizable and detectable.

(A2) $D_2 = 0$.

Assumption A1 is obviously necessary and sufficient for the existence of a stabilizing controller while assumption A2 is made for technical reason and does not cause lose of generality.

Let a desired controller be of the following form:

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \quad x_c(0) = 0 \tag{32}$$

$$u(t) = C_c x_c(t) + D_c y(t) \tag{33}$$

where $x_c(t) \in \mathcal{R}^{n_c}$ is the state, and A_c, B_c, C_c and D_c are constant matrices of appropriate dimensions. The order of controller may be different from the order of plant.

Similar to the analysis problem, we define the linear quadratic performance index $\mathcal{J}_1$ as following:

$$\begin{aligned}\mathcal{J}_1 &= \int_0^\infty \|z(t)\|^2 dt \\ &= \int_0^\infty \begin{bmatrix} x(t) \\ \xi(t) \\ u(t) \end{bmatrix}^T R \begin{bmatrix} x(t) \\ \xi(t) \\ u(t) \end{bmatrix} dt \end{aligned} \tag{34}$$

where

$$\begin{aligned} R &= [C_1 \;\; D_1 \;\; H_2]^T [C_1 \;\; D_1 \;\; H_2] \\ &= \begin{bmatrix} C_1^T C_1 & C_1^T D_1 & C_1^T H_2 \\ D_1^T C_1 & D_1^T D_1 & D_1^T H_2 \\ H_2^T C_1 & H_2^T D_1 & H_2^T H_2 \end{bmatrix}. \end{aligned} \tag{35}$$

In the above, H_2 is defined in (8).

The robust LQ synthesis problem associated with the uncertain system (28)-(30) is as follows: *Find a controller of the form (32)-(33) such that the closed-loop system of the uncertain system (28)-(30) with controller (32)-(33) is asymptotically stable and the performance index $\mathcal{J}_1$ is bounded by*

$$\mathcal{J}_1 < G(x_0), \qquad G(\cdot) \geq 0 \tag{36}$$

for all admissible uncertainties satisfying (31). Furthermore, the upper bound is minimized.

Besides the short-hand notation in (8)-(11), we define:

$$H_3 = [H_{31} \cdots H_{3p}]; \tag{37}$$

$$\bar{A} = \begin{bmatrix} A + BD_cC_2 & BC_c \\ B_cC_2 & A_c \end{bmatrix} \tag{38}$$

$$\bar{C} = [C_1 + D_1D_cC_2 \;\; D_1C_c] \tag{39}$$

$$\bar{H}_{1i} = \begin{bmatrix} H_{1i} + BD_cH_{3i} \\ B_cH_{3i} \end{bmatrix}; \quad \bar{H}_{2i} = H_{2i} + D_1D_cH_{3i} \tag{40}$$

$$\bar{E}_{1i} = [E_{1i} + E_{3i}D_cC_2 \;\; E_{3i}C_c]; \quad \bar{E}_{2i} = E_{2i} + E_{3i}D_cH_3 \tag{41}$$

$$\bar{H}_1 = [\bar{H}_{11} \ldots \bar{H}_{1p}]; \quad \bar{H}_2 = [\bar{H}_{21} \ldots \bar{H}_{2p}] \tag{42}$$

$$\bar{E}_1^T = [\bar{E}_{11}^T \ldots \bar{E}_{1p}^T]; \quad \bar{E}_2^T = [\bar{E}_{21}^T \ldots \bar{E}_{2p}^T] \tag{43}$$

It is straightforward to verify that the closed-loop system of (28)-(30) together with controller (32)-(33) is given by

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \sum_{i=1}^{p} \bar{H}_{1i}\xi_i(t), \quad \bar{x}_0 = [x_0^T \;\; 0]^T \tag{44}$$

$$z(t) = \bar{C}\bar{x}(t) + \sum_{i=1}^{p} \bar{H}_{2i}\xi_i(t) \tag{45}$$

with

$$\int_0^T ||\xi_i(t)||^2 dt \leq \int_0^T ||\bar{E}_{1i}\bar{x}(t) + \bar{E}_{2i}\xi(t)||^2 dt, \quad \text{as } T \to \infty, \ i = 1, \cdots, p \tag{46}$$

Applying theorem 2, the LQ synthesis problem is solvable using the controller in (32)-(33) if the following system is asymptotically stable and its H_∞-norm is less than 1:

$$\dot{\hat{x}}(t) = \bar{A}\hat{x}(t) + \bar{H}_1 J^{-1/2}\hat{w}(t) \tag{47}$$

$$\hat{z}(t) = \begin{bmatrix} \bar{C} \\ J^{1/2}\bar{E}_1 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} \bar{H}_2 J^{-1/2} \\ J^{1/2}\bar{E}_2 J^{-1/2} \end{bmatrix} \hat{w}(t) \tag{48}$$

Furthermore, (47)-(48) above is the closed-loop system of the controller (32)-(33) together with the auxiliary system defined below:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + H_1 J^{-1/2}\hat{w}(t) + Bu(t) \tag{49}$$

$$\hat{z}(t) = \begin{bmatrix} C_1 \\ J^{1/2}E_1 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} H_2 J^{-1/2} \\ J^{1/2}E_2 J^{-1/2} \end{bmatrix} \hat{w}(t) + \begin{bmatrix} D_1 \\ J^{1/2}E_3 \end{bmatrix} u(t) \tag{50}$$

$$y(t) = C_2\hat{x}(t) + H_3 J^{-1/2}\hat{w}(t) \tag{51}$$

for some $J > 0$.

