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# Guaranteed Cost Control of Uncertain Nonlinear Systems Via Polynomial Lyapunov Functions

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***Abstract*—In this note, we consider the problem of guaranteed cost control for a class of uncertain nonlinear systems. We derive linear matrix inequality conditions for the regional robust stability and performance problems based on Lyapunov functions which are polynomial functions of the state and uncertain parameters. The performance index is calculated over a set of initial conditions. Also, we discuss the synthesis problem for a class of affine control systems. Numerical examples illustrate our method.**

***Index Terms*—Convex optimization, guaranteed cost control, uncertain nonlinear systems.**

## I. INTRODUCTION

The development of robustness and performance analysis, as well as design techniques for nonlinear systems, is an important field of research. Despite the existence of powerful techniques to cope with these problems in the context of uncertain linear systems, the generalization to the nonlinear case is a difficult task that has motivated many researchers to study these problems. To deal with nonlinear systems, many control design methods use linear control methodologies applied to quasi-linear parameter varying (LPV) representations [1], or by means of polytopic differential inclusions [2]. For instance, the works of [1] and [3] consider LPV techniques (gain-scheduling), and [4] and [5] use robust controllers. However, these approaches may lead to conservativeness since the nonlinearities of the system are not taken into account and they only consider quadratic Lyapunov functions [6]. Moreover, there are some shortcomings related with the quasi-LPV form that may lead to an infinite-dimensional problem [7] or to the instability of the nonlinear closed-loop system [8]. On the other hand, it is well known that the nonlinear optimal control due to difficulties in the solution of the Hamilton–Jacob equation is not a practical approach [9].

Since the work [10] that showed a solution for rational systems in terms of linear matrix inequalities (LMIs) and based on quadratic Lyapunov functions, some authors have proposed more sophisticated Lyapunov functions to derive less conservative conditions using the LMI framework for uncertain nonlinear systems [11], [6]. The advantage of these techniques over the quasi-LPV or polytopic modeling is that they allow the use of polynomial Lyapunov functions by only requiring that the state and parameter vectors belong to a polytopic set instead of all (state and parameter) nonlinearity. As a result, the number of LMI tests is finite overcoming the problems associated with the quasi-LPV (and/or polytopic) methods for uncertain nonlinear systems.

In this note, we derive LMI conditions for the guaranteed cost control problem for a class of uncertain nonlinear systems. These conditions assure the regional stability of the unforced system and determine a bound on the energy of output signal for a given set of initial conditions. Via an iterative algorithm, this approach is extended to the synthesis problem. The main contributions of this technical note are two fold. First, we consider a polynomial Lyapunov function of the type $v(x,\delta) = x'\mathcal{P}(x,\delta)x$, where $\mathcal{P}(x,\delta)$ is a quadratic function of the state $x$ and uncertain parameters $\delta$, that may result in less conservative conditions. Second, the nonlinear system is modeled in an augmented space in which all nonlinearities are taken into account by using scaling matrices associated with them leading to a convex optimization problem in terms of LMI constraints.

The structure of this note is as follows. We state the problem of concern and derive an upper bound on the two-norm of the output performance for a set of initial conditions in Section II. Section III presents an application of the derived method to the guaranteed cost control problem. Section IV presents some concluding remarks.

The notation used in this work is standard. For a real matrix $S$, $S'$ denotes its transpose, $S > 0$ means that $S$ is symmetric and positive–definite, and $\mathrm{He}(S) = S + S'$. The constant matrices $I_n$, $0_{n\times m}$ and $0_n$ denote $n \times n$ identity matrix, $n \times m$ and $n \times n$ zero matrices respectively. The time derivative of a function $r(t)$ will be denoted by $\dot{r}(t)$ and the argument $(t)$ is often omitted. For two polytopes $\mathcal{B}_x \subset \mathbb{R}^n$ and $\mathcal{B}_\delta \subset \mathbb{R}^l$, the notation $\mathcal{B}_x \times \mathcal{B}_\delta$ represents that $(\mathcal{B}_x \times \mathcal{B}_\delta) \subset \mathbb{R}^{(n+l)}$ is a metapolytope obtained by the Cartesian product. The matrix and vector dimensions are omitted whenever they can be determined from the context.

## II. ROBUSTNESS AND PERFORMANCE OF NONLINEAR SYSTEMS

Consider the uncertain nonlinear system

$$\begin{aligned}\dot{x} &= f(x,\delta) = A(x,\delta)x, \qquad x(0) = x_0\\ z &= h(x,\delta) = C(x,\delta)x\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state vector, $\delta \in \mathbb{R}^l$ denotes the uncertain parameters and $z \in \mathbb{R}^r$ denotes the output performance vector.

With respect to the system (1), we consider the following assumptions:

A1) uncertain parameter vector $\delta$, and its time-derivative $\dot{\delta}$ lie in a given polytope $\mathcal{B}_\delta$, with known vertices, i.e., $(\delta, \dot{\delta}) \in \mathcal{B}_\delta$;

A2) origin $x = 0$ of the system is an equilibrium point;

A3) right-hand side of the differential equation is bounded for all values of $x, \delta, \dot{\delta}$ of interest;

A4) $\mathcal{B}_x$ is a given polytope specifying a desired neighborhood of the equilibrium point $x = 0$ of the system.

