Linear Quadratic Control with Input Saturation

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Abstract. This paper studies a new approach to linear quadratic control for linear systems with input saturation. Our work presents an optimal sector bound to model the mismatch between the unsaturated controller and saturated one and an optimised control design associated with this sector bound. This leads to a new characterisation of invariant sets and new switching controllers. The main outcome of this paper is that better performance can be guaranteed for the same region of attraction, or equivalently, a larger region of attraction is given for the same level of guaranteed performance.

1 Introduction

In this paper, we consider the problem of linear quadratic control for linear systems with input saturation. This problem has been widely studied and many design methods are available; see, e.g., [1–4].

When an optimal control input exceeds a given level of saturation, it is well-known that optimal performance can not be achieved by simply saturating the control input, unless the level of over-saturation is sufficiently small [4]. To so-called *anti-windup* technique is commonly used to overcome the saturation. The key to most anti-windup methods is to "de-tune" the optimal controller in some way. That is, a lower control gain is used when the state is large and the control gain is gradually increased when the state becomes small. Many ad-hoc methods were used in early days, but with little theoretical guarantee on stability. However, many rigorous design methods are available now to provide some guaranteed properties on stability [2–4].

To assure stability, most recent anti-windup design methods use the idea of nested ellipsoidal invariant sets. More precisely, a sequence of ellipsoids are given in the state space along with a sequence of controllers. The design is done such that each ellipsoid is an invariant set, the corresponding controller is asymptotically stabilising, and the ellipsoids are nested. The overall control law is of a switching type, i.e., the selected controller corresponds to the smallest ellipsoid in which the state resides. A common approach used to compute these ellipsoids and controllers is to "de-tune" the optimal controller. That is, the ellipsoids and the controllers are constructed by adding cost penalty on the control. The larger the cost penalty, the lower the control gain and the larger the ellipsoid. This idea is supported by the observation that low-gain controllers tend to improve the stability at the cost of performance. Although this idea is intuitive and practical, it is not clear in general how to design these ellipsoids (and the controllers) to give the best performance bound.

Despite the differences in various anti-windup design methods, most of them, if not all, use a sector bound on the mismatch between an unsaturated controller and a saturated one. Different design methods use different sector bounds and use them in different ways. No rigorous study has been done on how to optimally choose a sector bound and how to optimally use a given sector bound.

In this research, we consider the problem of designing a linear controller to optimise a given quadratic cost function. We also use a sector bound to model the mismatch between the unsaturated controller and the saturated one. However, we aim to use a least conservative sector bound and use it in a least conservative way. The main outcome of this research is that better performance can be guaranteed for the same region of attraction, or equivalently, a larger region of attraction is given for the same level of guaranteed performance.

This paper is organised as follows: Section 2 deals with the problem of designing a linear time-invariant state feedback controller to give the best performance bound. Section 3 studies the key properties of this optimised linear time-invariant controller. Section 4 deals with the problem of designing switching controller for the purpose of improving the performance. Section 5 gives a simple illustrative example. The conclusions are given in Section 6.

2 Linear Time-invariant Control

The system we consider in this paper is given by

$$\dot{x} = Ax + b\sigma(u), \quad x(0) = x_0 \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}$ is the input, $A \in \mathbf{R}^{n\times n}$ and $b \in \mathbf{R}^n$ are constant, and $\sigma(\cdot)$ is a saturation function with saturation level equal to 1. We assume that (A, b) is a controllable pair.

Given a control input u, the *level of over-saturation* is defined to be

$$d(u) = \max\{0,\ |u| - 1\} \tag{2}$$

Suppose the control law is such that the level of over-saturation is bounded by $\rho \geq 0$. Our first problem is to determine how to bound the nonlinearity caused by the saturation by a sector. More precisely, we rewrite $\sigma(u)$ as

$$\sigma(u) = \rho_1 u + \delta(u) \tag{3}$$

where

$$\delta(u) = \sigma(u) - \rho_1 u \tag{4}$$

We seek the optimal value for ρ_1 so that $\delta(u)$ has the smallest sector bound, i.e., ρ_2 below is minimised:

$$|\delta(u)| \leq \rho_2 |u|, \quad \forall |u| \leq 1+\rho \tag{5}$$

Lemma 1. *The optimal value for ρ_1 and the corresponding minimum ρ_2 are given below:*

$$\rho_1 = \frac{2+\rho}{2(1+\rho)}, \quad \rho_2 = \frac{\rho}{2(1+\rho)} \tag{6}$$

Proof. This is verified straightforwardly.

Next, we consider the following quadratic cost function

$$J(x_0, u) = \int_0^\infty (x^T Q x + r\sigma(u)^2) dt \tag{7}$$

for some $Q = Q^T > 0$ and $r > 0$, and linear control input

$$u = k^T x \tag{8}$$

for some $k \in \mathbf{R}^n$.

