Robust $\mathcal{H}_\infty$ Filter Design via Parameter-Dependent Lyapunov Functions*

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Abstract

This paper is concerned with the problem of $\mathcal{H}_\infty$ filtering for linear continuous-time systems with uncertain time-varying parameters in the matrices of the state-space signal model. The admissible values of the parameters and their rates of variation are assumed to belong to a given polyhedral region. Based on a parameter-dependent Lyapunov function, which is quadratic in the uncertain parameters, we develop an $\mathcal{LMI}$ method for designing a linear stationary, asymptotically stable filter, which ensures a prescribed performance in a $\mathcal{H}_\infty$ sense.

Keywords: Robust filtering, $\mathcal{H}_\infty$ filtering, uncertain systems, time-varying parameters, parameter-dependent Lyapunov functions.

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1 Introduction

Over the past two decades, there has been a lot of interest on the problem of $\mathcal{H}_\infty$ filtering; see, e.g. [12], [14], [15] and the references therein. In $\mathcal{H}_\infty$ filtering the noise signals are assumed to be deterministic with bounded energy (or average power) and the problem is to design a filter which ensures that the induced $\mathcal{L}_2$-gain from the noise signals to the filtering error remains bounded by a prescribed value. It has been known that the $\mathcal{H}_\infty$ filtering approach provides both a guaranteed noise attenuation level and robustness against unmodeled dynamics [15].

Recently, attention has been given to the problem of robust $\mathcal{H}_\infty$ filtering for linear systems with parameteric uncertainty, namely designing an asymptotically stable linear filter which ensures a prescribed performance in a $\mathcal{H}_\infty$ sense, irrespective of the uncertainty. Several robust $\mathcal{H}_\infty$ filtering approaches have been developed over the past few years. In the case of linear systems with norm-bounded parameter uncertainty, one may cite, the Riccati equation approaches developed in [2], [6] and [17], and the linear matrix inequalities ($\mathcal{LMI}$s) based technique in [10]. Very recently, an $\mathcal{LMI}$ approach for robust $\mathcal{H}_\infty$ filtering for linear systems with polytope type uncertainty has been proposed in [7], whereas [13] treated the design of a robust $\mathcal{H}_\infty$ filter with pole placement constraints.

A common feature of the aforementioned robust $\mathcal{H}_\infty$ filtering methods is that they are based on the notion of *quadratic stability*, i.e. a fixed parameter-independent Lyapunov function is used to guarantee robust stability and a prescribed $\mathcal{H}_\infty$ performance bound for the filtering error dynamics. Moreover, it turns out that stability and the guaranteed performance hold even when the parameters change arbitrarily fast, which can be quite conservative in many applications. Motivated by the recent developments on robust stability and control of linear uncertain systems via parameter-dependent Lyapunov functions (see, e.g. [5], [8], [9], [16], [18] and the references therein), which are known to provide less conservative results than the quadratic stability based methods, this paper investigates the design of a robust $\mathcal{H}_\infty$ filter based on a parameter-dependent Lyapunov function.

In this paper we develop $\mathcal{LMI}$ conditions to solve the problem of robust $\mathcal{H}_\infty$ filtering for linear systems with uncertain time-varying parameters which appear affinely in the matrices of the state-space model of the signal generating system. The parameters and their rates of variation are assumed to belong to a polytope with known vertices. The problem addressed is the design of a linear stationary, asymptotically stable, filter which provides a guaranteed $\mathcal{H}_\infty$ performance. A new robust $\mathcal{H}_\infty$ filter design methodology is proposed based on the notion of *bi-quadratic stability* [16], i.e. using a parameter-dependent Lyapunov function which is quadratic in the system state and in the parameters. The proposed method incorporates information on the bounds on the rates of change of the parameters and is less conservative than earlier methods [2], [6], [7], [10], [13], [17], which are based on quadratic stability. In particular, the new filtering method includes the quadratic stability based approach as a special case.

The paper is organized as follows. In Section 2, the statement of the robust $\mathcal{H}_\infty$ filtering problem addressed in this paper is presented and some preliminary results on linear time-varying systems are reviewed. In Section 3, we develop an $\mathcal{LMI}$ methodology for the design of a robust linear stationary $\mathcal{H}_\infty$ filter which uses a parameter-dependent Lyapunov function. Conclusions are given in Section 5.

Notation. $\Re^n$ denotes the n-dimensional Euclidean space, $\Re^{n\times m}$ is the set of $n\times m$ real matrices, I_n is the $n\times n$ identity matrix and diag$\{\cdots\}$ stands for a block-diagonal matrix. For a symmetric block matrix, the symbol $\star$ denotes the transpose of the symmetric blocks outside the main diagonal block, and the notation $S>0$ (respectively $S\geq 0$), for a real matrix S, means that S is symmetric and positive definite (respectively, positive semi-definite). $\mathcal{L}_2$ denotes the space of square integrable vector function on $[0,\infty)$ with norm $\|\cdot\|_2 := (\int_0^\infty \|\cdot\|^2 dt)^{\frac{1}{2}}$, where $\|\cdot\|$ denotes the Euclidean vector norm.