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The problem of concern in this work is to compute a bound $c$ on the two-norm of the performance output signal, i.e., $\|z\|_2^2 < c$ that holds for all values of $(\delta, \dot{\delta}) \in \mathcal{B}_\delta$ and any $x_0 \in \mathcal{R}_c$ where $\mathcal{R}_c$ is an invariant subset of $\mathcal{B}_x$. To this end, we shall define the output energy as

$$\|z\|_2^2 = \lim_{T\to\infty} \int_0^T z'z\,dt \qquad x_0 \in \mathcal{R}_c \subset \mathcal{B}_x. \tag{2}$$

In order to obtain a solution to the problem of concern in terms of LMIs, let us suppose that (1) can be decomposed as

$$\begin{aligned} \dot{x} &= A_1(x,\delta)x + A_2(x,\delta)\pi \\ z &= C_1(x,\delta)x + C_2(x,\delta)\pi \\ 0 &= \Omega_1(x,\delta)x + \Omega_2(x,\delta)\pi \end{aligned} \tag{3}$$

where the vector $\pi \in \mathbb{R}^m$ is a nonlinear function of $(x, \delta)$ and the matrices $A_1(x,\delta) \in \mathbb{R}^{n\times n}$, $A_2(x,\delta) \in \mathbb{R}^{n\times m}$, $C_1(x,\delta) \in \mathbb{R}^{r\times n}$, $C_2(x,\delta) \in \mathbb{R}^{r\times m}$, $\Omega_1(x,\delta) \in \mathbb{R}^{q\times n}$, and $\Omega_2(x,\delta) \in \mathbb{R}^{q\times m}$ are affine functions of $(x,\delta)$.

Note that the system representation (3) is based on an auxiliary state $\pi$ and the relationship between $\pi$ and $(x,\delta)$ is defined by means of the constraint $\Omega_1(x,\delta)x + \Omega_2(x,\delta)\pi = 0$. Then, we assume for (3) that

A5) the representations (3) and (1) are equivalent, i.e., if the auxiliary state $\pi$ is replaced by its corresponding function of $(x,\delta)$ we recover the original system representation (1).

To illustrate the aforementioned nonlinear decomposition, let us consider the following example borrowed from [10].

*Example 1:* Consider the following uncertain system:

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & \epsilon(x_1^2 - 1) \end{bmatrix} x \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad x \in \mathcal{B}_x \tag{4}$$

with the performance variable $z = [1 \quad 0]\, x$ and suppose that the nonlinear damping factor $\epsilon$ is constant, approximately known and represented by $\epsilon = \epsilon_0 + \epsilon_1\delta$ where $\epsilon_0 = 0.8$, $\epsilon_1 = 0.2$ and the unknown term $\delta$ satisfying $|\delta| \le 1$.

For convenience, we rewrite the previous system as follows:

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & \epsilon_0(x_1^2 - 1) \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & \epsilon_1(x_1^2 - 1) \end{bmatrix} \delta x$$

where $\delta$ is the uncertain time invariant parameter bounded by $\mathcal{B}_\delta = \{\delta : |\delta| \le 1\}$. Note that in this case we have $\dot{\delta} = 0$. The aforementioned system may be represented in terms of (3) as follows:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & -1 \\ 1 & -0.8 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0.8 & 0.2 \end{bmatrix} \pi \\ z &= [1 \quad 0]\, x \\ \pi &= \begin{bmatrix} \delta x_2 \\ x_1x_2 \\ x_1^2 x_2 \\ \delta x_1^2 x_2 \end{bmatrix} \\ \Omega_1(x,\delta) &= \begin{bmatrix} 0 & \delta \\ x_2 & 0 \\ 0 & x_1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \Omega_2(x,\delta) &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & x_1 & -1 & 0 \\ 0 & 0 & \delta & -1 \end{bmatrix}. \end{aligned}$$

*Remark 1:* The representation (1) of nonlinear systems and its nonlinear decomposition (3) are not unique and, until now, there is no a systematic way to compute them. As a result, a particular choice of $A(x,\delta)$ may provide a poor performance. However, for a representation $f(x) = A_0(x)x$ and some continuous matrix-valued function $A_0 : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$, any representation of $f(x)$ can be parameterized as $A(x) = A_0(x) + M(x)$, where $M : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ satisfies $M(x)x = 0$ [12]. Hence, we can use this parameterization to reduce the conservatism of choosing $A(x,\delta)$ via Finsler's lemma [2]. In particular, we apply this technique in Theorem 1 adding multipliers associated to the equality constraints, for further details see the proof of Theorem 1 and [11]. A similar technique was proposed in [13] for the $\mathcal{H}_\infty$ control of nonlinear systems with quasi-LPV representation. □

To analyze the stability of [3] and estimate its output energy, we use in this note a Lyapunov function which is more complex than those ones based on quadratic stability methods. With a more complex Lyapunov function we have more degrees of freedom to be exploited and the results are probably less conservative than the ones obtained from the usual quadratic stability notion such as the LFR modeling [10], gain-scheduling [1], and polytopic differential inclusions [4]. In the sequel, we define the class of Lyapunov functions to be considered in this note.

