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OPTIMAL CONTROL OF LINEAR SYSTEMS WITH INPUT SATURATION USING IQCS

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Abstract: In this paper, we propose a new approach to the optimal control problem for linear time invariant systems subject to input constraints. The input constraint is described by means of integral quadratic constraints in order to obtain an accurate model for the nonlinearity. The key point of this work is to use the IQC information on the design stage to improve the closed-loop performance. We apply this strategy to the input saturation problem by using a combination of multipliers and propose an algorithm to tune some parameters associated with the IQC. Numerical examples show that this strategy can significantly improve the performance of the system when compared with the circle and Popov criteria.

Keywords: saturation control, optimal control, absolute stability, convex optimization.

1. INTRODUCTION

The linear optimal control is a well-established topic of research and the textbooks (Anderson and Moore, 1989) and (Green and Limebeer, 1995) are good examples of this. However, in practical control problems the closed-loop system is subject to constraints on the control input converting the linear behavior of this system into a nonlinear one. The negligence of this nonlinearity at the control design stage can be a major source of performance degradation and even instability of the closed-loop system. There has been a lot of research devoted to the optimal input constrained control problem, see the survey (Bernstein and Michel, 1995). Basically, we have two approaches for solving this problem: the optimization-based methods, where the goal is to optimize the performance of the closed-loop system, and the *Ad-hoc* methods, like the *anti-windup* technique, that "*de-tune*" the optimal control in order to preserve the stability of the closed-loop system. The anti-windup methods only achieve the desired performance if the input is relatively small, i.e. when the system has a linear behavior, and loose their optimality as soon as the system turns nonlinear.

On the other hand, the regional stability problem of input constrained linear system has been addressed by some researchers via circle or Popov criteria and LMI constraints, e.g. (Hindi and Boyd, 1998; Kapila *et al.*, 1999; Kiyama and Iwasaki, 2000). In this case, it should be noted that the LMI framework is probably more appropriate than the Riccatti approach since it enables the use of non-quadratic Lyapunov functions and multipliers via numerical procedure reducing the conservatism. However, for the synthesis case, (Kiyama and Iwasaki, 2000) proved that controllers based on the circle criterion (saturated controllers) or linear control (unsaturated controllers) achieve the same domain of attraction. In other words, the circle criterion and perhaps the Popov criterion are potentially conservative for synthesis purposes.

The purpose of this paper is to develop a new strategy to the linear optimal control with input constraint using the *Absolute Stability Theory* and multipliers

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that characterizes the saturation nonlinearities in order to obtain less conservative results than the circle and Popov criteria. To this end, the "trouble-making" component is described by means of integral quadratic constraints (IQCs) (Megretski and Rantzer, 1997), and the IQC information is used to improve the performance of the closed-loop system. One of the most important properties of the IQC framework is the easy way to combine multipliers aiming an accurate description of the nonlinearity (Jonsson, 1996). As a result, we propose an IQC to describe the saturation nonlinearity that combines the circle criterion and the multiplier proposed by (Zames and Falb, 1968). Then, for a given IQC the optimal controller is numerically computed via a convex optimization problem (in terms of LMIs). To tune some of the parameters associated with the IQC, we propose an iterative algorithm to improve the closed-loop performance. Via numerical examples, we show that our methodology can significantly improve the performance without degradation of the domain of attraction when compared with the circle and Popov criteria.

This paper is organized as follows. Section 2 states the problem of concern and section 3 shows the main result of this work. Section 4 characterizes the input saturation by IQCs and proposes an algorithm to design a dynamical control law. Finally, section 5 gives illustrative examples and some conclusions are drawn in Section 6. Because of space limitation, some of references and results are omitted, for further details see (D. Coutinho and M. Fu, 2001) (available for download at ftp://warhol.newcastle.edu.au/pub/Reports/EE01046.ps.gz).

2. PROBLEM STATEMENT

In this work, we consider a system given by

$$\dot{x} \;=\; Ax \;+\; b\sigma(u)\;, \quad x(0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ are constant, and $\sigma(\cdot)$ is a nonlinear operator.

