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Computational Complexity of Real Structured Singular Value in ℓ_p Setting *

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Abstract

This paper studies the structured singular value (μ) problem with real parameters bounded by an ℓ_p norm. Our main result shows that this generalized μ problem is NP-hard for any given rational number $p \in [1, \infty]$, whenever k, the size of the smallest repeated block, exceeds 1. This result generalizes the known result that the conventional μ problem (with $p = \infty$) is NP-hard. However, our proof technique is different from the known proofs for the $p = \infty$ case as these proofs do not generalize to $p \neq \infty$. For $k = 1$ and $p = \infty$, the μ problem is known to be NP-hard. We provide an alternative proof of this result. For $k = 1$ and p finite the issue of NP-hardness remains unresolved. When every block has size 1, and $p = 2$ we outline some potential difficulties in computing μ.

1 Introduction

The problem of real structured singular value (real μ) arises in many robust control problems where the control system is subject to uncertain parameters. See, e.g., [3, 4, 5, 7, 2] for motivations and references.

Given a matrix $M \in \mathbf{C}^{n\times n}$ and a set $\mathbf{\Delta}$ described by

$$\mathbf{\Delta} = \{\Delta = \text{diag}\{\delta_1 I_{k_1}, \cdots, \delta_m I_{k_m}\} \mid \delta_i \in \mathbf{R}\}, \quad k_i > 0,\ \sum_{i=1}^{m} k_i = n \tag{1}$$

the real μ problem is to compute the value of $\mu_{\mathbf{\Delta}}(M)$. This value is defined to be 0 if $I_n - \Delta M$ is nonsingular for all $\Delta \in \mathbf{\Delta}$, or otherwise

$$\mu_{\mathbf{\Delta}}(M) = (\inf\{\alpha > 0 \mid \det(I_n - \Delta M) = 0, \ ||\delta||_\infty \leq \alpha\})^{-1} \tag{2}$$

where $||\cdot||_\infty$ denotes the ℓ_∞ norm and

$$\delta = \text{diag}\{\delta_1, \cdots, \delta_m\}. \tag{3}$$

Henceforth m will denote the *size of the problem.* It is known that the problem of determining if $\mu_{\mathbf{\Delta}}(M) < 1$ is NP hard, see Poljak and Rohn [9], Braatz *et. al.* [1], Nemirovskii [8], and Coxson and DeMarco [2]. This negative result means that finding an algorithm for computing $\mu_{\mathbf{\Delta}}(M)$ is very unlikely if the algorithm is forced to require a number of computations that rises at most polynomially in m, i.e. the problem can be solved in *polynomial time.*

In this paper, we study a generalized μ problem by allowing the norm on δ to be an ℓ_p norm for any $p \in [1, \infty]$. More precisely, given $M, \mathbf{\Delta}$, and $p \in [1, \infty]$, we define $\mu_{\mathbf{\Delta},p}(M)$ to be zero if $I_n - \Delta M$ is nonsingular for all $\Delta \in \mathbf{\Delta}$, or otherwise

$$\mu_{\mathbf{\Delta},p}(M) = (\inf\{\alpha > 0 \mid \det(I_n - \Delta M) = 0, \ ||\delta||_p \leq \alpha\})^{-1} \tag{4}$$

For simplicity, we will denote $\mu_{\mathbf{\Delta},p}(M)$ by μ_p. Our objective is to analyze the computational complexity of the μ_p problem.

One might hope that the techniques used in [9, 1, 2] for showing the NP-hardness of the μ_∞ problem is generalizable to the μ_p case. Unfortunately, this is not the case. Examining [9, 1, 8, 2], we find that all the NP-hardness proofs for the μ_∞ problem rely (directly or indirectly) on a well-known fact that the following quadratic program is NP-hard for $p = \infty$: Given a positive-definite and symmetric rational matrix Q, determine if

$$\max_{||x||_p \leq 1} x^T Q x < 1 \tag{5}$$

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See Vavasis [10, Exercise 4.3, p. 101]. However, when $p \neq \infty$, it is not clear whether the problem remains NP-hard. Further, it is known that for $p = 2$ the quadratic program above is in the class of P, i.e. a ploynomial time algorithm can be formulated to provide its solution; see Ye [11] for such an algorithm. Hence, a new technique is needed to investigate the computational complexity of the μ_p problem. The difficulty with the μ_p problem, $p \neq \infty$, is that there is a single constraint on δ rather than multiple constraints as in the μ_∞ case.