*Lyapunov Function Candidate:* Consider the following Lyapunov function candidate:

$$v(x,\delta) = x'\mathcal{P}(x,\delta)x \quad \mathcal{P}(x,\delta) = \begin{bmatrix} \Theta(x,\delta) \\ I_n \end{bmatrix}' P \begin{bmatrix} \Theta(x,\delta) \\ I_n \end{bmatrix} \tag{5}$$

where the matrix $\Theta(x,\delta) \in \mathbb{R}^{n_\theta \times n}$ is a given affine function of $(x,\delta)$ and $P$ is symmetric matrix to be determined. For convenience, let us introduce the auxiliary vector $\xi$ defined as follows:

$$\xi = \begin{bmatrix} \Theta(x,\delta) \\ I_n \end{bmatrix} x, \qquad \xi \in \mathbb{R}^{n+n_\theta}. \tag{6}$$

With the aforementioned notation, it follows that $v(x,\delta) = x'\mathcal{P}(x,\delta)x$ may be represented as $v(\xi) = \xi' P \xi$. Then, the time-derivative of $v(x,\delta)$ is given by $\dot{v}(x,\delta) = \dot{x}'\mathcal{P}(x,\delta)x + x'\mathcal{P}(x,\delta)\dot{x} + x'\dot{\mathcal{P}}(x,\delta)x = \dot{\xi}'P\xi + \xi'P\dot{\xi}$. To compute the term $\dot{\mathcal{P}}(x,\delta)$ or, equivalently, $\dot{\xi}$, observe that $d/dt(\Theta(x,\delta)x) = \dot{\Theta}(x,\delta)x + \Theta(x,\delta)\dot{x}$.

From the definition of the Lyapunov matrix in (5), the matrix $\Theta(x,\delta)$ is an affine matrix function of $(x,\delta)$. Then, we can represent it as $\Theta(x,\delta) = \sum_{j=1}^n T_j x_j + \sum_{j=1}^l U_j \delta_j + V$, where $x_i, \delta_i$ are the entries of the vectors $x$ and $\delta$ respectively, and $T_j, U_j, V$ are constant matrices of structure having the same dimensions of $\Theta(x,\delta)$.

Consequently, we can rewrite the term $\dot{\Theta}(x,\delta)x$ as follows:

$$\begin{aligned} \dot{\Theta}(x,\delta)x &= \left(\sum_{j=1}^n T_j \dot{x}_j + \sum_{j=1}^l U_j \dot{\delta}_j\right) \\ &= \sum_{j=1}^n T_j x s_j \dot{x} + \sum_{j=1}^l U_j \dot{\delta}_j x = \tilde{\Theta}(x)\dot{x} + \hat{\Theta}(\dot{\delta})x \end{aligned} \tag{7}$$

where $s_j$ is the $j$th row of the identity matrix $I_n$, $\tilde{\Theta}(x) = \sum_{j=1}^n T_j x s_j$ and $\hat{\Theta}(\dot{\delta}) = \sum_{j=1}^l U_j \dot{\delta}_j$.

From [2, Sec. 6.2] and [14], we can determine an upper-bound on the output energy of (1) by requiring that

$$\phi_1(\|x\|) \le v(x,\delta) \le \phi_2(\|x\|)$$

and

$$\dot{v}(x,\delta) < -z'z, \forall\, x \in \mathcal{B}_x, (\delta,\dot{\delta}) \in \mathcal{B}_\delta \tag{8}$$

where $\phi_i(\cdot)$, for $i = 1, 2$, are class $\mathcal{K}$ functions. In fact, these conditions imply that $\|z\|_2^2 < v(x_0, \delta(0))$. Since $\delta(0)$ may take any value in $\mathcal{B}_\delta$, we get $\|z\|_2^2 < v(x_0, \delta),\ \forall\, (\delta, \dot{\delta}) \in \mathcal{B}_\delta$. Moreover, $x_0$ has to belong to an invariant subset $\mathcal{R}_c$ of $\mathcal{B}_x$. To characterize the region $\mathcal{R}_c$, let us define

$$\mathcal{R}_c := \{x : v(x, \delta) \leq c \quad \forall\, \delta \in \mathcal{B}_\delta,\ 0 \leq c \leq c^*\} \tag{9}$$

where $c^*$ is a positive scalar given by

$$c^* = \max c \text{ such that } \mathcal{R}_c \subset \mathcal{B}_x. \tag{10}$$

Observe that the inclusion $\mathcal{R}_c \subset \mathcal{B}_x$ in (10) jointly with (8) imply the region $\mathcal{R}_c$ is positively invariant. In addition, the upper bound on $\|z\|_2^2$ will depend on the size of the particular invariant subset $\mathcal{R}_c$ to be considered. For this reason, in this note, we will consider the problem of estimating the upper bound for the largest possible $\mathcal{R}_c$, i.e., with $c$ being as close as possible to $c^*$.