We assume that the pair (A,b) is controllable and the nonlinear operator $\sigma(\cdot)$ has a bounded gain and can be described by

$$\sigma(u) \;=\; \rho_1 u + \delta(u) \tag{2}$$

for a given positive scalar ρ_1 and an operator $\delta(\cdot) \in \mathbb{R}$ that satisfies the following integral quadratic constraint (IQC)

$$\int_0^\infty f(x_\pi, u, \delta)\; dt \;\geq\; 0 \tag{3}$$

where $f(\cdot)$ is a quadratic form, and x_π is defined by

$$\dot{x}_\pi \;=\; A_\pi x_\pi + B_u u + B_\delta \delta\;, \quad x_\pi(0) = 0 \tag{4}$$

with $x_\pi \in \mathbb{R}^{n_\pi}$, and $A_\pi \in \mathbb{R}^{n_\pi\times n_\pi}$ is Hurwitz.

It is well-known that the above IQC has an equivalent frequency domain form as follows

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{u}(jw) \\ \widehat{\delta}(jw) \end{bmatrix}^* \Pi(jw) \begin{bmatrix} \widehat{u}(jw) \\ \widehat{\delta}(jw) \end{bmatrix} dw \;\geq\; 0 \tag{5}$$

in terms of a multiplier $\Pi(jw)$.

For convenience, let us represent the nonlinear system (1) and IQC (3) in an augmented space as follows:

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \hat{B}u + \tilde{B}\delta,\ \forall\, \delta\ :\ \int_0^\infty f(x_\pi, u, \delta)dt \geq 0 \tag{6}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ x_\pi \end{bmatrix};\ \tilde{A} = \begin{bmatrix} A & 0 \\ 0 & A_\pi \end{bmatrix};\ \hat{B} = \begin{bmatrix} \rho_1 b \\ B_u \end{bmatrix};\ \text{and}\ \ \tilde{B} = \begin{bmatrix} b \\ B_\delta \end{bmatrix}.$$

To analyze the regional stability of above system, we use the following result from the Lyapunov theory, as proposed in (Kiyama and Iwasaki, 2000) .

Lemma 1. Consider a nonlinear system $\dot{\tilde{x}} = f(\tilde{x})$ where $f : \mathbb{R}^{\tilde{n}} \mapsto \mathbb{R}^{\tilde{n}}$ is a continuous function such that $f(0) = 0$. Let $\tilde{X} \subset \mathbb{R}^{\tilde{n}}$ be a polyhedral set containing the origin. Assume that the differential equation satisfies the conditions for the existence and uniqueness of solution for any $\tilde{x}(0) \in \tilde{X}$. Suppose there exists a continuously differentiable function $V : \tilde{X} \mapsto \mathbb{R}$ satisfying the following inequalities, for some positive scalars ε_1, ε_2, ε_3 and μ.

$$\left.\begin{array}{rcl} \varepsilon_1 \tilde{x}'\tilde{x} \leq V(\tilde{x}) & \leq & \varepsilon_2 \tilde{x}'\tilde{x} \\ \dot{V}(\tilde{x}) & \leq & -\varepsilon_3 \tilde{x}'\tilde{x} \\ V(\tilde{x}) & \leq & \mu^2 \end{array}\right\}\ \forall \tilde{x} \in \tilde{X}$$

Then the set $\tilde{X} \triangleq \{\ \tilde{x} \in \tilde{X}\ :\ V(\tilde{x}) \leq \mu^2\ \}$ is a domain of attraction for the nonlinear system, i.e. $\tilde{x}(t) \to 0$ as $t \to \infty$ for any $\tilde{x}(0) \in \tilde{X}$. □

The proof of above lemma is verified straightforwardly from (Kiyama and Iwasaki, 2000) and the Lyapunov theory.

As Lyapunov function candidate, let us consider the following quadratic function in the augmented space

$$V(\tilde{x}) = V(x, x_\pi) = \tilde{x}'P\tilde{x} = \begin{bmatrix} x \\ x_\pi \end{bmatrix}' P \begin{bmatrix} x \\ x_\pi \end{bmatrix} \tag{7}$$

where $P = P' \in \mathbb{R}^{(n+n_\pi)\times(n+n_\pi)}$.

Now, consider the following quadratic cost function

$$J(x_0, u) = \int_0^\infty (x'Qx + r\sigma(u)^2)dt \tag{8}$$

for some matrix $Q = C'C$ ($C \in \mathbb{R}^{m\times n}$) and scalar $r \geq 0$.