Given controller (32)-(33), let $\bar{P}$ be a closed-loop matrix of (47)-(48), i.e. $\bar{P}$ is such that

$$\begin{bmatrix} \bar{A}^T\bar{P} + \bar{P}\bar{A} & \bar{P}\bar{H}_1 & \bar{C}^T & \bar{E}_1^T J \\ \bar{H}_1^T\bar{P} & -J & \bar{H}_2^T & \bar{E}_2^T J \\ \bar{C} & \bar{H}_2 & -I & 0 \\ J\bar{E}_1 & J\bar{E}_2 & 0 & -J \end{bmatrix} < 0$$

As a simple consequence of theorem 2, we have the following result:

Theorem 3. *Given the uncertain system (28)-(30), if for some $J > 0$ there exists a controller of the form (32)-(33) such that the closed-loop auxiliary system of (49)-(51) with the controller is stable and has H_∞-norm less than 1, then the robust LQ synthesis problem for system (28)-(30) is solvable with $\mathcal{J}_1 < \bar{x}_0^T\bar{P}\bar{x}_0$.*

Before proceeding further, we recall the following result:

Lemma 4. *[6] Consider the following system:*

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \tag{52}$$

$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \tag{53}$$

$$y(t) = C_2 x(t) + D_{21} w(t) \tag{54}$$

satisfying assumptions (A1)-(A2). Let N_R (respectively N_S) be any matrix whose columns form a basis of the null space of $[B_2^T \;\; D_{12}^T]$ (respectively $[C_2 \;\; D_{21}]$). Then, there exists a *controller of the form (32)-(33) such that the closed-loop system has H_∞ norm less than 1 if and only if there exist symmetric matrices R and S satisfying the following LMIs:*

$$\left[\begin{array}{c|c} N_R^T & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{cc|c} AR + RA^T & RC_1^T & B_1 \\ C_1 R & -I & D_{11} \\ \hline B_1^T & D_{11}^T & -I \end{array}\right] \left[\begin{array}{c|c} N_R & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{55}$$

$$\left[\begin{array}{c|c} N_S^T & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{cc|c} A^T S + AS & SB_1 & C_1^T \\ B_1^T S & -I & D_{11}^T \\ \hline C_1 & D_{11} & -I \end{array}\right] \left[\begin{array}{c|c} N_S & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{56}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \tag{57}$$

Using lemma 4 and theorem 3, we obtain the following result:

Theorem 5. *The following two conditions are equivalent:*

(a) *There exists a controller of the form (32)-(33) such that the closed-loop system of (49)-(51) is asymptotically stable and has H_∞-norm less than 1;*

(b): *Let $\mathcal{N}_R$ (respectively $\mathcal{N}_S$) be any matrix whose columns form a basis of the null space of $[B^T \;\; D_1^T \;\; E_3^T]$ (respectively $[C_2 \;\; H_3]$). There exist symmetric matrices $R, S \in \mathcal{R}^{n \times n}$ such that the following LMIs hold:*

$$\left[\begin{array}{c|c} \mathcal{N}_R^T & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{ccc|c} AR + RA^T & RC_1^T & RE_1^T & H_1 J^{-1} \\ C_1 R & -I & 0 & H_2 J^{-1} \\ E_1 R & 0 & -J^{-1} & E_2 J^{-1} \\ \hline J^{-1} H_1^T & J^{-1} H_2^T & J^{-1} E_2^T & -J^{-1} \end{array}\right] \left[\begin{array}{c|c} \mathcal{N}_R & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{58}$$

$$\left[\begin{array}{c|c} \mathcal{N}_S^T & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{cc|cc} A^T S + SA & SH_1 & C_1^T & E_1^T J \\ H_1^T S & -J & H_2^T & E_2^T J \\ \hline C_1 & H_2 & -I & 0 \\ JE_1 & JE_2 & 0 & -J \end{array}\right] \left[\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{59}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \tag{60}$$

Proof. The proof is a direct application of lemma 4 to the auxiliary system (49)-(51). We first note that the columns of the matrix

$$\text{diag}\{I, I, J^{-1/2}\}\mathcal{N}_R \text{ (respectively } \text{diag}\{I, J^{1/2}\}\mathcal{N}_S)$$

form a basis of the null space of $[B^T \;\; D_1^T \;\; E_3^T J^{1/2}]]$ (resp. $[C_2 \;\; H_3 J^{-1/2}]]$). Then, it is straightforward but tedious to verify that the LMI in (58) is the version of (55) for the auxiliary system, but pre- and post-multiplied by the following matrix:

$$\left[\begin{array}{c|c} I & 0 \\ \hline 0 & J^{-1/2} \end{array}\right]$$

Similarly, the LMI in (59) is the version of (56) for the auxiliary system, pre- and post-multiplied by the following matrix:

$$\left[\begin{array}{c|c} I & 0 \\ \hline 0 & \mathrm{diag}\{I, J^{1/2}\} \end{array}\right]$$

Q.E.D.