Now, with these definitions, we can state the main result of this work as follows.

*Theorem 1:* Consider that system (3) satisfies A1)–A5). Let $\mathcal{B}_x$ and $\mathcal{B}_\delta$ be polytopes with given vertices. Let $\Theta(x, \delta)$ be a given affine matrix function of $(x, \delta)$. Consider the definition of $\tilde{\Theta}(x)$ and $\hat{\Theta}(\dot{\delta})$ in (7) further define $G = [\,0_{q\times n_\theta} \quad \Omega_1(x, \delta)\,]$ and

$$E = \begin{bmatrix} I_{n_\theta} & -(\tilde{\Theta}(x) + \Theta(x, \delta)) \\ 0 & I_n \end{bmatrix}$$
$$N = \begin{bmatrix} 0 & M \\ I_{n_\theta} & -\Theta(x, \delta) \end{bmatrix}$$
$$H = \begin{bmatrix} 0 \\ A_2(x, \delta) \end{bmatrix}$$
$$F = \begin{bmatrix} 0 & \hat{\Theta}(\dot{\delta}) \\ 0 & A_1(x, \delta) \end{bmatrix}$$
$$M = \begin{bmatrix} x_2 & -x_1 & 0 & \cdots & 0 \\ 0 & x_3 & -x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_n & -x_{n-1} \end{bmatrix}.$$

Suppose there exist matrices $P = P'$, $R$, and $L_{ij}$ (for $i, j = 1, 2, 3$) that solve the optimization problem, shown in (11) and (12) at the bottom of the page, where the LMIs are satisfied at all vertices of $\mathcal{B}_x \times \mathcal{B}_\delta$. Then, $v(x, \delta)$ in (5) is a Lyapunov function for the system and the two-norm of the output signal satisfies

$$\|z\|_2^2 < v(x_0, \delta) \leq c^* \quad \forall\, x_0 \in \mathcal{R}_{c^*} \text{ and } (\delta, \dot{\delta}) \in \mathcal{B}_\delta \tag{13}$$

where $\mathcal{R}_{c^*}$ and $c^*$ are defined in (9) and (10), respectively.

By zeroing appropriate partitions of $P$, Theorem 1 can recover the results from quadratic stability [10]. However, we will increase the conservativeness. To illustrate this point, let us consider the following example.

TABLE I
DIFFERENT ESTIMATES OF THE UPPER-BOUND ON $\|z\|_2$ FOR (4)

| *Upper-bound on* $\|z\|_2^2$ | *Lyapunov Matrix* | | |
|---|---|---|---|
| | *(iii)* | *(ii)* | *(i)* |
| $c = \lambda^{-1}$ | *1.6* | *1.5* | *1.0* |

*Example 2:* Consider Example (1). Let $\mathcal{B}_x$ be defined as $\{x : |x_i| \leq 0.8,\ i = 1, 2\}$. Define the Lyapunov function candidate by considering $\Theta(x, \delta) = [\,x_1 I \quad x_2 I\,]'$. Consequently, $\hat{\Theta}(\dot{\delta}) = 0$ and $\tilde{\Theta}(x) = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$.

Now, consider the following partition of the constant matrix $P = \begin{bmatrix} P_2 & P_1' \\ P_1 & P_0 \end{bmatrix}$, with $P_1 \in \mathbb{R}^{2\times 4}$. Three different cases will be considered: i) $P_0$, $P_1$, and $P_2$ free, $\mathcal{P}(x, \delta)$ is quadratic in $(x, \delta)$; ii) $P_0$, $P_1$ free, and $P_2 = 0$, $\mathcal{P}(x, \delta)$ is affine in $(x, \delta)$; and iii) $P_1 = 0$, $P_2 = 0$, and $P_0$ is free, $\mathcal{P}(x, \delta)$ is a fixed matrix characterizing the quadratic stability. Table I shows estimated upper bounds on $\|z\|_2$ obtained with Theorem 1 and (10) for three types of Lyapunov functions. As expected, more complex Lyapunov functions achieve less conservative results, thus justifying the required extra computation.

Until now, we proposed a methodology for robust stability and performance analysis for a class of uncertain nonlinear systems. In Section III, we will consider the synthesis problem for a class of affine control systems.

## III. CONTROL

Consider the uncertain nonlinear system

$$\begin{aligned} \dot{x} &= A(x, \delta)x + B(x, \delta)u, \qquad x(0) = x_0 \\ z &= C(x, \delta)x + D(x, \delta)u \end{aligned} \tag{14}$$

where $u \in \mathbb{R}^p$ denotes the control input and $B(x, \delta)$ and $D(x, \delta)$ are affine matrix functions of $(x, \delta)$ with appropriate dimensions.

In this section, we are concerned with the problem of determining a control law of the type

$$u = K(x, \delta)x = K_1 x + K_2 \pi \tag{15}$$

where the matrices $K_1 \in \mathbb{R}^{p\times n}$ and $K_2 \in \mathbb{R}^{p\times m}$ are fixed gains to be determined in order to minimize an upper bound on the output energy of (14) (guaranteed cost control).