Then, the problem of concern in this work is to find a dynamical control law

$$u = K' \begin{bmatrix} x \\ x_\pi \end{bmatrix} \tag{9}$$

for some $K \in \mathbb{R}^{n+n_\pi}$ such that:

- the cost function (8) is minimized;

- the closed-loop system is regionally stable with a domain X, i.e $x(t) \to 0$ as $t \to \infty$ for any $x(0) \in X$.

In this paper, observe that we are considering a level surface of $V(\tilde{x})$, represented by $\tilde{X}$, as an estimate of the domain of attraction of the closed-loop augmented system (6) with $u = K'\tilde{x}$. For the original system (1), we will estimate its domain of stability as the projection of $\tilde{X}$ in the sub-space $(x, x_\pi = 0)$ that we denote by the set X.

3. CONTROL DESIGN

In this section, we propose a solution to the problem stated in section 2. To this end, let us consider the augmented system (1) and denote the following (to relax the infinite dimensional problem):

$$J(x_0,u,T) = \int_0^T (x'Qx + r\sigma(u)^2)dt,\ T \to \infty \quad (10)$$

From the definitions of the Lyapunov function (7) and the augmented system (1) and keeping in mind that $x(0) = x_0$ and $x_\pi(0) = 0$, we can rewrite above, for $T \to \infty$, as $J(x_0,u,T) = V(x_0,0) - V(x(T),x_\pi(T)) + \int_0^T (\dot{V}(x,x_\pi) + x'Qx + r\sigma(u)^2)dt \le V(x_0,0) + \int_0^\infty (\dot{V}(x,x_\pi) + x'Qx + r\sigma(u)^2)dt$.

It is clear that if $\int_0^\infty (\dot{V}(x,x_\pi) + x'Qx + r\sigma(u)^2)dt \le 0$, $\forall x, x_\pi$ satisfying (6), then

$$J(x_0,u) \le V(x_0,0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}' P \begin{bmatrix} x_0 \\ 0 \end{bmatrix}.$$

From the above analysis, we can reformulate the problem of concern in this paper as follows

Problem 2. Determine a Lyapunov function in the augmented space and design a control law u given by (9) such that: (i) minimize $V(x_0,0)$ subject to $\int_0^\infty (\dot{V}(x,x_\pi) + x'Qx + r(\rho_1 u + \delta(u))^2)dt \le 0$, for all x, x_π and $\delta(\cdot)$ satisfying (6); and (ii) the closed-loop system is regionally stable with a domain X. □

To solve the above problem, we have the following constraints $\int_0^\infty (\dot{V}(x,x_\pi) + x'Qx + r(\rho_1 u + \delta(u))^2)dt \le 0$ for all $\delta : \int_0^\infty f(x_\pi,u,\delta)dt \ge 0$.

Using the well-known $\mathcal{S}$-procedure, the above is satisfied if and only if there exists a scalar $\tau_1 > 0$ such that

$$\int_0^\infty (\dot{V}(x,x_\pi) + x'Qx + r(\rho_1 u + \delta)^2 + \tau_1 f(x_\pi,u,\delta))\,dt \le 0 \quad (11)$$

Note by definition that $f(x_\pi,u,\delta)$ is a quadratic form. In the sequel, we consider the following quadratic form for $f(x_\pi,u,\delta)$

$$\begin{bmatrix} x_\pi \\ u \\ \delta \end{bmatrix}' \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{12}' & F_{22} & F_{23} \\ F_{13}' & F_{23}' & F_{33} \end{bmatrix} \begin{bmatrix} x_\pi \\ u \\ \delta \end{bmatrix} \quad (12)$$

where $F_{11} = F_{11}' \in \mathbb{R}^{n_\pi \times n_\pi}$, $F_{12} \in \mathbb{R}^{n_\pi}$, $F_{13} \in \mathbb{R}^{n_\pi}$, $F_{22} \in \mathbb{R}$, $F_{23} \in \mathbb{R}$ and $F_{33} \in \mathbb{R}$.