2 Main Result

Despite the differences between the μ_∞ problem and the μ_p problem, $p \neq \infty$, as discussed above, we still have a negative result for the latter.

To explain this negative result, we define $k = \min\{k_1, \cdots, k_m\}$, which is the smallest size of the repeated blocks, and denote μ_p by $\mu_p(k)$, an explicit function of k. Then, our main result is as follows:

Theorem 1 *Given any (fixed) $k \geq 2$ and $p \in [1, \infty)$, the problem of determining if $\mu_p(k) < 1$ is NP-hard. Further, the problem of determining if $\mu_\infty(1) < 1$ is also NP-hard.*

Proof. As in almost all NP-hardness analysis cases, our basic idea is to polynomially transform a known NP-hard problem to the problem of determining if $\mu_p(k) < 1$. Two problems are said to be related by a polynomial transformation if (i) a polynomial number of operations can be used to transform the first problem to the second; and (ii) the size of the first depends polynomially on the size of the second. The μ_p and μ_∞ statements in the thoerem are proved separately in two parts.

Part 1: We start with the following NP-hard problem.

The 0-1 Knapsack Problem: Given an integer vector $c = (c_1, \cdots, c_m)^T$, determining if there exists a binary vector $x = (x_1, \cdots, x_m)^T \in \{-1,\ 1\}^m$ such that $c^T x = 0$.

It is well-known that the Knapsack problem is NP-hard; see, e.g., Garey and Johnson [6].

Let c be the integer vector in the Knapsack problem. Denote $p = \beta/\alpha$,where α and β are coprime positive integers. Without loss of generality, we assume that the size m of the Knapsack problem is such that the number $(2m)^{-1/p}$ is rational. If this is not the case, one can augment enough number of zero components to the vector c such that the new size, say $\hat{m}$, is such that $(2\hat{m})^{-1/p}$ is rational. One particular choice of $\hat{m}$ is given by $\hat{m} = (2m)^\beta/2$. Note that such augmentation does not alter the solvability of the Knapsack problem and that the new size $\hat{m}$ is polynomial in m.

Define $n = 2m$ and

$$f(\delta) = j\sum_{i=1}^{m} c_i(\delta_i - \delta_{m+i}) + \sum_{i=1}^{m}\left[(\delta_i + \delta_{m+i})^2 + (\delta_i\delta_{m+i} + d^2)^2\right], \quad (6)$$

where $j = \sqrt{-1}$, $\delta = (\delta_1, \cdots, \delta_n)^T$ and $d = (2m)^{-1/p} > 0$.

Obviously, necessary and sufficient conditions for $f(\delta) = 0$ for some $\delta \in \mathbf{R}^n$ are that

$$\delta_i = -\delta_{m+i}, \quad |\delta_i| = d, \quad i = 1, \cdots, m \quad (7)$$

and

$$\sum_{i=1}^{m} c_i\delta_i = 0 \quad (8)$$

Relating x_i in the Knapsack problem to δ_i, $i = 1, \cdots, m$, by

$$x_i = \frac{\delta_i}{d}$$

we know that the Knapsack problem has a solution $x \in \{-1,\ 1\}^m$ if and only if $f(\delta) = 0$ for some $||\delta||_p \leq 1$. Since the former problem is NP-hard, it follows that the problem of determining if $f_1(\delta) \neq 0$ for all $||\delta||_p \leq 1$ is NP-hard.

