# Parametric Lyapunov Functions for Uncertain Systems: the Multiplier Approach

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**Abstract.** In this chapter, we propose to use a *parametric multiplier approach* to deriving parametric Lyapunov functions for robust stability analysis of linear systems involving uncertain parameters. This new approach generalizes the traditional multiplier approach used in the absolute stability literature where the multiplier is independent of the uncertain parameters. Our main result provides a general framework for studying *multiaffine Lyapunov functions*. We show that these Lyapunov functions can be found using linear matrix inequality (LMI) techniques. Some known results on parametric Lyapunov functions are shown to be our special cases.

## 1 Introduction

This chapter considers robust stability analysis for linear time-invariant (LTI) uncertain systems. Our focus is on the existence of parametric Lyapunov functions. Our motivation for considering parametric Lyapunov functions stems from two facts. First, they can offer less conservative robust stability conditions than parameter-independent Lyapunov functions that are used in the quadratic stability theory. Secondly, they can be applied to stability analysis of systems with time-varying parameters. In the latter case, one possible approach is to find a parametric Lyapunov function which assures that the "frozen" uncertain system (i.e., the uncertain system with "frozen" parameters) is robustly stable with certain stability margin. This margin can then be used to determine the "average" time variation for the parameters that could be withstood without losing stability.

The use of parametric Lyapunov functions can be traced back to the work of Parks [8] who proves the Routh-Hurwitz stability condition for polynomials directly using a Lyapunov matrix called *Hermite matrix*. The unique feature of the Hermite matrix is that it is bilinear in the coefficients of the polynomial. The well-known Popov criterion [9], when specialized to dealing with a single constant uncertain parameter rather than sector-bounded nonlinearity, yields a Lyapunov function which depends on the uncertain parameter affinely. In fact, most absolute stability criteria given in the literature give, explicitly or implicitly, a parametric Lyapunov function when the uncertainty or nonlinearity they consider reduces to constant uncertain parameters.

Recently, Dasgupta *et. al.* [2] studies the existence of parametric Lyapunov functions for the following family of systems:

$$\dot{x}(t) = A(q)x(t) = (A_0 + bc^T(q))x(t), \quad q \in Q \tag{1}$$

where $b, c(q) \in \mathcal{R}^n$,

$$Q = \{q = [q_1, \cdots, q_m]^T : q_i^- \leq q_i \leq q_i^+\} \tag{2}$$

and $h(q)$ is affine in the elements of $q$ This type of uncertainty is called *rank-1 uncertainty*. It is shown in [2] that the uncertain system above is robustly stable, i.e., stable for all $q \in Q$, if and only if there exists some constant stable matrix $F$ and a compatibly dimensioned vector $w$ such that the augmented uncertain system

$$\dot{\bar{x}}(t) = \Pi(q)\bar{x}(t) = \begin{bmatrix} F & wc^T(q) \\ 0 & A(q) \end{bmatrix} \bar{x}(t), \quad q \in Q \tag{3}$$

admits a parametric Lyapunov function $P(q)$ that depends on $q$ *multiaffinely*. A procedure for constructing $F, w$ and $P(q)$ is given in [2]. It is also shown how this result is used in robust stability analysis for linear systems with time-varying parameters. In Feron *et. al.* [4] and Haddad and Bernstein [5], a more general class of uncertain matrices

$$A(q) = A_0 + \sum_{i=1}^{p} q_i A_i \tag{4}$$

are considered and the so-called *generalized Popov criterion* to generate parametric Lyapunov functions. This kind of uncertain systems are also studied by Gahinet *et. al.* [3] using an affine Lyapunov matrix

$$P(q) = P_0 + \sum_{i=1}^{p} q_i P_i \tag{5}$$

where $P_i, i = 0, 1, \cdots, p$ are symmetric. In contrast to [2], the robust stability tests given by the generalized Popov criterion and an affine Lyapunov matrix above are conservative in general.

In this chapter, we study a more general family of uncertain system described by

$$\dot{x}(t) = A(q)x(t) = (A_0 - BD^{-1}(q)C(q))x(t), \quad q \in Q \tag{6}$$

where the nominal matrix $A_0 \in \mathcal{R}^{n \times n}$ is assumed to be stable, $B \in \mathcal{R}^{n \times m}$ is a constant full rank matrix, $m \leq n$, $C(q) \in \mathcal{R}^{m \times n}$ and $D(q) \in \mathcal{R}^{m \times m}$ are affine in $q$ as follows:

$$C(q) = \sum_{i=1}^{p} q_i C_i, \quad D(q) = D_0 + \sum_{i=1}^{p} q_i D_i \tag{7}$$

and $D(q)$ is invertible for all $q \in Q$. Also, we seek for a multiaffine Lyapunov matrix $P(q) = P^T(q) > 0$ of the following form

$$P(q) = P_0 + \sum_{i=1}^{p} q_i P_i + \sum_{i \neq j} q_i q_j P_{ij} + \cdots \quad (8)$$

such that

$$(A_0 - BD^{-1}(q)C(q))^T P(q) + P(q)(A_0 - BD^{-1}(q)C(q)) < 0, \ \forall q \in Q \quad (9)$$