Ideally, we would like to provide an optimal control law, i.e., an optimal k, for each given initial state x_0 such that the cost function $J(x_0, u)$ is minimised. However, the optimal k is generally dependent on x_0, and the solution is difficult to give. To relax the problem, we aim to characterise an ellipsoid

$$X_\rho = \{x : x^T P_\rho x \leq \mu_\rho^2\}, \quad P_\rho = P_\rho^T > 0, \ \mu_\rho > 0 \tag{9}$$

and an associated suboptimal linear control gain k with the following properties for any parameter $\rho \geq 0$:

- The level of over-saturation $d(u) \leq \rho$;
- The set X_ρ is an invariant set, i.e., $x(t) \in X_\rho$ for all $t \geq 0$ if $x_0 \in X_\rho$;

It is well-known that if the control is not saturated, the optimal solution is given by

$$k = -r^{-1} P_0 b \tag{10}$$

where P_0 solves the following Ricatti equation:

$$A^T P_0 + P_0 A + Q - r^{-1} P_0 b b^T P_0 = 0 \tag{11}$$

Moreover, the optimal cost is given by $x_0^T P_0 x_0$.

We can rewrite (1) as follows:

$$\dot{x} = Ax + b(\rho_1 u + \delta(u)) \tag{12}$$

In view of (5), we relax the optimal control problem to designing a control gain k to minimize the worst-case cost for all $\delta(\cdot)$ satisfying the sector bound (5).

Now we give some analysis on the relaxed optimal control problem. Denote

$$J(x_0,u,T) = \int_0^T (x^T Q x + r\sigma(u)^2)dt \tag{13}$$

and consider the Lyapunov function candidate

$$V(x) = x^T P_\rho x \tag{14}$$

Also, define

$$\Omega_\rho = A^T P_\rho + P_\rho A + Q - r^{-1} P_\rho b b^T P_\rho \tag{15}$$

and

$$u^* = -r^{-1} b^T P_\rho x \tag{16}$$

Given any initial state x_0 and any $\delta(\cdot)$ satisfying (5), it is easy to check that $x(t)$ is finite for any $t > 0$ (i.e., there is no finite escape) and

$$\begin{aligned} J(x_0,u,T) &= V(x_0) - V(x(T)) + \int_0^T (\frac{d}{dt}V(x) + x^T Q x + r\sigma(u)^2)dt \\ &\le V(x_0) + \int_0^T (x^T \Omega_\rho x + r(\rho_1 u + \delta(u) - u^*)^2)dt \\ &= V(x_0) + \int_0^T f(x,u,\delta(u))dt \end{aligned}$$

where

$$f(x,u,\delta(u)) = x^T \Omega_\rho x + r(\rho_1 u + \delta(u) - u^*)^2 \tag{17}$$

It is clear that if $f(x,u,\delta(u)) \le 0$ for all $x \in \mathbf{R}^n$ and $\delta(\cdot)$ satisfying (5), then

$$J(x_0,u) \le V(x_0) \tag{18}$$

From the analysis above, we formulate the following relaxed optimal control problem:

P1. Design P_ρ and u to minimise $V_0(x_0)$ subject to $f(x,u,\delta(u)) \le 0$ for all $x \in \mathbf{R}^n$ and $\delta(\cdot)$ satisfying (5). Further, determine the (largest) invariant set X_ρ for which the cost $J(x_0,u)$ is bounded by $V_0(x_0)$.

Theorem 1. *Consider the system in (1) and the cost function in (7). For any level of over-saturation $\rho \ge 0$, the optimal P_ρ and u for Problem P1 are given by*

$$\begin{aligned} &u = k_\rho^T x = \rho_1^{-1} u^* = -\rho_1^{-1} r^{-1} b^T P_\rho x \\ &A^T P_\rho + P_\rho A + Q - (1-\rho_0^2) r^{-1} P_\rho b b^T P_\rho = 0, \quad P_\rho = P_\rho^T > 0 \end{aligned} \tag{19}$$

and the invariant set by

$$X_\rho = \left\{ x : x^T P_\rho x \le \frac{r^2}{(1-\rho_0)^2 b^T P_\rho b} \right\} \tag{20}$$

where

$$\rho_0 = \frac{\rho_2}{\rho_1} = \frac{\rho}{2+\rho} \tag{21}$$

Proof. Using the well-known S-procedure [5], the above holds iff there exists $\tau > 0$ such that

$$\bar{f}(x,u,\delta) = f(x,u,\delta) + \tau(\rho_2^2 u^2 - \delta^2) \le 0, \quad \forall x \in \mathbf{R}^n, \delta \in \mathbf{R}$$

For the above to hold, it is necessary that $\tau \ge r$. Assuming this and maximising $\bar{f}(x,u,\delta)$ with respect to δ yields

$$\delta = \frac{r}{\tau - r}(\rho_1 u - u^*)$$

which in turn yields

$$\bar{f}(x,u,\delta) = x^T \Omega_\rho x + \tau \rho_2^2 u^2 + \frac{\tau r}{\tau - r}(\rho_1 u - u^*)^2$$