2 Problem Formulation

Consider the following linear uncertain continuous-time system

$$\begin{aligned} \dot{x}(t) &= A(\theta)x(t) + Bw(t) \\ y(t) &= C(\theta)x(t) + Dw(t) \\ z(t) &= R(\theta)x(t) \end{aligned} \tag{1}$$

with $\theta := (\theta_1,\cdots,\theta_p)\in\Re^p$ and

$$A(\theta) = A_0 + \sum_{i=1}^{p}\theta_i(t)A_i \tag{2}$$

$$C(\theta) = C_0 + \sum_{i=1}^{p}\theta_i(t)C_i \tag{3}$$

$$R(\theta) = R_0 + \sum_{i=1}^{p}\theta_i(t)R_i \tag{4}$$

where $x(t)\in\Re^n$ is the state, $w(t)\in\Re^{n_w}$ is the noise signal (including process and measurement noises) which is assumed to belong to $\mathcal{L}_2$, $y(t)\in\Re^{n_y}$ is the measurement, $z(t)\in\Re^{n_z}$ is the signal to be estimated, B, D, A_i, C_i, R_i, $i=0,\ldots,p$, are known constant real matrices of appropriate dimensions and $\theta_i(t)$, $i=1,\ldots,p$, are uncertain bounded real time-varying parameters with bounded rates of variation $\dot{\theta}_i(t)$. It is assumed that $(\theta(t),\dot{\theta}(t))$, $\forall\, t\geq 0$, lie in a given polyhedral domain $\mathcal{B}$ with known ℓ vertices.

It is assumed, without loss of generality, that $\theta=0$ belongs to $\mathcal{B}$. Note that this condition can be always achieved by appropriately redefining the matrices A_0, C_0 and R_0.

In this paper we shall address the problem of designing a stationary linear filter which provides an estimate $\hat{z}$ of the signal z with a guaranteed performance in the $\mathcal{H}_\infty$ sense, irrespective of the uncertain parameters θ_i. Attention is focused on the design of a *linear* *time-invariant, asymptotically stable filter of order* n with state-space realization

$$\begin{aligned} \dot{\hat{x}}(t) &= A_f \hat{x}(t) + B_f w(t), \quad \hat{x}(0) = 0 \\ \hat{z}(t) &= C_f \hat{x}(t) \end{aligned} \tag{5}$$

where the matrices $A_f \in \Re^{n\times n}$, $B_f \in \Re^{n\times n_y}$ and $C_f \in \Re^{n_z\times n}$ are to be determined. Inspired by the work of [16] on robust stability analysis via parameter-dependent Lyapunov functions with quadratic dependence on the uncertain parameters, in this paper we shall consider a filter design methodology based on a parameter-dependent Lyapunov function which is quadratic in θ.

The robust $\mathcal{H}_\infty$ filter problem addressed in this paper is as follows: *Given a scalar $\gamma > 0$, find a filter of the form of (5) such that the gain from w to the estimation error $z - \hat{z}$ is smaller than γ for any $(\theta, \dot{\theta}) \in \mathcal{B}$, namely, under zero initial conditions and for any $(\theta, \dot{\theta}) \in \mathcal{B}$, the estimation error converges exponentially to zero and*

$$\sup_{(\theta,\dot{\theta})\in\mathcal{B}} \sup \left\{ \frac{\|z - \hat{z}\|_2}{\|w\|_2} \ : \ w \in \mathcal{L}_2, \ w \not\equiv 0 \right\} < \gamma. \tag{6}$$

We end this section by recalling a version of the bounded real lemma for linear time-varying systems which will be used in the paper. Let the linear time-varying system be

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)w(t) \\ z(t) &= C(t)x(t) \end{aligned} \tag{7}$$

where $x(t) \in \Re^n$ is the state, $w(t) \in \Re^{n_w}$ is the input, $z(t) \in \Re^{n_z}$ is output, and $A(t)$, $B(t)$ and $C(t)$ are real valued matrix functions with appropriate dimensions.

Lemma 2.1 ([3]) *Consider the system (7) and let $\gamma > 0$ be a given scalar. Then the following statements are equivalent:*

(1) The system (7) is exponentially stable and

$$\sup_{0\not\equiv w\in\mathcal{L}_2} \left\{ \frac{\|z\|_2}{\|w\|_2} \ : \ x(0) = 0 \right\} < \gamma.$$

(2) There exists a bounded positive definite matrix function $P(t)$ over $[0, \infty)$ such that

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + \gamma^{-2}P(t)B(t)B^T(t)P(t) + C^T(t)C(t) < 0, \quad \forall\, t \in [0, \infty) \tag{8}$$

3 Robust $\mathcal{H}_\infty$ Filter Design

In order to develop the $\mathcal{H}_\infty$ filter design method, the state equation of system (1) will be rewritten in the form

$$\dot{x}(t) = [A_0 + \mathcal{A}\Theta(t)]\, x(t) + Bw(t) \tag{9}$$

where $\mathcal{A} \in \Re^{n\times q}$ is a known matrix and $\Theta(t) \in \Re^{q\times n}$ is an uncertain matrix such that each row depends linearly on the uncertain parameters θ_i, and where the value of q depends on the structure of the matrices $A_i,\ i = 1, \ldots, p$, and on the structure chosen for Θ.