The tool we use for establishing parametric Lyapunov functions is the *multiplier approach*, which is popularly used in the absolute stability literature. The basic idea of this approach is to find a transfer matrix of certain structure, called multiplier, such that the cascade of the multiplier and some transfer matrix related to the uncertain system is strictly positive real. What makes our approach unique is the use of parametric multipliers while in the literature only constant multipliers are used. Our main result gives a sufficient condition for the existence of a multiaffine Lyapunov function in terms of the existence of an affine multiplier with a special structure. It turns out that this multiplier can be found by solving a set of linear matrix inequalities (LMIs). The solution to the LMIs automatically provides a multiaffine Lyapunov function for the given uncertain system. We then focus on the analysis of the conservatism of the method we propose. Two schemes have been analyzed in detail, namely, the so-called generalized Popov criterion for the multiparameter case, which has been studied by many authors, and the so-called affine quadratic stability test (AQS test) for the single parameter case. It is found that the former renders a constant multiplier, and the latter results in a constant multiplier in the single parameter case and an affine multiplier in general. Subsequently, a frequency domain interpretation of these two schemes is given.

## 2 Multipliers and Robust Stability

To motivate the multiplier approach, we consider the following uncertain transfer matrix associated with (6):

$$G(s,q) = C(q)(sI - A_0)^{-1}B + D(q) \quad (10)$$

**Lemma 1.** *The system (6) is stable if and only if $G^{-1}(s,q)$ is stable.*

*Proof.* The characteristic polynomial of the system (6) is given by

$$\begin{aligned} &\det(sI - A_0 + BD^{-1}(q)C(q)) \\ &= \det(sI - A_0)\det(I + (sI - A_0)^{-1}BD^{-1}(q)C(q)) \\ &= \det(sI - A_0)\det(I + D^{-1}(q)C(q)(sI - A_0)^{-1}B) \\ &= \det(sI - A_0)\det(D^{-1}(q))\det(D(q) + C(q)(sI - A_0)^{-1}B) \end{aligned}$$

Hence, the eigenvalues of the system (6) coincides with the poles of $G^{-1}(s,q)$ modulo the stable eigenvalues of $A_0$.

In view of the result above, we are interested in finding an affine transfer matrix, called affine multiplier, of the following form

$$\begin{aligned} K(s,q) &= C_k(q)(sI-A_0)^{-1}B + D_k(q) \\ &= (C_{k0} + \textstyle\sum_{i=1}^{p} q_i C_{ki})(sI-A_0)^{-1}B + (D_{k0} + \textstyle\sum_{i=1}^{p} q_i D_{ki}) \end{aligned} \tag{11}$$

such that the transfer matrix below

$$H(s,q) = K(s,q)G^{-1}(s,q) \tag{12}$$

is strictly positive real (SPR) for all $q \in Q$.

A few remarks are in order.

1. A direct consequence of the existence of such a multiplier is that $G^{-1}(s,q)$ and hence the system (6) are robustly stable. This follows from the trivial fact that an SPR function is stable.
2. If $B$ and $C(q)$ are rank 1, which is slightly more general than the uncertain system (1) where $D(q) = 1$, the corresponding $G(s,q)$ is a single-input single-output transfer function depending on $q$ affinely. That is, the numerator coefficients of $G(s,q)$ are affine in $q$ and the denominator of $G(s,q)$ is parameter independent. In this case, it is known that $G^{-1}(s,q)$ is robustly stable if and only if a parameter-independent multiplier $K(s)$ (possibly with a high order) exists to render $K(s)G^{-1}(s,q)$ robustly SPR; see Anderson *et. al.* [1]. The parametric Lyapunov functions given in [2] are actually based on the existence of such a multiplier. The multipliers we allow in this chapter are more general than [2] in the sense that they are paremeter dependent but more restrictive in the sense that they share the same $A_0$ as $G(s,q)$. As we will see later, this restriction is used to assure that the multiplier will yield a multiaffine Lyapunov matrix (8).
3. Although the multiplier approach is a frequency domain method, it has a natural state space domain interpretation. This interpretation is achieved using a version of the well-known Kalman-Yakubovic-Popov (KYP) Lemma called *Parametric KYP Lemma* that is to be introduced in the next section. This result is the key device for deriving a multiaffine Lyapunov function from a affine multiplier. It is this lemma that motivates the use of parametric multipliers.

## 3 Parametric KYP Lemma

This section presents a a version of the well-known KYP Lemma which is instrumental to dealing with systems with uncertain parameters. To prepare for the result, we first introduce a known generalized KYP lemma by Willems [12].