Before we state the main result of this section, let us define the following auxiliary variables for simplicity of notation.

$$\tilde{F}_{11} = \begin{bmatrix} 0 \\ \tau_1 F_{11} \end{bmatrix};\ \tilde{F}_{12} = \begin{bmatrix} 0 \\ F_{12} \end{bmatrix};\ \tilde{F}_{13} = \begin{bmatrix} 0 \\ F_{13} \end{bmatrix};$$
$$\tilde{C} = \begin{bmatrix} C & 0 \end{bmatrix};\ \tau_1 = -\frac{r}{F_{33}};\ \alpha = r\rho_1^2 + \tau_1 F_{22} \quad (13)$$
$$\text{and}\quad \beta = r\rho_1 + \tau_1 F_{23}.$$

Theorem 3. Consider the augmented system (6), the IQC (3) satisfying (12), the cost function (8) and the auxiliary notation (13). Let $\tilde{X}$ be a given polyhedral set containing the origin. Let $(x_0, 0) \in \tilde{X}$ be a given initial condition for the augmented system (6). Let λ and S be the solution to the following problem:

$$\min \lambda \ \text{subject to:}$$
$$S > 0,\ S = S' \quad (14)$$
$$\begin{bmatrix} \lambda & \begin{bmatrix} x_0' & 0 \end{bmatrix} \\ \begin{bmatrix} x_0 \\ 0 \end{bmatrix} & S \end{bmatrix} \ge 0 \quad (15)$$
$$\begin{bmatrix} \Omega(S) + \Psi(S) & S\tilde{C}' & S\tilde{F}_{13} \\ \tilde{C}S & -I_m & 0 \\ \tilde{F}_{13}'S & 0 & -\frac{\beta^2}{\alpha\tau_1^2} \end{bmatrix} \le 0 \quad (16)$$

where $\Omega(S)$ and $\Psi(S)$ are given by

$$\Omega(S) = S\tilde{A}' + \tilde{A}S - \frac{1}{\beta}\left(\tilde{B}\hat{B}' + \tau_1 S\tilde{F}_{13}\hat{B}' + \hat{B}\tilde{B}' + \tau_1 \hat{B}\tilde{F}_{13}'S\right) + \frac{\alpha}{\beta^2}\left(\tilde{B}\tilde{B}' + \tau_1\tilde{B}\tilde{F}_{13}'S + \tau_1 S\tilde{F}_{13}\tilde{B}'\right) - \frac{\tau_1}{\beta}\left(S\tilde{F}_{12}\tilde{B}' + \tilde{B}\tilde{F}_{12}'S\right); \quad (17)$$

$$\Psi(S) = S\tilde{F}_{11}S - \frac{\tau_1^2}{\beta}(S\tilde{F}_{12}\tilde{F}_{13}'S + S\tilde{F}_{13}\tilde{F}_{12}'S) \quad (18)$$

Then, the cost function (8) is minimized and satisfies $J(x_0,u) \le \lambda$ for the control law $u = -\frac{1}{\beta}\left(\tilde{B}'P + \tau_1\tilde{F}_{13}'\right)\tilde{x}$. Moreover, if $x(0) \in X$ then the trajectory $x(t) \to 0$ as $t \to \infty$ for some $\mu > 0$, where X is the intersection of $(x, x_\pi = 0)$ sub-space with the set $\tilde{X} \triangleq \{\tilde{x} \in \tilde{X} : \tilde{x}'S^{-1}\tilde{x} \le \mu^2\}$. □

The proof of above theorem was omitted because of space limitation. See (D. Coutinho and M. Fu, 2001) for further details.

Observe that the optimization problem in theorem 3 is not convex on the decision variable S. However, as we will see subsequently, it is possible for some classes of IQCs that this problem is converted into a convex one. To illustrate this point and the potential of the proposed controller, in the next section we apply this framework to the control of linear systems with input saturation.

4. IQC CHARACTERIZATION

Consider in system (1) that the nonlinear operator $\sigma(\cdot)$ is the unit saturation function, i.e.

$$\sigma(u) \ = \ \begin{cases} 1 & u > 1 \\ u \quad \text{for} & |u| \leq 1 \\ -1 & u < -1 \end{cases}$$

By assumption, the operator $\sigma(\cdot)$ can be decomposed into linear and uncertain nonlinear parts. Thus, the first problem of concern in this section is how to bound by a sector the uncertain nonlinear part caused by the saturation.