Note that the representation of (9) always exits and does not impose any loss of generality. For example, a direct, and intuitive, choice of $\mathcal{A}$ and Θ is

$$\mathcal{A} = \begin{bmatrix} A_1 & \ldots & A_p \end{bmatrix}, \qquad \Theta = \begin{bmatrix} \theta_1 I_n & \ldots & \theta_p I_n \end{bmatrix}^T.$$

It turns out that $q = np$ for the above Θ.

It should be noted that the decomposition $\mathcal{A}\Theta$ of (9) is not unique. Further, it turns out that the choice of the dimension q of Θ should be based on the tradeoff between the conservatism and the computational effort required by the filter design method. Indeed, an increase of q is likely to reduce the conservatism of the method as it will increase the number of decision variables in the underlying optimization problem, however this will increase the required computational effort.

It follows from (5) and (9) that the estimation error $e := z - \hat{z}$ can be described by the following state-space model

$$\begin{aligned} \dot{\xi}(t) &= A_e(\theta)\xi(t) + B_e w(t), \qquad \xi(0) = 0 \\ e(t) &= C_e(\theta)\xi(t) \end{aligned} \tag{10}$$

where $\xi = \begin{bmatrix} x^T & \hat{x}^T \end{bmatrix}^T$ and

$$A_e(\theta) = \begin{bmatrix} A_0 + \mathcal{A}\Theta & 0 \\ B_f C(\theta) & A_f \end{bmatrix}, \quad B_e = \begin{bmatrix} B \\ B_f D \end{bmatrix}, \quad C_e(\theta) = \begin{bmatrix} R(\theta) & -C_f \end{bmatrix}. \tag{11}$$

In this section we shall develop an $\mathcal{LMI}$ based method for determining a filter of the form of (5) such that the estimation error system (10) satisfies the bounded real lemma inequality of (8) with a parameter-dependent matrix $\mathcal{P}(\theta)$ that depends quadratically on θ, i.e. there exists a real symmetric matrix function $\mathcal{P}(\theta) \in \Re^{2n\times 2n}$, which is quadratic in θ and satisfies:

$$\mathcal{P}(\theta) > 0, \ \ \forall\, \theta \in \mathcal{B} \tag{12}$$

$$\dot{\mathcal{P}}(\theta) + A_e^T(\theta)\mathcal{P}(\theta) + \mathcal{P}(\theta)A_e(\theta) + \gamma^{-2}\mathcal{P}(\theta)B_e B_e^T\mathcal{P}(\theta) + C_e^T(\theta)C_e(\theta) < 0, \ \ \forall\, (\theta, \dot{\theta}) \in \mathcal{B}. \tag{13}$$

In view of structure of the matrix $A_e(\theta)$, which has the feature that only its (1,1) block depends on Θ, the matrix $\mathcal{P}(\theta)$ will be assumed to be of the following form:

$$\mathcal{P}(\theta) = \begin{bmatrix} P_0 + P_1\Theta + \Theta^T P_1^T + \Theta^T P_2 \Theta & P_3 \\ P_3^T & P_4 \end{bmatrix} \tag{14}$$

where $P_0,\ P_3,\ P_4 \in \Re^{n\times n}$, $P_1 \in \Re^{n\times q}$ and $P_2 \in \Re^{q\times q}$.

Note that as $\mathcal{P}(\theta) > 0$ for $\Theta = 0$, it follows that the matrices P_0, P_3 and P_4 should satisfy:

$$\begin{bmatrix} P_0 & P_3 \\ P_3^T & P_4 \end{bmatrix} > 0. \tag{15}$$

Further, without loss of generality, we shall assume that P_3 is nonsingular.

In view that the inequalities of (12) and (13) are not convex in θ, the problem of testing if these inequalities are feasible is not tractable, in general, neither analytically nor numerically, even when the filter matrices A_f, B_f and C_f are given. In the sequel we shall develop sufficient conditions for checking the feasibility of (12) and (13), including finding suitable filter matrices A_f, B_f and C_f.

The first result provides a necessary and sufficient condition for the existence of a matrix $\mathcal{P}(\theta)$ of the form of (14) that satisfies the conditions of (12) and (13). To this end, introduce the notation

$$A_a(\theta) = \begin{bmatrix} A_0 & \mathcal{A} & 0 \\ \Theta A_0 + \dot{\Theta} & \Theta\mathcal{A} & 0 \\ B_f C(\theta) & 0 & A_f \end{bmatrix}, \quad B_a(\theta) = \begin{bmatrix} B \\ \Theta B \\ B_f D \end{bmatrix} \tag{16}$$

$$C_a(\theta) = \begin{bmatrix} R(\theta) & 0 & -C_f \end{bmatrix}, \quad H(\theta) = \begin{bmatrix} \Theta & -I_q & 0 \end{bmatrix}. \tag{17}$$