**Lemma 2. (Generalized KYP Lemma)** *Given $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times k}$ and symmetric $\Omega \in \mathbf{R}^{(n+k)\times(n+k)}$, there exists a symmetric matrix $P \in \mathbf{R}^{n\times n}$ such that*

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \Omega < 0 \tag{13}$$

*if and only if there exists some $\epsilon > 0$ such that*

$$[B^T((sI-A)^{-1})^* \;\; I]\Omega \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix} < 0, \quad \forall \; \mathrm{Re}[s] \geq \epsilon \tag{14}$$

*Further, if $A$ is Hurwitz stable and the upper left $n \times n$ block of $\Omega$ is positive-semidefinite, then $P$ above, when exists, is positive definite.*

Our desired result is given as follows:

**Lemma 3. (Parametric KYP Lemma)** *Given matrices $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}$, $m \leq n$, a hyperrectangular set $Q \subset \mathbf{R}^p$, a parametric matrix $\Omega(q) \in \mathbf{R}^{(n+m)\times(n+m)}$ described by*

$$\Omega(q) = \Omega^T(q) = \Omega_M(q) + \sum_{i=1}^{p} q_i^2 \Omega_{ii} \tag{15}$$

*where $\Omega_M(q)$ is multiaffine in $q$ and*

$$\Omega_{ii} \geq 0, \quad i = 1, \cdots, p \tag{16}$$

*Then the following conditions are equivalent:*

**i)** *There exists $\epsilon > 0$ such that*

$$[B^T((sI-A)^{-1})^* \quad I]\Omega(q) \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix} < 0, \; \forall \; \mathrm{Re}[s] \geq -\epsilon \tag{17}$$

*for all $q \in Q$.*

**ii)** *There exists a multiaffine matrix*

$$P(q) = P^T(q) \in \mathbf{R}^{n\times n}, \quad q \in Q \tag{18}$$

*such that*

$$\Pi(q) = \begin{bmatrix} A^T P(q) + P(q)A & P(q)B \\ B^T P(q) & 0 \end{bmatrix} + \Omega(q) < 0 \tag{19}$$

*for all $q \in Q$.*

**iii)** *The inequality (17) holds at all vertices of $Q$.*

**iv)** *The inequality (19) holds at all vertices of $Q$.*

*Proof.* The equivalences i)<=>iii) and ii)<=>iv) are obvious due to the assumption in (16).

The implication ii)=>i) follows directly from Lemma 2. More precisely, for each fixed $q \in Q$, there exists $\epsilon(q) > 0$ such that (17) holds for all $\mathrm{Re}[s] \geq -\epsilon(q)$. Since $Q$ is a compact set,

$$\epsilon = \min_{q\in Q} \epsilon(q)$$

exists and is positive. Subsequently, (17) holds for $\mathrm{Re}[s] \geq -\epsilon$.

To show iii)=>iv), we apply Lemma 2 and induction. Without loss of generality, we assume $Q = [0,1]^p$.

Suppose iii) holds for $p = 1$, i.e, $Q = [0,\ 1]$. Let $P_0$ and $P_1$ be any solutions to (19) at $q = 0$ and $q = 1$, respectively. (The solutions are guaranteed to exist by Lemma 2.) Also denote the corresponding $\Pi(q)$ by $\Pi_0$ and $\Pi_1$. Define

$$P(q) = (1 - q)P_0 + qP_1 \tag{20}$$

which is affine in $q$, symmetric and positive definite. It is obvious that the corresponding $\Pi(q)$ is negative definite at $q = 0$ or 1. That is, iv) holds for $p = 1$.

Suppose iii)=>iv) for $p = k$, we need to prove it for $p = k + 1$. Assume iii) holds for $p = k + 1$ and write $q = (q^k, q_{k+1})$. Then, (17) holds for all $q^k \in [0,\ 1]^k, q_{k+1} = 0$ and $q_{k+1} = 1$. By the assumption, there exists $P_0(q^k)$ and $P_1(q^k)$, both multiaffine, symmetric and positive definite, such that the corresponding $\Pi(q)$, denoted by $\Pi_0(q^k)$ and $\Pi_1(q^k)$, are negative definite for all $q^k \in [0,\ 1]^k$. Now we apply the same "trick" for $p = 1$ again. That is, define

$$P(q) = (1 - q_{k+1})P_0(q^k) + q_{k+1}P_1(q^k) \tag{21}$$

Then $P(q)$ is multiaffine, symmetric and positive definite. Also, it is straightforward to verify (19) at $q^k \in [0,\ 1]^k$, $q_{k+1} = 0$ and 1, when the above $P(q)$ is applied. That is, iv) holds for $p = k + 1$.

*Remark.* It is important to know that the proof above gives an algorithm for constructing the multiaffine Lyapunov matrix $P(q)$ required for the Parametric KYP Lemma. Alternatively, $P(q)$ can be searched using SDP algorithms because (19) is an LMI at each vertex of $Q$. The latter is efficient especially when only a few terms of $P_i$, $P_{ij}$, $\cdots$ are sought for.

## 4 Main results on parametric Lyapunov functions

Under a "convexity condition" , our main result below provides a necessary and sufficient condition for the existence of a multiplier of the form (11). This condition automatically renders a multiaffine Lyapunov matrix.