Now we need to transform $f(\delta)$ to some $\det(I - \Delta M)$ with $k = 2$. Define for all $i \in \{1, \cdots, m\}$:

$$g_i = [-1 \quad 1 - d^4 + jc_1 \quad \delta_i \quad 1 - d^4]'$$

$$h_i = [1 \quad \delta_{m+i} \quad -1 - 2d^2 - jc_1 \quad 1]'$$

and

$$D_i = \begin{pmatrix} 1 & -\delta_{m+i} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \delta_i & 1 & 1 & -2d^2 - 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Further define

$$A(\delta) = \left(\begin{array}{c|cccc} 0 & g_1' & g_2' & \cdots & g_m' \\ \hline h_1 & D_1 & & & \\ h_2 & & D_2 & & \\ \vdots & & & \ddots & \\ h_m & & & & D_m \end{array}\right).$$

Observe, for each $i \in \{1, \cdots, m\}$

$$det(D_i) = 1.$$

Further

$$\begin{aligned} & g_i' D_i^{-1} h_i \\ = \; & g_i' \begin{pmatrix} 1 & \delta_{m+i} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\delta_i & -\delta_i\delta_{m+i} - 2d^2 - 2 & 1 & 2d^2 + 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} h_i \\ = \; & c_i(\delta_i - \delta_{m+i}) + (\delta_i + \delta_{m+i})^2 + (\delta_i\delta_{m+i} + d^2)^2. \end{aligned}$$

Thus,

$$\begin{aligned} det(A(\delta)) &= -\sum_{i=1}^{m} g_i' D_i^{-1} h_i \\ &= j \sum_{i=1}^{m} c_i(\delta_i - \delta_{m+i}) \\ &\quad + \sum_{i=1}^{m} \left[(\delta_i + \delta_{m+i})^2 + (\delta_i \delta_{m+i} + d^2)^2 \right] \\ &= f(\delta). \end{aligned}$$

Since each δ_i appears in only two rows in $A(\delta)$ and that too in an affine fashion, for each $i \in \{1, \cdots, 2m\}$,

$$\text{rank} \ \left[\frac{\partial A(\delta)}{\partial \delta_i} \right] \leq 2.$$

Thus we can write $A(\delta)$ as follows:

$$A(\delta) = A_0 - \sum_{i=1}^{2m} \delta_i B_i C_i^T \tag{9}$$

where B_i and C_i are matrices with two columns only. Further,

$$\det A(0) = f(0) = \sum_{i=1}^{m} d^4 = md^4 \neq 0$$

It follows that

$$f(\delta) = \det A(0) \det \left(I - \sum_{i=1}^{2m} \delta_i (A_0^{-1} B_i) C_i^T \right)$$

Let

$$B = \left[A_0^{-1} B_1 \ \cdots \ A_0^{-1} B_{2m} \right]; \quad C = \left[C_1 \ \cdots \ C_{2m} \right];$$

$$M = C^T B; \quad \Delta = \text{diag}\{\delta_1 I_2, \ \cdots, \ \delta_{2m} I_2\}$$

Then,

$$f(\delta) = md^4 \det(I - B\Delta C^T) = md^4 \det(I - \Delta M)$$

So, $\det(I - \Delta M) \neq 0$ for all $||\delta||_p \leq 1$ if and only if $f(\delta) \neq 0$ for all $||\delta||_p \leq 1$, which is NP-hard to determine. Hence, the problem of determining $\mu_p(k) < 1$ is NP-hard for all $k_i = 2$ and rational $p \geq 1$. The fact that the same applies when the $k_i \geq 2$, follows by taking the above constructed M and suitably augmenting it with zero rows and columns.

Part 2: The proof that the problem of determining if $\mu_\infty(1) < 1$ is NP-hard can be found in [8]. A similar proof is included below for completeness. Take the quadratic program in (5). We first argue that for any positive semi-definite symmetric Q,

$$\max_{||x||_\infty \leq 1} x^T Q x = \max_{||x||_\infty \leq 1, ||y||_\infty \leq 1} x^T Q y \tag{10}$$

This is easily shown because

$$\begin{aligned} \max_{||x||_\infty \leq 1} x^T Q x &\leq \max_{||x||_\infty \leq 1, ||y||_\infty \leq 1} x^T Q y \\ &\leq \max_{||x||_\infty \leq 1, ||y||_\infty \leq 1} \left(x^T Q x \right)^{1/2} \left(y^T Q y \right)^{1/2} \\ &\leq \max_{||x||_\infty \leq 1} (x^T Q x)^{1/2} \max_{||y||_\infty \leq 1} (y^T Q y)^{1/2} \\ &= \max_{||x||_\infty \leq 1} x^T Q x \end{aligned}$$