Lemma 3.1 *Given a scalar $\gamma > 0$, there exists a matrix $\mathcal{P}(\theta)$ of the form of (14) such that the bounded real conditions of (12) and (13) are satisfied if and only if there exists a symmetric matrix $P \in \Re^{(2n+q)\times(2n+q)}$ of the form*

$$P = \begin{bmatrix} P_0 & P_1 & P_3 \\ P_1^T & P_2 & 0 \\ P_3^T & 0 & P_4 \end{bmatrix}, \quad P_0, P_3, P_4 \in \Re^{n\times n}, \quad P_1 \in \Re^{n\times q}, \quad P_2 \in \Re^{q\times q} \tag{18}$$

such that for all $\eta \in \Re^{2n+q}$ the following conditions hold

$$\eta^T P \eta > 0, \ \ \forall \eta \neq 0 : H(\theta)\eta = 0 \tag{19}$$

$$\eta^T \left[A_a^T(\theta)P + PA_a(\theta) + \gamma^{-2}PB_a(\theta)B_a^T(\theta)P + C_a^T(\theta)C_a(\theta)\right]\eta < 0, \ \forall \eta \neq 0 : H(\theta)\eta = 0 \tag{20}$$

for all $(\theta, \dot{\theta}) \in \mathcal{B}$

Proof. First we shall factorize the matrix $\mathcal{P}(\theta)$ as

$$\mathcal{P}(\theta) = \Psi^T(\theta) P \Psi(\theta) \tag{21}$$

where P is a symmetric matrix as in (18) and

$$\Psi(\theta) = \begin{bmatrix} I_n & 0 \\ \Theta & 0 \\ 0 & I_n \end{bmatrix}. \tag{22}$$

Further, observe that

$$\eta = \Psi(\theta)\xi, \ \ \xi \in \Re^{2n} \quad \Leftrightarrow \quad \eta : H(\theta)\eta = 0. \tag{23}$$

Hence, it follows that $\mathcal{P}(\theta) > 0$ for all $\theta \in \mathcal{B}$ if and only if

$$\eta^T P \eta > 0, \quad \eta = \Psi(\theta)\xi, \quad \forall\, \xi \in \Re^{2n}, \quad \xi \neq 0$$

for all $\theta \in \mathcal{B}$, which is equivalent to (19).

Now, the equivalence of (13) and (20) will be established. Considering (16), (17), (21) and (22), it can be easily shown that the left-hand side of (13), denoted by $\Lambda(\theta)$, can be written as

$$\Lambda(\Theta) = \Phi^T(\theta)\Big[A_a^T(\theta)P + PA_a(\theta) + \gamma^{-2}PB_a(\theta)B_a^T(\theta)P + C_a^T(\theta)C_a(\theta)\Big]\Psi(\theta).$$

In view of (23), it follows that $\Lambda(\theta) < 0$ for all $(\theta, \dot{\theta}) \in \mathcal{B}$ if and only if

$$\eta^T\Big[A_a^T(\theta)P + PA_a(\theta) + \gamma^{-2}PB_a(\theta)B_a^T(\theta)P + C_a^T(\theta)C_a(\theta)\Big]\eta < 0,\ \eta = \Psi(\theta)\xi,\ \forall\, \xi \in \Re^{2n}, \xi \neq 0$$

for all $(\theta, \dot{\theta}) \in \mathcal{B}$, which is equivalent to (20). $\nabla\nabla\nabla$

In view of Lemma 3.1, the bounded real conditions of (12) and (13) with a matrix $\mathcal{P}(\theta)$ as in (14) can be verified via the positive definiteness of certain matrices in the null space of the matrix $H(\theta)$. Although the latter conditons could be tested using the so-called (D, G) scaling method ([4], [11]), in this paper we shall use a technique proposed in [16], which follows from Finsler's lemma and will be referred to as the *generalised multiplier approach*. As it will be shown in the next section, it turns out that the (D, G) scaling is more conservative than the generalised multiplier approach.

The next result presents the generalised multiplier approach for testing the conditions (19) and (20) of Lemma 3.1.

Lemma 3.2 *Given a scalar $\gamma > 0$, the bounded real conditions of (12) and (13) are satisfied with a matrix $\mathcal{P}(\theta)$ of the form of (14) if there exists a symmetric matrix $P \in \Re^{(2n+q)\times(2n+q)}$ of the form of (18) and matrices L and $M \in \Re^{(2n+q)\times q}$ such that the following inequalities are satisfied*

$$P + LH(\theta) + H^T(\theta)L^T > 0, \quad \forall\, \theta \in \mathcal{B} \tag{24}$$

$$\begin{bmatrix} A_a^T(\theta)P + PA_a(\theta) + MH(\theta) + H^T(\theta)M^T & C_a^T(\theta) & PB_a(\theta) \\ C_a(\theta) & -I & 0 \\ B_a^T(\theta)P & 0 & -\gamma^2 I \end{bmatrix} < 0, \ \forall\, (\theta, \dot{\theta}) \in \mathcal{B} \tag{25}$$