**Theorem 4.** *Given the uncertain system in (6), suppose there exists an affine multiplier $K(s,q)$ of the form (11) such that the transfer matrix $H(s,q)$ in (15) is SPR at all vertices of $Q$. In addition, the* convexity condition *below is satisfied:*

$$\mathbf{He}\left[\begin{bmatrix} C_i^T \\ D_i^T \end{bmatrix} [C_{ki} \quad D_{ki}]\right] \leq 0, \quad i = 1, \cdots, p \tag{22}$$

*Then, the following properties hold:*

**i)** *$H(s,q)$ is SPR for all $q \in Q$.*
**ii)** *$H(s,q)$ has the following $n$-th order realization*

$$\begin{aligned} Hs,q) = {} & (C_k(q) - D_k(q)D^{-1}(q)C(q))(sI - A_0 + BD^{-1}(q)C(q))^{-1} \\ & \times BD^{-1}(q) + D_k(q)D^{-1}(q) \end{aligned} \tag{23}$$

**iii)** *There exists a multiaffine $P(q) = P^T(q)$ of (8) to establish the robust SPR property of $H(s,q)$, i.e.,*

$$\Pi(q) = \begin{bmatrix} A^T(q)P(q) + P(q)A(q) & \Pi_{12}(q) \\ \Pi_{12}^T(q) & \Pi 22(q) \end{bmatrix} < 0 \tag{24}$$

*holds for all $q \in Q$, where*

$$\Pi_{12}(q) = P(q)BD^{-1}(q) - C_k^T(q) + C^T(q)D^{-T}(q)D_k^T(q)$$
$$\Pi 22(q) = -(D_k(q)D^{-1}(q) + D^{-T}(q)D_k^T(q))$$

**iv)** *(24) holds for all $q \in Q$ if and only if it holds at all corners of $Q$.*

**v)** *The same $P(q)$ above is a Lyapunov matrix for establishing the robust stability of (6).*

*Conversely, if there exists $P(q)$ of the form (8) and a multiplier $K(s,q)$ of the form (11) such that the convexity condition (22) is satisfied and that the LMI (24) holds at all vertices of $Q$. Then, $K(s,q)G^{-1}(s,q)$ is SPR for all $q \in Q$.*

*Remark.* The property **iv)** shows that the multiaffine Lyapunov matrix $P(q)$ can be searched using LMI techniques. Indeed, the matrix in (24) is linear in $P_0, P_i$, $P_{ij}, \cdots$ ( defined in (8)). Subsequently, the existence of such $P(q)$ is equivalent to that the $2^p$ LMIs (24), one for each corner of $Q$, are feasible.

Now the proof of Theorem 4 follows.

*Proof.* Define

$$\Omega(q) = -\begin{bmatrix} C^T(q) \\ D^T(q) \end{bmatrix} [C_k(q) \;\; D_k(q)] - \begin{bmatrix} C_k^T(q) \\ D_k^T(q) \end{bmatrix} [C(q) \;\; D(q)]$$

Then, $K(s,q)G^{-1}(s,q)$ being SPR for all $q \in Q$ is equivalent to the existence of $\epsilon > 0$ such that

$$-2\mathbf{He}[G^*(s,q)K(s,q)] = [B^T((sI - A_0)^{-1})^* \;\; I]\Omega(q) \begin{bmatrix} (sI - A_0)^{-1}B \\ I \end{bmatrix} < 0$$

for all $\mathrm{Re}[s] \geq -\epsilon$ and $q \in Q$. Note that the quadratic terms in $\Omega(q)$ are given by

$$\Omega_{ii} = -2\mathbf{He}\left[\begin{bmatrix} C_i^T \\ D_i^T \end{bmatrix} [C_{ki} \;\; D_{ki}]\right] \geq 0, \quad i = 1, \cdots, p$$

Using Lemma 3, the SPR condition above is equivalent to the existence of a multiaffine $P(q)$ such that the following holds at all vertices of $Q$:

$$\begin{bmatrix} A_0^T P(q) + P(q)A_0 & P(q)B \\ B^T P(q) & 0 \end{bmatrix} - 2\mathbf{He}\left[\begin{bmatrix} C^T(q) \\ D^T(q) \end{bmatrix} [C_k(q) \;\; D_k(q)]\right] < 0$$

Pre- and post-multiplying both sides by

$$\begin{bmatrix} I & 0 \\ -D^{-1}(q)C(q) & D^{-1}(q) \end{bmatrix}$$

and its transpose, respectively, will yield (24).

*Remark.* **(LMI Solution)** The inequalities (22) and (24) represent a finite set of LMIs ($p$ from (22) and $2^p$ from (24)). These LMIs are jointly convex in $P_0, P_i, P_{ij\ldots}, C_{ki}$ and $D_{ki}$. Thus, efficient LMI algorithms, such as interior point algorithms, can be applied to compute the affine multiplier and and the associated multiaffine Lyapunov matrix.

The following result shows that the form of the affine multiplier can be simplified to have one less pair of $C_{ki}, D_{ki}$. In particular, for single parameter uncertain systems, the existence of affine multipliers is equivalent to the existence of a constant multiplier, provided that the convexity condition mentioned above is satisfied.