With this aim, let us define the *level of over-saturation* as $d(u) = \max\{0, \ |u|-1\}$ for a given control u, and suppose that the control law is such that $d(u) \leq \rho$ and $|\delta(u)| \leq \rho_2|u|$ for $\rho, \rho_2 \geq 0$. From these definition and assumption, we can state the following problem.

Problem 4. What is the optimal sector that bounds the nonlinearity caused by the saturation ?, i.e. how to find $\min \rho_2$ such that $\delta(u) = \sigma(u) - \rho_1 u$, $|\delta(u)| \leq \rho_2|u|$ for all $|u| \leq 1+\rho$. □

Recently, in the work (Fu, 2000), M. Fu proposed a solution for this problem that is summarized as follows.

Lemma 5. (Optimal sector bound, (Fu, 2000)) The optimal sector bound to the problem 4 is obtained for the following values of ρ_1 and ρ_2

$$\rho_1 \ = \ \frac{2 \ + \ \rho}{2\,(1 \ + \rho)}\,, \quad \rho_2 \ = \ \frac{\rho}{2\,(1 \ + \rho)}\,. \tag{19}$$

Now, we have to find a suitable description in terms of integral quadratic constraints for the saturation nonlinearity with the optimal sector bound (19). To this end, we will use the strong result obtained by (Zames and Falb, 1968) for the unit saturation function. By their statement, the stability problem is described by the IQC:

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{u}(jw) \\ \widehat{\sigma}(jw) \end{bmatrix}^* f_{ZF}(H) \begin{bmatrix} \widehat{u}(jw) \\ \widehat{\sigma}(jw) \end{bmatrix} dw \geq 0$$

where $f_{ZF}(H)$ is given by

$$f_{ZF}(H) = \begin{bmatrix} 0 & \tau_2 + H(jw) \\ \tau_2 + H(-jw) & -2(\tau_2 + ReH(jw)) \end{bmatrix},$$

$\tau_2 \geq 0$ is a scalar to be determined and the transfer function $H(jw)$ is such that $ReH(jw) \leq \tau_2$.

From the above IQC, the following holds

$$\int_{-\infty}^{\infty} Re \begin{bmatrix} \widehat{u}(jw) \\ \widehat{\sigma}(jw) \end{bmatrix}^* g_{ZF} \begin{bmatrix} \widehat{u}(jw) \\ \widehat{\sigma}(jw) \end{bmatrix} dw \geq 0 \tag{20}$$

where g_{ZF} is given by

$$g_{ZF} = \begin{bmatrix} 0 & \tau_2 + H(jw) \\ 0 & -(\tau_2 + H(jw)) \end{bmatrix}$$

Define a new signal as $y(jw) = (\tau_2 + H(jw))\sigma(jw)$ and consider the following realization of the transfer function $\tau_2 + H(s)$

$$\tau_2 + H(s) \ = \ C_\pi(sI - A_\pi)^{-1}B_\pi + D_\pi + \tau_2 \tag{21}$$

Thus the time-domain signal $y(t)$ is given by the following state equations

$$\begin{cases} \dot{x}_\pi \ = \ A_\pi x_\pi \ + \ B_\pi \sigma(u) \\ y \ = \ C_\pi x_\pi \ + \ (D_\pi + \tau_2)\sigma(u) \end{cases} \tag{22}$$

where $x_\pi \in \mathbb{R}^{n_\pi}$. Note from (4) that $B_u = \rho_1 B_\pi$ and $B_\delta = B_\pi$.

With the signal $y(t)$, we can write the following time-domain form of the condition (20) $\int_0^\infty \{(u' - \sigma(u)')$ $(C_\pi x_\pi + (\tau_2 + D_\pi)\sigma(u)) + (x_\pi' C_\pi' + (\tau_2 + D_\pi)\sigma(u))\ (u - \sigma(u))\}\ dt \geq 0$.