In the state feedback case, we can similarly show that one of the LMIs is void and the resulted LMI is fully convex in the variables (matrices). This point is clarified in the following corollary:

Corollary 6. *Given the uncertain system (28)-(30) with uncertainty satisfying IQCs (31). The following two conditions are equivalent:*

(a) *There exists a controller of the following form*

$$u(t) = K_c x(t) \tag{61}$$

such that the closed-loop system for the auxiliary system (49)-(51) is asymptotically stable and has H_∞-norm less than 1;

(b)*: Let $\mathcal{N}_R$ be any matrix whose columns form a basis of the null space of $[B^T \;\; D_1^T \;\; E_3^T]$. There exist symmetric matrices $R \in \mathcal{R}^{n\times n}$, $R > 0$ and $J > 0$ such that the LMI (58) holds.*

Remark 2. Note that from [6], if the LMI problem in Theorem 5 is solvable, the closed-loop matrix is given by

$$\bar{P} = \left[\begin{array}{cc} S & N \\ N^T & * \end{array}\right]$$

where N is to be chosen. In this situation, the robust LQ problem is solvable with $\mathcal{J}_1 < x_0^T S x_0$. This is again an EVP. As a special case, assuming $\mathcal{E} x_0 x_0^T = I$ and denoting $\mathcal{E}$ the mathematical expectation, we have $\mathcal{J}_1 < tr(S)$. Thus, an optimization can be set up to minimize $tr(S)$. In the state feedback case (Corollary 6), the above optimization problem in terms of scaling constants J is in fact convex.

4 Conclusion

This paper studies the robust LQ control problem for a class of linear systems with uncertainty subject to integral quadratic constraints. We show that the solvability of a LMI which is linear in matrix P and scaling parameters J will guarantee the solvability of the robust LQ analysis problem.

For the robust LQ synthesis problem, we show that the solvability of two constrained LMIs which are linear in matrix R and scaling parameters J and linear in matrix S and the inverse of scaling parameters J respectively will guarantee the existence of the desired dynamic output feedback controller. However, the two LMIs are not jointly linear in J, therefore, iterative procedure must be employed to search the solution of the two LMIs. For the case that a state feedback controller is desired, we show that one of the two LMIs is void and we have a fully convex solution to the robust LQ synthesis problem. Optimization of the performance bound has also been discussed.

The results reported in this paper are all sufficient due to the nature of the $\mathcal{S}$-procedure. Further research is encouraged to exploit the possibility of necessary and sufficient results.

REFERENCES

[1] D.S. Bernstein and W.M. Haddad, "Robust stability and performance analysis for state-space systems via quadratic Lyapunov bounds," *SIAM Journal on Matrix Analysis,* vol. 11, pp. 239-271, 1990.

[2] S.S.L. Chang and T.K.C. Peng, "Adaptive guaranteed cost control of systems with uncertain parameters," *IEEE Trans. on Automatic Control,* vol. 17, pp. 474-483, 1972.

[3] I.R. Petersen and D.C. McFarlane, "Optimizing the guaranteed cost in the control of uncertain linear systems," *Robustness of Dynamic Systems with parameter uncertainties*, M. Mansour, S. Balemi and W. Truol (eds.), pp. 241-250, 1992.

[4] A.A. Stoorvogel, "The robust H_2 control problem," *IEEE Trans. on Automatic Control,* vol. 38, pp. 1358-1370, 1993.

[5] P. Gahinet, "Explicit controller formulas for LMI-based H_∞ synthesis." Submitted to *Automatica*, 1994. Also in *Proc. Amer. Contr. Conf.*, pp. 2396-2400, Baltimore, Maryland, June, 1994.

[6] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to H_∞ control," *Int. J. Robust and Nonlinear Contr.*, vol. 4, pp. 421–428, 1994.

[7] Y. Nesterov and A. Nemirovsky, *Interior Point Polynomial Methods in Convex Programming.* Philadelphia: SIAM, 1993.

[8] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory.* Philadelphia: SIAM, 1994.

[9] V. A. Yakubovich, "$\mathcal{S}$-procedure in nonlinear control theory," *Vestnik Leninggradskogo Universiteta, Ser. Matematika*, pp. 62–77, 1971.

[10] A. Megretsky and S. Treil, "Power distribution inequalities: A new method in optimization and robustness of uncertain systems." *Journal of Mathematical Systems, Estimation and Control,* vol. 3, no.3, pp. 301-319, 1993.

[11] P.P. Khargonekar and K. Zhou, "An algebraic Riccati equation approach to H_∞ optimization," *Systems & Control Letters,* vol. 11, pp. 85-91,1988.

[12] P.P. Khargonekar and M.A. Rotea, "Mixed H_2/H_∞ control: A convex optimization approach," *IEEE Trans. on Automatic Control,* vol. 36, pp. 824-837, 1991.