**Theorem 5.** *Given the uncertain system in (6), suppose there exists an affine multiplier $K(s,q)$ of the form (11) such that the transfer matrix $H(s,q)$ in (15) is SPR at all vertices of $Q$ and the convexity condition in (22) is satisfied. In addition, assume that $[C_p, \quad D_p]$ has rank $m$ (full rank). Then, there exists $E_p \in \mathcal{R}^{m\times m}$ such that*

$$[C_{kp} \quad D_{kp}] = E_p[C_p, \quad D_p], \quad E_p + E_p^T \leq 0 \tag{25}$$

*Further, Let*

$$\bar{C}_{ki} = C_{ki} - E_pC_i, \quad \bar{D}_{ki} = D_{ki} - E_pD_i, \quad i = 0, 1, \cdots, p-1 \tag{26}$$

*and*

$$\bar{K}(s,q) = (\bar{C}_{k0} + \sum_{i=1}^{p-1} q_i\bar{C}_{ki})(sI - A_0)^{-1}B + (\bar{D}_{k0} + \sum_{i=1}^{p-1} q_i\bar{D}_{ki}) \tag{27}$$

*Then, $\bar{K}(s,q)G^{-1}(s,q)$ is SPR for all $q \in Q$. In particular, if $p = 1$ then the resulting multiplier $\bar{K}(s,q)$ is parameter independent.*

The proof of the theorem above hinges upon the following lemma:

**Lemma 6.** *Suppose $A, B \in \mathcal{R}^{n\times k}, k \leq n$ with rank$(A) = k$ and*

$$AB^T + BA^T \geq 0 \tag{28}$$

*Then, there exists some $H \in \mathcal{R}^{n\times k}$ such that*

$$B = AH, \quad \text{rank}(H) = \text{rank}(A), \quad H + H^T \geq 0 \tag{29}$$

*Proof.* Let $T \in \mathcal{R}^{n\times n}$ be a nonsingular transformation matrix such that

$$TA = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$$

and denote

$$\bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} = TB$$

Then, (28) implies

$$\begin{bmatrix} I_k \\ 0 \end{bmatrix} [\bar{B}_1^T \quad \bar{B}_2^T] + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} [I_k \quad 0] = \begin{bmatrix} \bar{B}_1 + \bar{B}_1^T & \bar{B}_2^T \\ \bar{B}_2 & 0 \end{bmatrix} \geq 0 \tag{30}$$

This results in $\bar{B}_2 = 0$. Consequently,

$$TB = \begin{bmatrix} I_k \\ 0 \end{bmatrix} \bar{B}_1 = TA\bar{B}_1$$

So, $H = \bar{B}_1$ and $B = AH$ and $\mathrm{rank}(B) = \mathrm{rank}(H)$. Finally, $H + H^T \geq 0$ follows from (28) and the full rank property of $A$. So (29) holds.

Now the proof of Theorem 5 follows.

*Proof.* Using Lemma 6, the convexity condition (22) and the full rank property of $[C_p, \quad D_p]$ imply that $E_p$ exists to satisfy (25). Since $K(s,q)G^{-1}(s,q)$ is SPR for all $q \in Q$, there exists $\epsilon > 0$ such that

$$\mathbf{He}[G^*(s,q)K(s,q)] > 0, \quad \mathrm{Re}[s] \geq -\epsilon, \ q \in Q$$

It follows that

$$\begin{aligned} \mathbf{He}[G^*(s,q)\bar{K}(s,q)] &\geq \mathbf{He}[G^*(s,q)\bar{K}(s,q)] + G^*(s,q)\mathbf{He}[E_p]G(s,q) \\ &= \mathbf{He}[G^*(s,q)(\bar{K}(s,q) + E_pG(s,q))] \\ &= \mathbf{He}[G^*(s,q)K(s,q)] > 0, \quad \mathrm{Re}[s] \geq -\epsilon, \ q \in Q \end{aligned}$$

*Remark.* The assumption that $[C_p, \quad D_p]$ is full rank is always satisfied for the single parameter case. This condition, however, may not be satisfied in general. Whenever this condition holds, the result above shows that the LMIs (22) and (24) are simplified without any harm by setting $[C_{kp} \quad D_{kp}] = 0$. It is however, difficult to simplify further because the resulting $[\bar{C}_{ki}, \bar{D}_{ki}]$ may not meet the convexity condition.

*Remark.* The result above shows how to simplify the multiplier. But it makes no implication that similar simplification can be made for the Lyapunov matrix. For example, it is not claimed that a constant multiplier renders a constant Lyapunov matrix.

## 5 Comparison with other schemes

Theorem 4 gives a sufficient condition for the existence of multiaffine Lyapunov matrix that is of the same order as the plant. In this section, we show that two other known schemes for obtaining parametric Lyapunov functions are special cases of Theorem 4. One of these two schemes is a generalized Popov criterion derived based on the so-called S-procedure [4, 3, 6, 5, 11], and the other is called *affine quadratic stability* (AQS) test by Gahinet *et. al.* [3]. We show that the former gives a parameter independent multiplier, while the latter gives a parameter independent multiplier in the single parameter case and an affine multiplier in general.