To use Theorem 1 for synthesis purposes, we can replace $A_i(x, \delta)$ (for $i = 1, 2$) by their corresponding closed-loop form given by $A_i(x, \delta) + B(x, \delta)K_i$. As a consequence, the control gains $K_i$ will appear only in the LMI (12) multiplying the scaling variables $L_{j2}$ (for $j = 1, 2, 3$). Based on this property, we propose an iterative design procedure in which the gains $K_i$ appear explicitly as decision variables in a convex LMI subproblem to be solved. More details about this design procedure will be given later in this section. Now, let us draw some remarks concerning the structure of the gains $K_i$.

$$\min \operatorname{trace}(P + RN + N'R') \text{ subject to:}$$
$$P + RN + N'R' > 0 \tag{11}$$
$$\operatorname{He}\begin{bmatrix} -L_{12}E & L_{11}G + L_{12}F + L_{13}N & L_{11}\Omega_2(x, \delta) + L_{12}H & 0 \\ P - L_{22}E & L_{21}G + L_{22}F + L_{23}N & L_{21}\Omega_2(x, \delta) + L_{22}H & 0 \\ -L_{32}E & L_{31}G + L_{32}F + L_{33}N & L_{31}\Omega_2(x, \delta) + L_{32}H & 0 \\ 0 & [\,0_{r\times n_\theta} \quad C_1(x, \delta)\,] & C_2(x, \delta) & -0.5 I_r \end{bmatrix} < 0 \tag{12}$$

TABLE II
UPPER-BOUNDS ON THE COST FUNCTION FOR ALGORITHM 1 WITH $x(0) = x_0$

| quadratic $\mathcal{P}(x,\delta)$ | iterations | | |
|---|---|---|---|
| | 1st | 2nd | 3rd |
| $v(x(0))$ | 5.23 | 2.26 | 2.26 |

Notice that the auxiliary state $\pi$ is a vector function containing nonlinear terms in $(x,\delta)$. Let us consider the situation in which some of the states are not available for feedback and the system has some uncertain parameters. In this case, the control gains corresponding to the entries of $\pi$ depending on these unavailable states and uncertain parameters must be zeroed to remove them from the control law. In addition, we may consider the control law (15) as parameter dependent if the parameters are available online as in LPV control [7]. Theorem (1) provides the foundation for our synthesis framework. Suppose that $K_1$ and $K_2$ are given and consider the notation

$$\begin{aligned} A_{K1}(x,\delta) &= A_1(x,\delta) + B(x,\delta)K_1 \\ A_{K2}(x,\delta) &= A_2(x,\delta) + B(x,\delta)K_2 \\ C_{K1}(x,\delta) &= C_1(x,\delta) + D(x,\delta)K_1 \\ C_{K2}(x,\delta) &= C_2(x,\delta) + D(x,\delta)K_2. \end{aligned} \quad (16)$$

As a first step, apply Theorem 1 for closed-loop stability analysis to obtain the upper bound on the output energy (guaranteed cost) for we given control gains. Next, with the scaling matrices $L_{j2}$ (for $j = 1,2,3$) fixed obtained in the previous step, use again Theorem 1 to improve the guaranteed cost by recomputing these control gains and remaining free decision variables. The idea of the design procedure is to iterate on these two above steps until the convergence to a local optimum or to the achievement of an acceptable guaranteed cost. Notice that each step consists of solving convex LMI problems. Based on this observation, even if the design procedure is not globally convex[1] the convergence to local optimum is guaranteed. This type of design procedure is not new from the literature, see, for instance, [16]. In the following, we summarize the design algorithm.

*Algorithm 1:*

Step 1) Determine a local stabilizing controller for the nominal system (14) with any standard stabilization technique.

Step 2) Replace the matrices $A_1(x,\delta), A_2(x,\delta), C_1(x,\delta), C_2(x,\delta)$ in Theorem 1 by their corresponding closed loop form in (16). Compute the guaranteed cost (13) by solving Theorem 1 for suitable polytopes $\mathcal{B}_x$ and $\mathcal{B}_\delta$.

Step 3) For fixed matrices $L_{12}$, $L_{22}$, and $L_{32}$ obtained from the previous step, consider the control gains $K_1$ and $K_2$ as decision variables. Then, recompute the control gains and the guaranteed cost by solving the optimization problem in Theorem 1.

Step 4) Iterate over Steps 2) and 3) until convergence or the achievement of a satisfactory guaranteed cost.

[1]Notice that the matrix inequality in (11), for the design case, is indeed a bilinear matrix inequality (BMI) [15].