Consider the optimal sector bound condition, i.e. $\sigma(u) = \rho_1 u + \delta(u)$ and $|\delta(u)| \leq \rho_2 u$, and above inequality. Then, we can state the following time-domain IQC for saturated systems

$$\int_0^\infty \begin{bmatrix} x_\pi \\ u \\ \delta \end{bmatrix}' f_a(\dot{x}_\pi, y, \tau_2, \tau_3) \begin{bmatrix} x_\pi \\ u \\ \delta \end{bmatrix} dt \ \geq \ 0 \tag{23}$$

where $f_a(\dot{x}_\pi, y, \tau_2, \tau_3)$ is given by

$$\begin{bmatrix} 0 & (1-\rho_1)C_\pi' & -C_\pi' \\ (1-\rho_1)C_\pi & \begin{pmatrix} 2\rho_1\rho_2(D_\pi + \tau_2) \\ +\tau_3\rho_2^2 \end{pmatrix} & \begin{pmatrix} (1-2\rho_1) \\ (D_\pi + \tau_2) \end{pmatrix} \\ -C_\pi & (1-2\rho_1)(D_\pi + \tau_2) & \begin{pmatrix} -2(D_\pi + \\ \tau_2 + 0.5\tau_3) \end{pmatrix} \end{bmatrix}$$

and $\tau_3 \geq 0$ is a free parameter to be tuned.

Remark 1. Controllers based on circle and Popov criteria can be viewed as special cases of theorem 3. To illustrate this point, consider the following IQC to describe the saturation:

$$\int_0^\infty \begin{bmatrix} u \\ \delta \end{bmatrix}' \begin{bmatrix} \rho_2^2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ \delta \end{bmatrix} dt \ \geq \ 0 \tag{24}$$

i.e. the IQC (23) with $\tau_2 = 0$, $\tau_3 = 1$ and $H(jw) = 0$. Note that controllers designed with above IQC correspond to the ones obtained by the technique proposed in (Fu, 2000) (using the circle criterion). In a similar way, we can also recover controllers based on Popov (e.g. (Kapila *et al.*, 1999)) and Zames & Falb criteria if we choose respectively: (a) $\tau_2 = 1$, $\tau_3 = 1$ and $H(s) = s$; and (b) $\tau_2 = 1$ and $\tau_3 = 0$. □

In order to design the control law, we need to obtain a convex form for the optimization problem (16). From the IQC (23) observe that $\tilde{F}_{11} = 0$, $\tilde{F}_{12} = -(1-\rho_1)\tilde{F}_{13}$ and define an auxiliary matrix as $\tilde{F} = [\,0 \ C_\pi\,]'$. Then, we can write $\tilde{F}_{12}\tilde{F}_{13}' = -(1-\rho_1)\tilde{F}\tilde{F}'$ and state the following theorem.

Theorem 6. Consider the same conditions of theorem 3, the IQC (23) and the above analysis. Let x_0 be the initial condition of the system (1). Let λ and S be the solution to the following convex optimization problem

$$\begin{aligned}
&\min \lambda \quad \text{subject to:} \\
&S > 0,\ S = S^{'}\ ; && (25)\\
&\begin{bmatrix} \lambda & \begin{bmatrix} x_0^{'} & 0 \end{bmatrix} \\ \begin{bmatrix} x_0 \\ 0 \end{bmatrix} & S \end{bmatrix} \geq 0\ ; && (26)\\
&\begin{bmatrix} \Omega(S) & S\tilde{F} & S\tilde{C}^{'} & S\tilde{F}_{13} \\ \tilde{F}^{'}S & -\dfrac{\beta}{\tau_1(1-\rho_1)} & 0 & 0 \\ \tilde{C}S & 0 & -I_m & 0 \\ \tilde{F}_{13}^{'}S & 0 & 0 & -\dfrac{\beta^2}{\alpha\tau_1^2} \end{bmatrix} \leq 0 && (27)
\end{aligned}$$

where $\Omega(S)$ is given by (17).

Define a set X_ρ as follows:

$$X_\rho = \left\{ x : \begin{bmatrix} x \\ 0 \end{bmatrix}^{'} S^{-1} \begin{bmatrix} x \\ 0 \end{bmatrix} \leq \mu_\rho^2 \right\} \qquad (28)$$

where $\mu_\rho^2 = \frac{\beta^2(1+\rho)^2}{\tilde{B}^{'}S^{-1}\tilde{B}+\tau_1(B_\delta^{'}F_{13}+F_{13}^{'}B_\delta)+\tau_1^2\tilde{F}_{13}^{'}S\tilde{F}_{13}}$.