Proof. First note that by using Schur's complements, (25) is equivalent to

$$A_a^T(\theta)P + PA_a(\theta) + \gamma^{-2}PB_a(\theta)B_a^T(\theta)P + C_a^T(\theta)C_a(\theta) + MH(\theta) + H^T(\theta)M^T < 0, \ \forall\, (\theta, \dot{\theta}) \in \mathcal{B}. \tag{26}$$

Hence, the result follows immediately by noting that (24) and (26) imply the inequalities (19) and (20) of Lemma 3.1. $\nabla\nabla\nabla$

Remark 3.1 Note that the inequalies of (24) and (25) are affine in θ and $\dot{\theta}$, and as such, they only need to be verified at the vertices of the polytope $\mathcal{B}$. In the case where the filter matrices A_f, B_f and C_f are given, (24) and (25) are $\mathcal{LMI}$s in P, L and M, and thus the feasibility of these inequalities can be tested via standard $\mathcal{LMI}$ algorithms ([1]). □

The filter design methodology proposed in this paper involves finding filter matrices A_f, B_f and C_f along with matrices P, L and M such that the inequalities of (24) and (25) are satisfied at all vertices of the polytope $\mathcal{B}$. Observe that these inequalities are not jointly convex in P, A_f and B_f. However, as it will be shown in the next theorem, it turns out that by performing appropriate transformations and changing new variables, these inequalities can be transformed into $\mathcal{LMI}$s.

Theorem 3.1 *Given a scalar $\gamma > 0$, the robust $\mathcal{H}_\infty$ filtering problem for system (1) is solvable via a parameter-dependent Lyapunov function with a $\mathcal{P}(\theta)$ of the form of (14) if there exist symmetric matrices $P_0 > 0$, P_2 and Ω_P, and matrices P_1, L_1, L_2, M_1, M_2, Ω_A, Ω_B, Ω_C, Ω_L and Ω_M, such that the following $\mathcal{LMI}$s are satisfied at all the vertices of the polytope $\mathcal{B}$:*

$$\begin{bmatrix} P_0 - \Omega_P + (L_1 - \Omega_L)\Theta + \Theta^T(L_1 - \Omega_L)^T & P_1^T - L_1 + (L_2\Theta)^T + \Omega_L & \Theta^T\Omega_L^T \\ P_1 - L_1^T + L_2\Theta + \Omega_L^T & P_2 - L_2 - L_2^T & -\Omega_L^T \\ \Omega_L\Theta & -\Omega_L & \Omega_P \end{bmatrix} > 0 \tag{27}$$

$$\begin{bmatrix} \Phi_{11} & \Phi_{21}^T & \Phi_{31}^T & R^T(\theta) & \Phi_{51}^T \\ \Phi_{21} & \Phi_{22} & \Phi_{32}^T & 0 & \Phi_{52}^T \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & -\Omega_C^T & \Phi_{53}^T \\ R(\theta) & 0 & -\Omega_C & -I & 0 \\ \Phi_{51} & \Phi_{52} & \Phi_{53} & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{28}$$

where

$$\Phi_{11} = A_0^T P_0 + P_0 A_0 + \Omega_B C(\theta) + C^T(\theta)\Omega_B^T + P_1\Upsilon + \Upsilon^T P_1^T + M_1\Theta + \Theta^T M_1^T \tag{29}$$

$$\Phi_{21} = P_1^T A_0 + P_2\Upsilon + (P_0\mathcal{A} + P_1\Theta\mathcal{A})^T - M_1^T + M_2\Theta \tag{30}$$

$$\Phi_{22} = \mathcal{A}^T P_1 + P_1^T\mathcal{A} + P_2\Theta\mathcal{A} + (P_2\Theta\mathcal{A})^T - M_2 - M_2^T \tag{31}$$

$$\Phi_{31} = \Omega_P A_0 + \Omega_B C(\theta) + \Omega_A^T + \Omega_M\Theta \tag{32}$$

$$\Phi_{32} = \Omega_P\mathcal{A} - \Omega_M \tag{33}$$

$$\Phi_{33} = \Omega_A + \Omega_A^T \tag{34}$$

$$\Phi_{51} = B^T(P_0 + P_1\Theta)^T + D^T\Omega_B^T \tag{35}$$

$$\Phi_{52} = B^T(P_1 + \Theta^T P_2) \tag{36}$$

$$\Phi_{53} = B^T\Omega_P + D^T\Omega_B^T \tag{37}$$

$$\Upsilon = \Theta A_0 + \dot{\Theta}. \tag{38}$$

Moreover, under the above conditions, the transfer function matrix of a suitable filter is given by

$$H_{\hat{z}y}(s) = (\Omega_C\Omega_P^{-1})(sI - (\Omega_A\Omega_P^{-1}))^{-1}\Omega_B \tag{39}$$

Proof. It will be shown that the inequalities of (24) and (25) are equivalent to the $\mathcal{LMI}$s of (27) and (28), respectively. First, we partition the matrices L and M accordingly to P and $H(\theta)$, namely

$$L = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}, \quad M = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} \tag{40}$$

where $L_1,\ L_3,\ M_1,\ M_3 \in \Re^{n\times q}$ and $L_2,\ M_2 \in \Re^{q\times q}$.