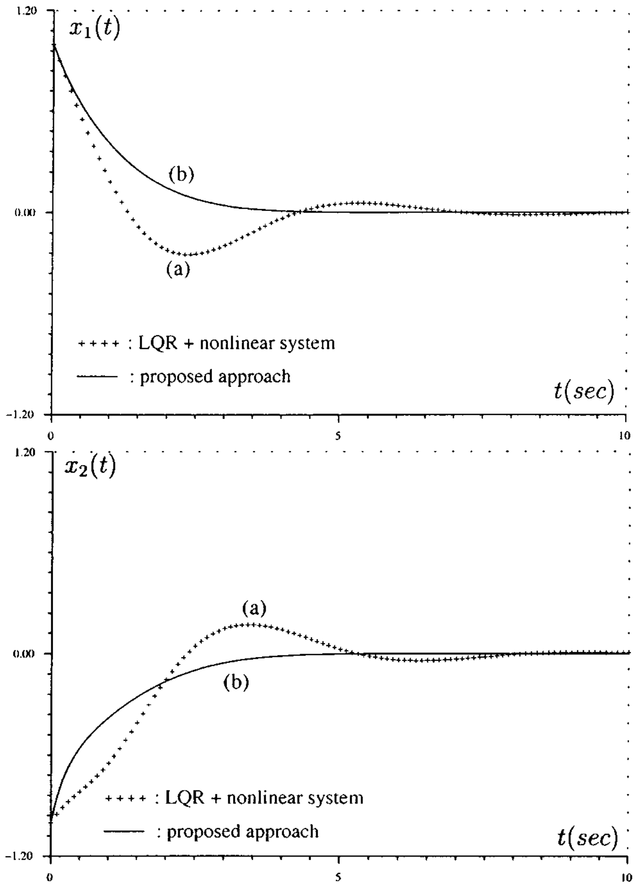


Fig. 1. (a) LQR (plus sign). (b) Proposed controller (solid line).

At each step of the previous algorithm, the regional stability of the closed-loop system and the nonincreasing of the guaranteed cost are assured. As previously mentioned, this algorithm will always converge to a local minimum. To overcome the problem of finding an initial stabilizing controller [Step 1)], we may use the classical LQR technique [2] applied to the linearized model of the nominal nonlinear system. Then, in the next step, we use Theorem 1 to estimate the polytopes $\mathcal{B}_x, \mathcal{B}_\delta$ for the uncertain nonlinear system.

Let us illustrate the aforementioned design algorithm through a numerical example where we consider the cost function for a given initial condition $x(0) = x_0$ and a linear state feedback.

*Example 3:* Consider the following uncertain nonlinear system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & (0.8+0.2\delta)(1-x_1^2) \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ x &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{B}_x \\ z &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ x_0 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad |\delta| \le 1 \end{aligned} \quad (17)$$

where $\delta$ is a time-invariant parameter and $\mathcal{B}_x = \{x : |x_i| \le 5,\ i = 1,2\}$.

Also, consider the following nonlinear decomposition of (17):

$$\dot{x} = A_1(\delta)x + A_2(\delta)\pi + Bu \qquad z = C_1 x + Du \quad (18)$$

where the matrices and vectors are given by $B = [0 \quad 1]'$, $D = [0 \quad 1]'$, and

$$A_1(\delta) = \begin{bmatrix} 0 & 1 \\ -1 & (0.8 + 0.2\delta) \end{bmatrix}$$
$$A_2(\delta) = \begin{bmatrix} 0 & 0 \\ 0 & -(0.8 + 0.2\delta) \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \pi = \begin{bmatrix} x_1 x_2 \\ x_1^2 x_2 \end{bmatrix}.$$

To stabilize (18), we used LMI-LQR techniques from [2] applied to the linearized model of the nominal system ($\delta = 0$) yielding $K = [-0.4 \quad -2.1 \quad]$. Then, this state-feedback is used as starting point of Algorithm 1. Table II shows the value of $v(x_0)$ for each iteration of algorithm 1, by considering $\Theta(x) = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \\ 0 & x_2 \end{bmatrix}$.

After three iterations, we obtained $K_1 = [-0.8 \quad -3.5]$ in which $v(x_0) = 2.26$. For comparison purposes, Fig. 1 shows the trajectories $x_1(t)$ and $x_2(t)$ for the following controllers: a) LQR (obtained from the linearized model) and b) proposed method.

## IV. Concluding Remarks

The purpose of this note was to show that by using polynomial Lyapunov functions we get less conservative LMI conditions for analysis and design of guaranteed cost control. To ascertain the system stability and performance criterion, we used Lyapunov functions of the type $v(x, \delta) = x'\mathcal{P}(x, \delta)x$, where the matrix $\mathcal{P}(x, \delta)$ is a quadratic function of $x$ and $\delta$. Based on the analysis results, an iterative technique for the synthesis problem was also presented. An interesting point of the design technique is that it can be used to determine different types of control laws, such that nonfragile, gain-scheduling and output feedback controllers. Numerical examples showed the potential of this approach as well as a comparative study among constant, affine and quadratic Lyapunov matrices demonstrating that more complex Lyapunov functions can lead to less conservative results. Future research will be concentrated on the design problem in order to obtain globally convex LMI conditions.

## Appendix

### A. Proof of Theorem 1

Suppose that the (11) and (12) have a solution at all vertices of $\mathcal{B}_x \times \mathcal{B}_\delta$. Then, by convexity, it is also satisfied $\forall\, x \in \mathcal{B}_x$ and $\forall\, (\delta, \dot{\delta}) \in \mathcal{B}_\delta$. For readability, this proof is divided in the following steps.