Then, for $u = -\frac{1}{\beta}\left(\tilde{B}^{'}P + \tau_1\tilde{F}_{13}^{'}\right)\tilde{x}$ and any $x(0) \in X_\rho$, the following conditions hold: (i) $J(x_0,0) \leq \lambda$; (ii) $x(t) \to 0$ as $t \to \infty$; and (iii) $|u(t)| \leq 1+\rho$. □

PROOF. The proof of item (i) follows directly from theorem 3 and the *Schur* complement applied to (27). Now, define the following polyhedral $\tilde{\mathcal{X}}_\rho = \{\tilde{x} : |K\tilde{x}| \leq 1+\rho\}$, where $K = \frac{1}{\beta}(\tilde{B}^{'}P + \tau_1\tilde{F}_{13}^{'})$. Let $\tilde{X}_\rho$ be a level surface of the Lyapunov function $V(\tilde{x}) = \tilde{x}^{'}S^{-1}\tilde{x}$, i.e. $\tilde{X}_\rho = \{\tilde{x} : \tilde{x}^{'}S^{-1}\tilde{x} \leq \mu_\rho^2\}$. From the Lyapunov theory, the set $\tilde{X}_\rho$ will be invariant if the relation $\tilde{X}_\rho \subset \tilde{\mathcal{X}}_\rho$ is satisfied. From (Luenberger, 1989) and (Doná, 2000), lemma 3.6.1., the minimum $V(\tilde{x})$ that satisfies above is given by $\mu_\rho^2 = \frac{(1+\rho)^2}{K^{'}SK} = \frac{\beta^2(1+\rho)^2}{\tilde{B}^{'}S^{-1}\tilde{B}+\tau_1(B_\delta^{'}F_{13}B_\delta)+\tau_1^2\tilde{F}_{13}^{'}S\tilde{F}_{13}}$. Therefore, for any $\tilde{x}(0) \in \tilde{X}_\rho$ the trajectory $\tilde{x}(t) \to 0$ as $t \to \infty$. Keep in mind that $x_\pi(0) = 0$ by definition. Then, for any $x(0) = x_0$ that belongs to the set X_ρ as defined in (28), the trajectory $x(t) \to 0$ as $t \to \infty$. Moreover, for any $x_0 \in X_\rho$ the over-saturation constraint $|u(t)| \leq 1+\rho$ is always satisfied (since $X_\rho \subset \tilde{X}_\rho \subset \tilde{\mathcal{X}}_\rho$). □

The key point of the proposed method is the choice of the transfer function $H(s)$. Unfortunately, until now, there is no a systematic way to compute it. To overcome this difficulty, we propose the following Algorithm to help on the control design.

Algorithm 1. Consider theorem 6, the IQC (23) and define the structure of S as follows:

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

where the partition corresponds to the partition of $\tilde{x}$.

STEP 1 Choose $H(s)$ to have the following structure

$$H(s) = -\frac{c_{n-1}s^{n-1} + \ldots + c_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} \qquad (29)$$

and the following state space realization

$$A_\pi = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix},\ B_\pi = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix},$$

$C_\pi = [\, c_0 \ \cdots \ c_{n-1} \,]$, and $D_\pi = 0$. Initialize $a_0,\ldots,a_{n-1}$ and $c_0,\ldots,c_{n-1}$ arbitrarily but with A being Hurwitz;

STEP 2 Choose the scalar τ_2 such that $\tau_2 \geq \int_0^\infty |h(t)|dt$;

STEP 3 Determine τ_3 and S, to minimize λ;

STEP 4 Fix S_{12}, S_{21}, S_{22}, and tune $a_0,\ldots,a_{n-1}$ and $c_0,\ldots,c_{n-1}$, to minimize λ;

STEP 5 Iterate between steps 2 and 4. □

For each iteration i, note in above algorithm that the closed-loop system is regionally stable and the performance bound satisfies $\lambda_i \leq \lambda_{(i-1)}$. As a result, this algorithm converges on a local minimum.

5. NUMERICAL RESULTS

To demonstrate the potential of our approach, we consider two examples to compare the closed-loop performance, where the first one is from (Fu, 2000) (circle criterion) and the second one is from (Kapila *et al.*, 1999) (Popov criterion).