In light of the definitions of $A_a(\theta)$, $B_a(\theta)$, $C_a(\theta)$, $H(\theta)$ and P in (16)-(18), the inequality of (25) can be rewritten as:

$$\begin{bmatrix} W_{11} & W_{21}^T & W_{31}^T & R^T(\theta) & W_{51}^T \\ W_{21} & W_{22} & W_{32}^T & 0 & W_{52}^T \\ W_{31} & W_{32} & W_{33} & -C_f^T & W_{53}^T \\ R(\theta) & 0 & -C_f & -I & 0 \\ W_{51} & W_{52} & W_{53} & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{41}$$

where

$$W_{11} = A_0^T P_0 + P_0 A_0 + P_3 B_f C(\theta) + [P_3 B_f C(\theta)]^T + P_1 \Upsilon + \Upsilon^T P_1^T + M_1 \Theta + \Theta^T M_1^T \tag{42}$$

$$W_{21} = P_1^T A_0 + P_2 \Upsilon + (P_0 \mathcal{A} + P_1 \Theta \mathcal{A})^T - M_1^T + M_2 \Theta \tag{43}$$

$$W_{22} = \mathcal{A}^T P_1 + P_1^T \mathcal{A} + P_2 \Theta \mathcal{A} + (P_2 \Theta \mathcal{A})^T - M_2 - M_2^T \tag{44}$$

$$W_{31} = P_3^T A_0 + P_4 B_f C(\theta) + A_f^T P_3^T + M_3 \Theta \tag{45}$$

$$W_{32} = P_3^T \mathcal{A} - M_3 \tag{46}$$

$$W_{33} = P_4 A_f + A_f^T P_4 \tag{47}$$

$$W_{51} = B^T (P_0 + P_1 \Theta)^T + D^T (P_3 B_f)^T \tag{48}$$

$$W_{52} = B^T (P_1 + \Theta^T P_2) \tag{49}$$

$$W_{53} = B^T P_3 + D^T B_f^T P_4 \tag{50}$$

where Υ is as in (38).

Next, introduce the matrix

$$J_1 = \mathrm{diag}\{\, I_n, \ \ I_q, \ \ P_4^{-1} P_3^T, \ \ I_n, \ \ I_{n_w} \,\}.$$

Note that as the matrices P_3 and P_4 are nonsingular, J_1 is well defined and nonsingular. Pre- and post-multiplying (41) by J_1^T and J_1, respectively, and defining the following new variables

$$\Omega_A = P_3 A_f P_4^{-1} P_3^T, \quad \Omega_B = P_3 B_f, \quad \Omega_C = C_f P_4^{-1} P_3^T \tag{51}$$

$$\Omega_M = P_3 P_4^{-1} M_3, \quad \Omega_P = P_3 P_4^{-1} P_3^T \tag{52}$$

the inequality of (41) becomes the $\mathcal{LMI}$ of (28). Further, (51)-(52) imply that the mapping from (A_f, B_f, C_f, M, P_4) to $(\Omega_A, \Omega_B, \Omega_C, \Omega_M, \Omega_P)$ is invertible. Hence, (41) is equivalent to (28).

We now prove the equivalence of the inequalities of (24) and (27). In view of the definition of $H(\theta)$ in (17) and the partition of L in (40), the inequlity of (24) becomes

$$\begin{bmatrix} P_0 + L_1\Theta + \Theta^T L_1^T & P_1^T - L_1 + \Theta^T L_2^T & P_3 + \Theta^T L_3^T \\ P_1 - L_1^T + L_2\Theta & P_2 - L_2 - L_2^T & -L_3^T \\ P_3^T + L_3\Theta & -L_3 & P_4 \end{bmatrix} > 0. \tag{53}$$

Pre- and post-multiplying (53) by J_2^T and J_2, respectively, where

$$J_2 = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_q & 0 \\ -P_4^{-1}P_3^T & 0 & P_4^{-1}P_3^T \end{bmatrix}$$

and introducing the the variabable $\Omega_L = P_3 P_4^{-1} L_3$, we readily obtain that the inequality of (53) is equivalent to the $\mathcal{LMI}$ of (27).

Next, considering (51) and (52), it follows that

$$A_f = P_3^{-1}(\Omega_A \Omega_P^{-1})P_3, \quad B_f = P_3^{-1}\Omega_B, \quad C_f = (\Omega_C \Omega_P^{-1})P_3$$

This implies that the transfer function matrix $H_{\hat{z}y}(s)$ of the filter is given by

$$H_{\hat{z}y}(s) = C_f(sI - A_f)^{-1}B_f = (\Omega_C \Omega_P^{-1}) \left[sI - (\Omega_A \Omega_P^{-1}) \right]^{-1} \Omega_B. \tag{54}$$

This completes the proof. $\nabla\nabla\nabla$

Theorem 3.1 provides an $\mathcal{LMI}$ based method for solving the problem of robust $\mathcal{H}_\infty$ filtering for system (1) based on a parameter-dependent Lyapunov function which is quadratic in the uncertain parameters. The proposed method incorporates information on the bounds on the rates of variation of the parameters and has the feature that both stability and $\mathcal{H}_\infty$ performance are dependent on the uncertain parameters. Note that any feasible solution to the $\mathcal{LMI}$s of (27) and (28) yields a suitable robust filter. Further, the robust filter with the smallest γ attenuation level obtainable from Theorem 3.1 can be easily determined by solving the following convex optimization problem:

$$\begin{aligned} &\text{minimize } \kappa \\ &\text{subject to (27) and (28) with } \gamma^2 = \kappa \end{aligned}$$

The optimal filter transfer function matrix is as in (39), whereas the minimum value of γ, namely γ^*, is given by $\gamma^* = \sqrt{\kappa^*}$, where κ^* is the optimal value of κ.