### 5.1 Generalized Popov Criterion

Consider the following system:

$$\dot{x} = (A_0 + \sum_{i=1}^{p} q_i A_i)x \tag{31}$$

Or equivalently,

$$\begin{aligned} \dot{x} &= A_0 x + \sum_{i=1}^{p} A_i w_i \\ y_i &= x \\ w_i &= q_i y_i, \quad i = 1, \cdots, p \end{aligned} \tag{32}$$

where $q = (q_1, \cdots, q_p) \in Q = [-1,\ 1]^p$ (without loss of generality), $A_0, A_i \in \mathbf{R}^n$, $i = 1, \cdots, p$, and $A_0$ is asymptotically stable.

The generalized Popov criterion seeks for a Lyapunov function of the following form [3]:

$$V(x,t) = \frac{1}{2} x^T (P_0 + \sum_{i=1}^{p} q_i P_i) x + \sum_{i=1}^{p} (1 - q_i^2) \int_0^t y_i^T S_i y_i d\tau \tag{33}$$

where $S_i = S_i^T \geq 0$ and $(P_0 + \sum_i q_i P_i) = (P_0 + \sum_i q_i P_i)^T > 0$ for all $q_i \in [-1,\ 1]$. Note that this function appears to be more general than an affine quadratic Lyapunov function due to the integral terms.

The following sufficient condition is established [4, 3] which is generalized from [5, 6, 11]:

**Lemma 7.** *The system (31) is robustly stable with the Lyapunov function (33) if there exists symmetric matrices $P_0$, $P_d = \mathrm{diag}\{P_i\}$, $S = \mathrm{diag}\{S_i\} \geq 0$ and skew symmetric matrix $T = \mathrm{diag}\{T_i\}$ such that the following LMI is satisfied:*

$$\begin{bmatrix} A_0^T P_0 + P_0 A_0 + C^T S C & P_0 B + A_0^T C^T P_d + C^T T \\ B^T P_0 + P_d C A_0 - T C & B^T C^T P_d + P_d C B - S \end{bmatrix} < 0 \tag{34}$$

*where $B = [A_1, A_2, \cdots, A_p]$ and $C^T = [I, I, \cdots, I]$.*

What we intend to show here is that the condition in Lemma 7, if satisfied, will lead to a parameter independent multiplier $K(s)$ which in turn leads to a multiaffine Lyapunov function *without* an integral term (*c.f.* (33)). The following is the result:

**Theorem 8.** *Consider the uncertain system (32) and suppose that the condition in Lemma 7 is satisfied. Define $\Delta = \mathrm{diag}\{q_i I_n\}$, $C(q) = \Delta C$,*

$$G(s,q) = I - C(q)(sI - A_0)^{-1} B \tag{35}$$

$$K(s) = -(TC + P_d C A_0)(sI - A_0)^{-1} B + (S - P_d C B) \tag{36}$$

*Then, $K(s)$ is invertible and $K(s) G^{-1}(s,q)$ is SPR for all $q \in Q$.*

*Proof.* Suppose the condition in Lemma 7 holds. A straightforward application of Lemma 3 yields

$$G^*(s)SG(s) - S + (sP_d + T)G(s) + G^*(s)(sP_d + T)^* < 0, \quad \forall \ \mathrm{Re}[s] \geq -\epsilon$$

for some $\epsilon > 0$, where $G(s) = C(sI - A_0)^{-1}B$.

Define

$$X(s) = S + G^*(s)(sP_d + T)$$

$$Y(s) = G^*(s)SG(s) - S + (sP_d + T)G(s) + G^*(s)(sP_d + T)^*$$

It is straightforward to verify that

$$\begin{aligned} & X(s)G(s,q) + G^*(s,q)X^*(s) \\ = & -Y(s) + G^*(s)S^{1/2}(I - \Delta^*\Delta)S^{1/2}G(s) \\ & +(S^{1/2} - \Delta S^{1/2}G(s))^*(S^{1/2} - \Delta S^{1/2}G(s)) \\ \geq & -Y(s) > 0, \qquad \forall \ \mathrm{Re}[s] \geq -\epsilon, \ \Delta \end{aligned}$$

That is,

$$\mathbf{He}[(S + G^*(s)(sP_d + T))G(s,q)] > 0, \quad \forall \ \mathrm{Re}[s] \geq -\epsilon \tag{37}$$

holds for all $\Delta$. The idea above is in fact borrowed from [7]. It is straightforward to verify that an alternative representation of $K(s)$ in (36) is given by

$$K(s) = S - (sP_d + T)C(sI - A_0)^{-1}B$$

Noting that $S$ and $P_d$ are symmetric and $T$ is skew symmetric, (37) is equivalent to

$$\mathbf{He}[G^*(s,q)K(s)] = \mathbf{He}[K^*(s)G(s,q)] > 0, \quad \forall \ \mathrm{Re}[s] \geq -\epsilon$$

That is, $K(s)G^{-1}(s,q)$ is SPR for all $q \in Q$.

*Remark.* Note that the Lyapunov function in (33) involves integral terms. However, the existence of a multiplier $K(s)$ in (36) implies that an alternative Lyapunov function can be found which does not involve integral terms. This claim follows from the property **iv)** in Theorem 4. The tradeoff is that the new Lyapunov function may involve multiaffine terms. But in the special case where $p = 1$, we can conclude that the Lyapunov function obtained using Theorems 8 and 4 is affine.