Example 1. Consider the following open-loop unstable linear systems with input saturation (Fu, 2000):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1.25 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(u) \qquad (30)$$

and define the cost function $J(x_0,u)$ with $r = 1$ and $C = [\ 1 \ \ 0\ ]$.

To allow a comparative study, we will consider that the saturation is described by the circle criterion with optimal bound sector, i.e. IQC (24), and by IQC (23). Also, we will use three different levels of over-saturation. Note for the circle criterion that we can apply directly Theorem 6 with $F_{12} = 0$, $F_{13} = 0$, $F_{23} = 0$, $\tau_2 = 0$ and $H(s) = 0$.

Now, we start algorithm 1 from the following conditions:

$$A_\pi = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix};\ C_\pi = [\ 1 \ \ 0\ ];\ \tau_2 = 1 \ \text{ and } \ \tau_3 = 1.$$

After five iterations, we got the following

$$H(s) = -0.5\frac{s+2}{s^2+0.5s+5.1},\ \tau_2 = 1.3 \ \text{ and } \ \tau_3 = 10\ .$$

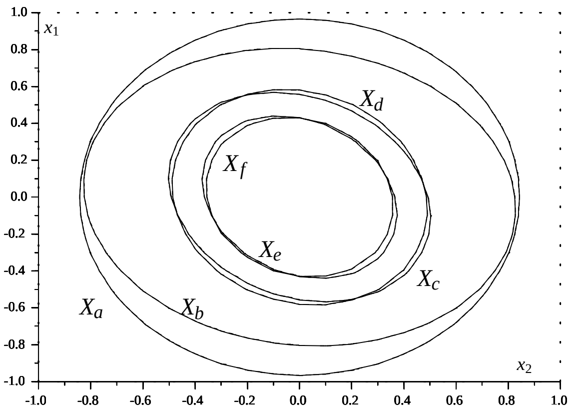

Fig. 1. Domain of stability for different levels of over-saturation considering circle criterion and IQC (23).

The table 1 and figure 1 summarize the results (cost

Level of over-saturation	Multiplier	
	IQC (24)	IQC (23)
$\rho = 0$	1.58 / X_e	1.05 / X_f
$\rho = 1$	1.73 / X_c	1.18 / X_d
$\rho = 10$	4.18 / X_a	2.43 / X_b

Table 1. Optimal costs and domains of stability.

function and domain of stability), for an initial condition $x_0 = [\,0.5 \;\; 0.5\,]'$. The results showed clearly that the proposed dynamical control law improved the performance of the closed-loop system without degradation of the domain of stability when compared with the circle criterion (even the linear controller, i.e. $\rho = 0$).

Example 2. Consider the following LTI system with input saturation (Kapila *et al.*, 1999):

$$\dot{x} = \begin{bmatrix} -0.2 & 1 & 0 \\ 0 & -0.2 & 1 \\ 0 & 0 & 0.1 \end{bmatrix} x - \begin{bmatrix} 0 \\ 0 \\ 0.35 \end{bmatrix} \sigma(u), \; x(0) = x_0 \tag{31}$$

and that the cost function (8) is defined by $Q = I_3$ and $r = 1$. The objective in this example is to design a state-feedback controller that minimizes the cost function defined above for a level of over-saturation $\rho = 0.25$ and initial condition $x_0 = [\,2 \;\; -4.5 \;\; 1.3\,]'$. Using theorem 6, we obtained a cost of 47.63. To illustrate the potential of our approach, the controller proposed in (Kapila *et al.*, 1999) (using the Popov criterion) obtained a cost of 652 for the same conditions.

6. CONCLUSION

This paper presented a new methodology to design optimal controllers for linear time invariant systems with input constraints that achieves a better performance than the circle and Popov criteria. The saturation is described by means of integral quadratic constraints. The key contributions of this paper are the use of IQC information to improve the closed-loop performance (in other words, we construct a dynamical control law using the time-domain form of the IQC), and the extension of the Zames & Falb criterion to the optimal sector bound problem proposing an iterative algorithm to make easier for the designer to choose the suitable multiplier. Future research will be concentrate on the tuning of parameters associated with the IQC.

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