Remark 3.2 The robust $\mathcal{H}_\infty$ filter design method of Theorem 3.1 can be readily extended to the case where the filter transfer function matrix $H_{\hat{z}y}(s)$ is required to satisfy additional structure constraints. A typical example is the case where a "block-decoupled" filter is required, i.e. when $H_{\hat{z}y}(s)$ is required to be block-diagonal.

The design of a robust $\mathcal{H}_\infty$ filter with constraints on the structure of its transfer function matrix can be easily achieved by imposing the desired structure on the matrices Ω_A, Ω_B and Ω_C and a corresponding block-diagonal structure on the matrix Ω_P.

As for example, suppose that the transfer function matrix $H_{\hat{z}y}(s)$ of the robust $\mathcal{H}_\infty$ filter is required to have the following block-triangular structure

$$H_{\hat{z}y}(s) = \begin{bmatrix} \blacksquare & 0 \\ \blacksquare & \blacksquare \end{bmatrix}$$

where $\blacksquare$ stands for a block of appropriate dimensions with no additional restriction on its entries. This can be achieved by imposing the following structure constraints to the matrices Ω_A, Ω_B, Ω_C, and Ω_P

$$\Omega_A = \begin{bmatrix} \blacksquare & 0 \\ \blacksquare & \blacksquare \end{bmatrix}, \quad \Omega_B = \begin{bmatrix} \blacksquare & 0 \\ \blacksquare & \blacksquare \end{bmatrix},$$

$$\Omega_C = \begin{bmatrix} \blacksquare & 0 \\ \blacksquare & \blacksquare \end{bmatrix}, \quad \Omega_P = \begin{bmatrix} \blacksquare & 0 \\ 0 & \blacksquare \end{bmatrix}.$$

Note that the above constraints are convex and can be incorporated to the filter design method of Theorem 3.1. □

4 (D, G) Scaling vs. the Generalised Multiplier Approach

It turns out that the robust filtering problem that we have addressed in this paper can also be solved using the so-called (D, G)-scaling method which was initially introduced in [4]. Essentially, this method replaces the uncertain parameters Θ by two scaling matrices D and G (to be specified later). Consequently, the robust filter design problem reduces to solving a linear matrix inequality, which is somewhat simpler than the generalised multiplier approach that we have proposed in this paper.

However, the (D, G)-scaling method is known to be conversative in general [11]. The purpose of this section is to show that the generalised multiplier approach is indeed less conservative than the (D, G)-scaling method, thus justifying the required extra computation.

The fundamental problem to be addressed considers the following problem: Given a complex matrix

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} = \Omega^*, \ \Omega_{11} \in \mathcal{C}^{n\times n}, \Omega_{22} \in \mathcal{C}^{m\times m} \tag{55}$$

and a matrix set

$$\mathcal{B} = \{\mathrm{diag}\{\theta_1 I_1, \cdots, \theta_p I_p\} : |\theta_i| \le 1, i = 1, \cdots, p\} \tag{56}$$

we want to determine if the following quadratic condition holds:

$$(x^* \ y^*)\Omega \begin{pmatrix} x \\ y \end{pmatrix} < 0 \ \forall \ y = \Theta x, \ \ \Theta \in \mathcal{B}, x \neq 0, x \in \mathcal{C}^n \tag{57}$$

For simplicity reasons, we do not consider derivatives of θ_i.

The (D, G)-scaling method is simply stated as follows: The condition in (57) holds if there exist matrices D and G satisfying the following:

$$D = D^* > 0, \ D\Theta = \Theta D \ \forall \Theta \in \mathcal{B} \tag{58}$$

$$G = -G^*, \ G\Theta = \Theta G \ \forall \Theta \in \mathcal{B} \tag{59}$$

$$\Omega + \begin{bmatrix} D & G \\ G^* & -D \end{bmatrix} < 0 \tag{60}$$

On the other hand, the generalised multiplier approach uses the following: The condition in (57) holds if there exists $N \in \mathcal{C}^{(n+m)\times m}$ such that

$$\Omega + N[\Theta \ \ -I] + \begin{bmatrix} \Theta^* \\ -I \end{bmatrix} N^* < 0, \ \forall \Theta \in \mathcal{B} \tag{61}$$

Theorem 4.1 *If the conditions (58)-(60) hold for some D and G, then the condition (61) holds for some N. That is, the (D, G)-scaling method is at least as conservative as the generalised multiplier approach.*