Next, we define

$$\delta = \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$A(\delta) = \begin{pmatrix} 1 & x^T \\ y & Q^{-1} \end{pmatrix},$$

with Q positive definite symmetric. Then,

$$\det A(\delta) = \frac{1 - x^T Q y}{\det Q}.$$

It follows that (5) holds if and only if

$$\det A(\delta) \neq 0, \quad \forall \ ||\delta||_\infty \leq 1 \tag{11}$$

As done in **Step 1**, it is straightforward to construct M and Δ such that

$$\det A(\delta) = \det A(0) \det(I - \Delta M)$$

Further, since each x_i or y_i appears in at most one row in $A(\delta)$, the resulting Δ has $k_1 = \cdots = k_{2m} = 1$. Then by suitably augmenting the M matrix by zero rows and columns, if need be, it follows that the problem of determining if $\mu_\infty(1) < 1$ is NP-hard. □

3 Some Remarks

The result in Theorem 1 leaves one question unanswered: Is the problem of determining whether $\mu_p(1) < 1$ NP-hard for $p \neq \infty$? In the following, we offer some remarks on this problem when $p = 2$ and every $k_i = 1$.

First, we note that $f(\delta) = \det(I - \Delta M)$ is a multilinear function in δ when $k = 1$. If either i) $f(\delta)$ is real and bilinear in δ_i or ii) $f(\delta)$ is complex and linear in δ_i, then checking if $\mu_2(1) < 1$ is a special quadratic problem with $p = 2$. In these cases, the problem has polynomial complexity (provided that M is rational), as pointed out in Section 1. Unfortunately, this observation does not generalize.

Secondly, we define the unit ball

$$B = \{\delta : ||\delta||_2 \leq 1\} \tag{12}$$

$$F(B) = \{\det(I - \Delta M) : \delta \in B\} \in \mathbf{R}^2 \tag{13}$$

and use the symbol $\partial(X)$ to denote the boundary of a set X. We ask the following question: Is $\partial F(B) \subset F(\partial B)$?

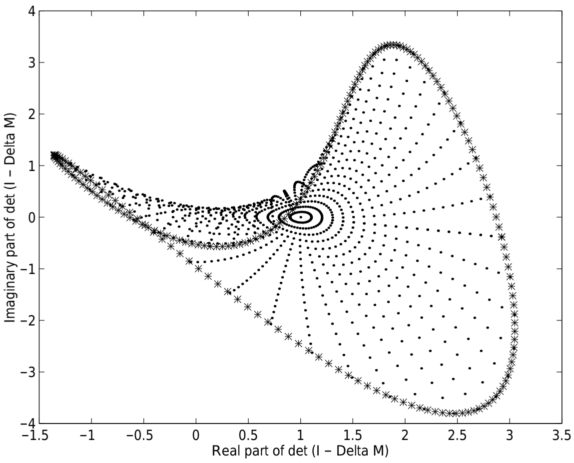
The motivation of this question is simple because an affirmative answer to this question would make it sufficient to consider ∂B alone to solve the $\mu_2(1)$ problem. Unfortunately, the following example shows that the answer is negative.

Example: Take

$$M = \begin{pmatrix} 0.3846 + 1.9231i & 0.0769 + 1.3846i \\ 0.3846 + 1.9231i & -1.9231 + 0.3846i \end{pmatrix} \quad (14)$$

$$\Delta = \text{diag}\{\delta_1,\ \delta_2\} \quad (15)$$

Both $F(B)$ and $F(\partial B)$ are illustrated in Figure 1 by dots and asterisks, respectively. It is clear that $\partial F(B) \not\subset F(\partial B)$ in this case. Also observed in this example is that $0 \in F(B)$ but $0 \notin F(\partial B)$.



4 Conclusion

Our main result shows that the generalized μ problem is NP-hard regardless what l_p norm is used to measure the uncertainty block, as long as the minimum block size is 2. Thus, the difficulty in computing μ is not unique to the l_∞ measure of the uncertainty. The case when $k = 1$ remains unsolved apart from the usual μ problem ($p = \infty$). A nice feature of a subset of this case, namely when each $k_i = 1$, is that $\det(I - \Delta M)$ is multilinear in δ. Nevertheless, difficulties exist in dealing even with this special case, as illustrated in the example in Section 3.

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