Step 1) Consider LMI (11). Define a matrix $S = [0 \quad I_n]$. For a sufficient small-positive scalar $\epsilon_1$, it is possible to add $\epsilon_1 S'S$ to (11) without changing its signal, i.e., $P + RN + N'R' \geq \epsilon_1 S'S$. Pre- and post-multiplying it by $\xi'$ and its transpose, respectively, we get $v(x, \delta) = v(\xi) = \xi' P \xi \geq \epsilon_1 x'x$ for all $x \in \mathcal{B}_x$ and $\delta \in \mathcal{B}_\delta$, since $N\xi = 0$ by construction. From Assumption A3), the entries of $N$ are bounded. Then, there exists a sufficient large positive scalar $\epsilon_2$ such that $\epsilon_2 I_{n+n_1} \geq P + RN + N'R'$. Thus, multiplying it by $\xi'$ and $\xi$, respectively, yields $\epsilon_2(x'x + x'\Theta(x,\delta)'\Theta(x,\delta)x) \geq \xi' P \xi$. Keeping in mind that $x$ and $\delta$ belong to polytopes, there exists a positive scalar $\epsilon_3$ such that $\epsilon_3 I_n \geq \Theta(x,\delta)'\Theta(x,\delta)$. Hence, $v(x, \delta) = \xi' P \xi \leq \epsilon_2(1 + \epsilon_3)x'x$ for all $x \in \mathcal{B}_x$ and $\delta \in \mathcal{B}_\delta$.

Step 2) Applying the *Schur* complement on (12), we can rewrite it as follows:

$$\begin{bmatrix} 0 & P & 0 \\ P & \begin{bmatrix} 0 & 0 \\ 0 & C_1'C_1 \end{bmatrix} & \begin{bmatrix} 0 \\ C_1'C_2 \end{bmatrix} \\ 0 & [0 \quad C_2'C_1] & C_2'C_2 \end{bmatrix} + \mathrm{He}\left( [\,L_{ij}\,]_{i,j=1,2,3} \begin{bmatrix} 0 & G & \Omega_2 \\ -E & F & H \\ 0 & N & 0 \end{bmatrix} \right) < 0.$$

Pre- and postmultiplying the aforementioned LMI by $[\,\dot{\xi}' \quad \xi' \quad \pi'\,]$ and its transpose, respectively, leads to

$$\begin{bmatrix} \dot{\xi} \\ \xi \\ \pi \end{bmatrix}' \begin{bmatrix} 0 & P & 0 \\ P & \begin{bmatrix} 0 & 0 \\ 0 & C_1'C_1 \end{bmatrix} & \begin{bmatrix} 0 \\ C_1'C_2 \end{bmatrix} \\ 0 & [0 \quad C_2'C_1] & C_2'C_2 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \xi \\ \pi \end{bmatrix} < 0$$
$$\forall\, (x, \delta, \dot{\delta}) \in \mathcal{B}_x \times \mathcal{B}_\delta : \begin{cases} -E\dot{\xi} + F\xi + H\pi = 0 \\ N\xi = 0 \\ \Omega_1 x + \Omega_2 \pi = 0 \end{cases} \quad (19)$$

Consider the following partition of the vector $\xi = [\xi_a' \quad \xi_b']'$, where $\xi_a = \Theta x$ and $\xi_b = x$. Taking the time derivative of $\xi_a$ yields $\dot{\xi}_a = \hat{\Theta}x + (\tilde{\Theta} + \Theta)\dot{x}$. It is easy to verify that the above equality has the following compact form $-E\dot{\xi} + F\xi + H\pi = 0$. Then, it is possible to write (19) as follows: $\dot{\xi}' P \xi + \xi' P \dot{\xi} < -(x'C_1'C_1 x + 2x'C_1'C_2\pi + \pi' C_2'C_2\pi)$, $\forall\, (x, \delta, \dot{\delta}) \in \mathcal{B}_x \times \mathcal{B}_\delta$. Since $z = C_1 x + C_2 \pi$, the above expression is equivalent to $\dot{v}(x, \delta) = x' \left(A(x, \delta)'\mathcal{P}(x, \delta),\right.$ $\left. +\mathcal{P}(x, \delta)A(x, \delta) + \dot{\mathcal{P}}(x, \delta)\right) x < -z'z$, $\forall\, (x, \delta, \dot{\delta}) \in \mathcal{B}_x \times \mathcal{B}_\delta$.

Step 3) From the previous analysis and [14], (1) is locally exponentially stable and $v(x, \delta) = x'\mathcal{P}(x, \delta)x$ is a Lyapunov function for the origin. Keep in mind that $\epsilon_a x'x \leq v(x, \delta) \leq \epsilon_b x'x$ and $\dot{v}(x, \delta) < -z'z$ for all $x \in \mathcal{B}_x$ and $(\delta, \dot{\delta}) \in \mathcal{B}_\delta$. Moreover, from (9) and (10), $\mathcal{R}_{c^*}$ is a positively invariant set. In other words, for all $x_0 \in \mathcal{R}_{c^*}$, $x(t)$ approaches the origin as $t \to \infty$.