Proof. Consider (61) and apply the (D, G)-scaling method to it to eliminate the variable Θ, yielding the following:

$$\left[\begin{array}{c|c} \Omega & 0 \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{c} N \\ \hline 0 \end{array}\right] [0 \ \ -I | I] + \left[\begin{array}{c} 0 \\ -I \\ \hline I \end{array}\right] [N^* \ |0] + \left[\begin{array}{cc|c} D & 0 & G \\ 0 & 0 & 0 \\ \hline G^* & 0 & -D \end{array}\right] < 0 \tag{62}$$

That is, (61) holds if there exist D, G and N satisfying (58)-(59) and (62). Applying the elimination method [1], (62) holds for some N iff the following two conditions hold:

$$[0 \ |I] \left(\left[\begin{array}{c|c} \Omega & 0 \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{cc|c} D & 0 & G \\ 0 & 0 & 0 \\ \hline G^* & 0 & -D \end{array}\right] \right) \left[\begin{array}{c} 0 \\ \hline I \end{array}\right] < 0 \tag{63}$$

$$\left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & I & I \end{array}\right] \left(\left[\begin{array}{c|c} \Omega & 0 \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{cc|c} D & 0 & G \\ 0 & 0 & 0 \\ \hline G^* & 0 & -D \end{array}\right] \right) \left[\begin{array}{cc} I & 0 \\ 0 & I \\ \hline 0 & I \end{array}\right] < 0 \tag{64}$$

It is easy to verify that (63) is equivalent to $D > 0$, which is void, and that (64) is the same as (60). Hence, (58)-(60) imply (61). $\nabla\nabla\nabla$

It is well-known that the (D, G)-scaling is non-conservative when there is a single real paramter [11]. Hence, we have the following corollary.

Corollary 4.1 *If $\mathcal{B} = \{\theta_1 I : \ |\theta_1| \leq 1\}$, then the condition (61) is equivalent to (57).*

Next, we give an example to show that the generalised multiplier approach is actually less conservative than the (D, G)-scaling method.

Example 4.1 *Let*

$$M = \begin{bmatrix} 0 & 1 \\ j & 0 \end{bmatrix}; \ \Theta = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}; \ \Omega = -\begin{bmatrix} I \\ \alpha M \end{bmatrix} [I \ \alpha M^*] \tag{65}$$

where $\alpha > 0$ is a tuning parameter.

It is easy to check that (57) is equivalent to

$$(I + \alpha\Theta M)(I + \alpha M^*\Theta) > 0, \ \forall |\theta_i| \leq 1$$

which in turn is the same as $I + \alpha\Theta M$ being nonsingular for all $|\theta_i| \leq 1$. This actually holds for any $\alpha > 0$.

If we apply the (D, G)-scaling method, we have

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \ G = \begin{bmatrix} 0 & g \\ -g^* & 0 \end{bmatrix} \tag{66}$$

for some $d_1 > 0, d_2 > 0$ and g. Further, (60) becomes

$$\begin{bmatrix} I \\ \alpha M \end{bmatrix} [I \ \alpha M^*] + \begin{bmatrix} D & G \\ G^* & -D \end{bmatrix} < 0$$

Pre-multiplying and post-multiplying the above by $[-\alpha M \ I]$ and its complex conjugate, respectively, we get

$$\alpha^2 MDM^* + \alpha MG - \alpha G^* M^* - D < 0 \tag{67}$$

It can verified that (67) has a solution (d_1, d_2, g) if and only if $\alpha < 1$. Hence, (58)-(60) have a solution only if $0 < \alpha < 1$.

However, if we apply the generalised multiplier approach and take

$$N = \begin{bmatrix} -\rho\alpha M \\ 0 \end{bmatrix}$$

for some $\rho > 0$, (61) becomes

$$\begin{bmatrix} -I - \rho\alpha M\Theta - \rho\alpha\Theta M^* & -\alpha(1-\rho)M \\ -\alpha(1-\rho)M^* & -\alpha^2 MM^* \end{bmatrix} < 0$$

Using Schur complement and the fact that M is nonsingular, the above holds if and only if

$$-I - \rho\alpha M\Theta - \rho\alpha\Theta M^* + (1-\rho)^2 I < 0$$

or equivalently,

$$-(I + \alpha M\Theta) - (I + \alpha M\Theta)^* + \rho I < 0$$

Choosing $\rho > 0$ to be arbitrarily small, it can be verified that the above holds iff $\alpha < \sqrt{2}$. Hence, (61) has a solution N for all $0 < \alpha < \sqrt{2}$.

From the above, it is clear that the (D, G)-scaling method is more conservative than the generalised multiplier approach for this example.

5 Conclusions

In this paper, we have proposed a design method for robust $\mathcal{H}_\infty$ filters for systems with time-varying parameters. This method employs a linear time-invariant filter but uses a parameter-dependent Lyapunov function, thus reducing conservatism compared with previous design methods. To deal with the difficulties introduced by the parameter-dependent Lyapunov function, the so-called generalised multiplier approach has been introduced. The resulting design method becomes solving a set of linear matrix inequalities. It is also shown that this approach yields less conservative results compared with the (D, G)-scaling method at the expense of some more linear matrix inequalities.

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