### 5.2 The Affine Quadratic Stability (AQS) Test

Consider the uncertain system (31) and the affine Lyapunov matrix

$$P(q) = P_0 + \sum_{i=1}^{p} q_i P_i \tag{38}$$

The AQS test proposed in Gahinet *et. al.* [3] amounts to finding such $P(q)$ that the following two conditions are satisfied:

$$(A_0 + \sum_{i=1}^{p} q_i A_i)^T P(q) + P(q)(A_0 + \sum_{i=1}^{p} q_i A_i) < 0, \quad \forall\ q_i = \pm 1; \tag{39}$$

$$A_i^T P_i + P_i A_i \geq 0, \quad \forall\ i = 1, \cdots, p \tag{40}$$

The use of the constraint (40), which adds to the conservatism of the method, is to assure that (39) holds for all $q \in Q$. The AQS test requires to solve the set of LMIs above.

The result below gives an interpretation of the AQS in terms of the multiplier approach.

**Theorem 9.** *Given the uncertain system (31). Suppose there exists an affine Lyapunov matrix of the form (38) such that (39) holds. Define*

$$G(s, q) = I - (\sum_{i=1}^{p} q_i A_i)(sI - A_0)^{-1} \tag{41}$$

*and*

$$K(s, q) = (P_0 + \sum_{i=1}^{p} q_i P_i)(sI - A_0)^{-1} \tag{42}$$

*Then, $K(s,q)G^{-1}(s,q)$ is SPR for all $q \in Q$. The convexity condition for $K(s,q)$ as stated in Theorem 4 holds when (40) is satisfied.*

*Further, for the single parameter case, let $A_1$ be decomposed into*

$$A_1 = BC \tag{43}$$

*where $B \in \mathbf{R}^{n \times m}$, $C \in \mathbf{R}^{m \times n}$ are full rank matrices and $m \leq n$. Then, there exists some $H \in \mathbf{R}^{k \times k}$ such that*

$$B^T P_1 = HC, \quad \text{rank}(H) = \text{rank}(B^T P_1), \quad H + H^T \geq 0 \tag{44}$$

*and $\bar{K}(s)\bar{G}(s,q)^{-1}$ is SPR for all $q \in [-1,\ 1]$, where*

$$\bar{G}(s, q) = I - qC(sI - A_0)^{-1}B \tag{45}$$

*and*

$$\bar{K}(s) = B^T P_0 (sI - A_0)^{-1} B + H \tag{46}$$

*Proof.* The properties about $K(s,q)G^{-1}(s,q)$ are trivial. The existence of $H$ follows from Lemma 6. To show that $\bar{K}(s)\bar{G}^{-1}(s,q)$ is SPR for all $q \in [-1,\ 1]$, we take $p = 1$ and note that (39) implies the existence of $\epsilon > 0$ such that

$$\mathbf{He}[(sI - A_0 - qBC)^*(P_0 + qP_1)] > 0, \quad \forall\ \text{Re}[s] \geq -\epsilon,\ q \in [-1,\ 1]$$

Pre- and post-multiplying the above by the full rank matrix $(sI - A_0)B$ and its Hermitian gives

$$\mathbf{He}[\bar{G}^*(s,q)(B^T P_0(sI - A_0)^{-1}B + qB^T P_1(sI - A_0)^{-1}B))] > 0$$

Using (44), the above becomes

$$\begin{aligned} 0 &< \mathbf{He}[\bar{G}^*(s,q)(B^T P_0(sI - A_0)^{-1}B + qHC(sI - A_0)^{-1}B)] \\ &= \mathbf{He}[\bar{G}^*(s,q)(B^T P_0(sI - A_0)^{-1}B + H - H\bar{G}(s,q))] \\ &\le \mathbf{He}[\bar{G}^*(s,q)(B^T P_0(sI - A_0)^{-1}B + H)] \\ &= \mathbf{He}[(\bar{G}^*(s,q)\bar{K}(s)], \quad \forall\ \mathrm{Re}[s] \ge -\epsilon,\ q \in [-1,1] \end{aligned}$$

So $\bar{K}(s)\bar{G}^{-1}(s)$ is SPR for all $q \in [-1,\ 1]$.

## 6 Generalization to Discrete-time Systems

Consider the uncertain discrete-time system:

$$x(k+1) = A(q)x(k) = (A_0 + BD^{-1}(q)C(q))x(k) \tag{47}$$

where all the entries are the same as in the continuous time case. Searching for parametric Lyapunov functions in the discrete-time case seems more involved than in the continuous-time case. The reason is that the stability requirement in the discrete-time case becomes finding $P(q) = P^T(q) > 0$ such that

$$A^T(q)P(q)A(q) - P(q) < 0, \quad \forall\ q \in Q \tag{48}$$

Even when $A(q)$ and $P(q)$ are both affine, the inequality above involves cubic terms. Alternatively, the following equivalent condition to (48) can be analyzed:

$$\begin{bmatrix} -P(q) & A^T(q) \\ A(q) & -P^{-1}(q) \end{bmatrix} < 0 \tag{49}$$

This condition is often used for quadratic stability analysis when $P(q)$ is restricted to be constant. When parametric Lyapunov matrices are in consideration, the condition (49) again seems to be powerless because $P^{-1}(q)$ is nonlinear in $q$.