Step 4) Integrating $\dot{v}(x, \delta) < -z'z$ from 0 to $T$, we have $v(x(T), \delta(T)) - v(x(0), \delta(0)) < -\int_0^T z'z\, dt$, $\forall\, T > 0$ and $x_0 \in \mathcal{R}_{c^*}$. As $T \to \infty$, the previous expression leads to $\|z\|_2^2 = \lim_{T \to \infty} \int_0^T z'z\, dt < v(x_0, \delta) \leq c^*$, $\forall\, x_0 \in \mathcal{R}_{c^*}$ and $(\delta, \dot{\delta}) \in \mathcal{B}_\delta$.

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# Task-Space Adaptive Control of Robotic Manipulators With Uncertainties in Gravity Regressor Matrix and Kinematics

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***Abstract*—Thus far, most research in adaptive control of robotic manipulators has assumed that models of regressor matrix and kinematics are known exactly. To overcome these drawbacks, we propose in this note a task-space adaptive law for setpoint control of robots with uncertainties in gravity regressor matrix and kinematics. In addition, we investigate the stability problem when an estimated task-space velocity is used in the feedback loop. Sufficient conditions for choosing the feedback gains, gravity regressor, and Jacobian matrix are presented to guarantee the stability.**

***Index Terms*—Setpoint control, stability, task space, uncertain kinematics, uncertain regressor.**

## I. Introduction

In most applications of robots, a desired path of the end effector is usually specified in task coordinates. However, a majority of the robot controllers in the literature were joint-space controllers [1]–[10]. In order to control the robot with these controllers, an inverse kinematics problem should be solved to generate a desired path in joint coordinates.

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A more effective control strategy without using the inverse kinematics is the task-space control method [1], [11]–[18]. In this method, a task oriented information is used directly in the feedback control law. Takegaki and Arimoto [1] proposed a transposed Jacobian controller for setpoint control in Cartesian coordinates. Later, this study is analyzed further by Kelly *et al.* [11]–[14]. A local feedback control law with imperfect Jacobian matrix from Cartesian space to visual space is proposed by Miyazaki and Masutani [15]. In these controllers [1]–[15], an exact knowledge of the robot kinematics from joint space to task space is required. However, since the robot is interacting with its environment, its kinematics changes for different tasks when it picks up different objects. To overcome the problem of uncertain kinematics, Cheah *et al.* [16]–[18] proposed task-space feedback laws with uncertain kinematics and Jacobian matrix from joint space to task space.

In most of the setpoint controllers, an exact knowledge of a gravitational force is used in the controllers. When the gravitational force is uncertain, several adaptive control laws [2], [3], [6], [10], [16], [17] using a gravity regressor are proposed for compensating the gravitational force. However, the exact knowledge of the gravity regressor matrix is assumed to be known in these controllers. Unfortunately, no model can be obtained precisely. In addition, the gravity regressor also changes when the robot picks up different objects.

In this note, we propose a task-space adaptive law for setpoint control of robot with uncertainties in both the gravity regressor matrix and kinematics. In addition, we investigate the stability problem when an estimated task-space velocity is used in the feedback loop. To the best of our knowledge, such problem has not been studied before. Therefore, it is unknown whether the stability of the robot's motion can still be guaranteed in the presence of such uncertainties. We shall present sufficient conditions for choosing the feedback gains, gravity regressor, and Jacobian matrix to guarantee the stability.

## II. Problem Formulation

We consider a class of robotic manipulators with all revolute joints. These are sometimes said to be articulated robots since their configuration of links and joints corresponds to that of a human arm. In most applications, a desired path for the robot end effector is specified in task space such as visual space or Cartesian space. Let $X \in R^m$ represents a task-space vector [16]

$$X = h(q) \tag{1}$$

where $m \leq n$ and $h(\cdot) \in R^n \rightarrow R^m$ is generally a nonlinear transformation describing the relation between the joint space and task space. The task-space velocity $\dot{X}$ is related to joint-space velocity $\dot{q}$ as [16]

$$\dot{X} = J(q)\dot{q} \tag{2}$$

where $J(q)$ is a Jacobian matrix of the mapping from joint space to task space. Note that $h(q)$ and $J(q)$ are trigonometric functions of $q$.

The equation of motion for the robotic manipulator is given in joint space as [6]

$$M(q)\ddot{q} + \left(B_0 + \frac{1}{2}\dot{M}(q) + S(q,\dot{q})\right)\dot{q} + g(q) = \tau \tag{3}$$

where $q \in R^n$ denotes joint angles, $n$ denotes degrees of freedom of the robot, $M(q) \in R^{n\times n}$ is an inertia matrix, $B_0 \in R^{n\times n}$ denotes a diagonal viscous friction matrix, $S(q,\dot{q})\dot{q} = (1/2)\dot{M}(q)\dot{q} - (1/2)\{(\partial/\partial q)\dot{q}^T M(q)\dot{q}\}^T$, $g(q) = (\partial P(q)/\partial q)^T \in R^n$ is a gravitational force, $\tau \in R^n$ denotes control inputs, and $P(q)$ is the potential energy due to gravitational force. The gravitational force can be com-