In this section, we show that the multiplier idea studied in previous sections can be generalized to discrete-time systems with ease. We first introduce a counterpart of the Parametric KYP Lemma for the discrete-time.

**Lemma 10. (Discrete-time Parametric KYP Lemma)** *Given matrices $A, B$, $\Omega(q), Q$ as in Lemma 2. Then the following two conditions are equivalent:*

**i)** *There exists $0 < \epsilon < 1$ such that*

$$\begin{aligned} [B^T((zI - A)^{-1})^* \quad I]\Omega(q) \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix} < 0, \\ \forall\ |z| \ge 1 - \epsilon,\ q \in Q \end{aligned} \tag{50}$$

**ii)** *There exists a multiaffine matrix*

$$P(q) = P^T(q) \in \mathbf{R}^{n\times n}, \quad q \in Q \tag{51}$$

*such that*

$$\Pi(q) = \begin{bmatrix} A^T P(q)A - P(q) & A^T P(q)B \\ B^T P(q)A & B^T P(q)B \end{bmatrix} + \Omega(q) < 0 \tag{52}$$

*for all $q \in Q$.*

**iii)** *The inequality (50) holds at all vertices of $Q$;*

**iv)** *The inequality (52) holds at all vertices of $Q$.*

*Proof.* The proof is essentially the same and is based on a discrete-time version of Lemma 2. So the details are omitted.

Applying the lemma above, the counterpart of Theorem 4 is easily obtained.

**Theorem 11.** *Given the uncertain system in (47), suppose there exists an affine multiplier $K(z,q)$ of the form (11) such that the transfer matrix $H(z,q)$ in (12) is SPR at all vertices of $Q$. In addition, the convexity condition in (22) is satisfied. Then, the following properties hold:*

**i)** *$H(z,q)$ is SPR for all $q \in Q$.*

**ii)** *$H(z,q)$ has the following $n$-th order realization*

$$\begin{aligned} H(z,q) = {} & (C_k(q) - D_k(q)D^{-1}(q)C(q))(zI - A_0 + BD^{-1}(q)C(q))^{-1} \\ & \times BD^{-1}(q) + D_k(q)D^{-1}(q) \end{aligned} \tag{53}$$

**iii)** *There exists a multiaffine $P(q) = P^T(q)$ to establish the robust SPR property of $H(z,q)$, i.e.,*

$$\Pi(q) = \begin{bmatrix} A^T(q)P(q)A(q) - P(q) & \Pi_{12}(q) \\ \Pi_{12}^T(q) & \Pi_{22}(q) \end{bmatrix} < 0 \tag{54}$$

*holds for all $q \in Q$, where*

$$\Pi_{12}(q) = A^T(q)P(q)BD^{-1}(q) - C_k^T(q) + C^T(q)(D^T)^{-1}(q)D_k^T(q)$$
$$\Pi_{22}(q) = (D^T)^{-1}B^T P(q)BD^{-1}(q) - 2\mathbf{He}[D_k(q)D^{-1}(q)]$$

**iv)** *The same $P(q)$ above is a Lyapunov matrix for establishing the robust stability of (47).*

*Conversely, suppose there exists $P(q)$ of the form (8) and a multiplier $K(z,q)$ of the form (11) such that the convexity condition (22) is satisfied and that the LMI (54) holds at all vertices of $Q$. Then, $K(z,q)G^{-1}(z,q)$ is SPR for all $q \in Q$.*

*Proof.* The proof is virtually identical to the continuous-time case. The details are thus omitted.

*Remark.* Note that, as in the continuous time case, (54) is affine in $P_0, P_i, P_{ij}$, $\cdots$, $C_{ki}, D_{ki}$. Hence, finding a multiaffine Lyapunov matrix $P(q)$ amounts to solving a finite number of LMIs ($p$ for (22) and $2^p$ for (54) at the vertices of $Q$).

## 7 Conclusions

In this chapter, we have studied the use of the multiplier approach to generate parametric Lyapunov functions for linear systems with parameter uncertainty. In the process of doing so, we have derived an extended version of the KYP lemma, parametric KYP lemma, as a general tool to study the robust stability with parameter uncertainty. Using this lemma, we have provided conditions under which an affinely parameterized multiplier exists to establish the robust stability of the uncertain system. This type of parametric multiplier then naturally leads to a multiaffine Lyapunov function that can be used to establish robust stability in the state space domain. Although not studied in this chapter, we point out that parametric Lyapunov functions can also be used in dealing with time-varying parameters; see [2]. Also shown in this chapter is that some previous results in the literature on parametric Lyapunov functions lead to special multipliers. This analysis confirms the generality of our approach. We have also demonstrated the ease of adapting our approach to discrete-time systems despite of the observation that uncertain parameters in the discrete-time case appear to be harder to deal with.

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