Then, minimising $\bar{f}(x,u,\delta)$ with respect to u results in

$$u = \frac{r\rho_1}{(\tau - r)\rho_2^2 + r\rho_1^2} u^*$$

and

$$\bar{f}(x,u,\delta) = x^T \Omega_\rho x + \frac{\tau r \rho_2^2}{(\tau - r)\rho_2^2 + r\rho_1^2}(u^*)^2$$

Note that $\rho_1 > \rho_2$. Finally, minimising $\bar{f}(x,z,\delta)$ with respect to τ yields $\tau = r$,

$$u = \rho_1^{-1} u^*$$

and

$$\bar{f}(x,u,\delta) = x^T \Omega_\rho x + r\frac{\rho_2^2}{\rho_1^2}(u^*)^2 = x^T \bar{\Omega}_\rho x$$

where

$$\bar{\Omega}_\rho = A^T P_\rho + P_\rho A + Q - (1-\rho_0^2) r^{-1} P_\rho b b^T P_\rho$$

with ρ_0 given by (21). To assure $\bar{f}(x,u,\delta) \ge 0$, it is necessary that $\bar{\Omega}_\rho \le 0$. It is easily verified that P_ρ is a monotonically decreasing function of $\bar{\Omega}_\rho$.

Therefore, the optimal $\bar{\Omega}_\rho = 0$. Consequently, we obtain the optimal solution in (19).

Next, we try to characterise an invariant set X_ρ as in (9) for which the control law above applies. All we need to do is to find the largest μ_ρ in (9) such that

$$\max_{x \in X_\rho} |\delta(u)| = \rho_2 |u|$$

Equivalently,

$$\max_{x \in X_\rho} \left| \rho_1^{-1} r^{-1} b^T P_\rho x \right| = 1 + \rho$$

The solution is given by

$$x = \frac{\mu}{\sqrt{b^T P_\rho b}} b$$

and

$$\mu = \frac{r(2+\rho)}{2\sqrt{b^T P_\rho b}} = \frac{r}{(1-\rho_0)\sqrt{b^T P_\rho b}}$$

That is, X_ρ is given by (20).

Since $f(x, u, \delta(u)) \leq 0$ for all $x \in X_\rho$,

$$\frac{d}{dt} V(x) \leq -(x^T Q x + r\sigma(u)^2), \quad \forall x \in X_\rho \tag{22}$$

Hence, we know that X_ρ is an invariant set.

Remark 1. In Theorem 1, we assume that the sector bound characterised in Lemma 1 is used. This assumption can be relaxed to any sector bound satisfying the following condition: Parameters ρ_1 and ρ_2 are such that $\rho_1 > \rho_2 \geq 0$ and $|\delta(u)| \leq \rho_2 |u|$ for all u with the level of over-saturation bounded by ρ, where $\delta(u)$ is defined in (4). It can be shown (with somewhat more effort) that the optimal solution to Problem P1 is still given by (19) as in Theorem 1, with $\rho_0 = \rho_2/\rho_1$. It can be further shown that the parameters ρ_1 and ρ_2 that minimise $V(x_0)$ and maximise X_ρ are those given in Lemma 1.

3 Properties of the Proposed Controller

In this section, we study two key properties of the proposed controller in Theorem 1. The first property shows the improvement of the saturation control compared with an unsaturated controller. The second property is to do with nesting of invariant sets and monotonicity of Lyapunov matrices.

Returning to (19), we see that the Ricatti equation for P_ρ corresponds to the solution to an optimal control where the weight (or penalty) for the control in the cost function is changed to $(1-\rho_0^2)^{-1}r$. To achieve the cost $J(x_0,u,T)=J^*(x_0,T)$, u must be such that

$$\sigma(u)=-(1-\rho_0^2)r^{-1}b^TP_\rho x \tag{23}$$

To make the above feasible (i.e., to avoid saturation), the invariant set must be

$$\bar{X}_\rho=\left\{x:x^TP_\rho x=\frac{r^2}{(1-\rho_0^2)^2b^TP_\rho b}\right\} \tag{24}$$

Compared with (20), we have

$$X_\rho=(1+\rho_0)\bar{X}_\rho \tag{25}$$

This illustrates that the control law in (19) gives a substantially larger invariant set for the same cost, compared with an unsaturated control law.

On the other hand, we may consider choosing the control law such that

$$\sigma(u)=-\frac{1-\rho_0^2}{2}r^{-1}b^TP_\rho x \tag{26}$$

and choosing an invariant set such that the above control law is feasible (i.e., no saturation). Note that this control law is stabilising, due to the well-known gain margin of an optimal linear quadratic control (also seen directly from the Ricatti equation in (19)). In this case, the invariant set is given by

$$\hat{X}_\rho=\left\{x:x^TP_\rho x=\frac{4r^2}{(1-\rho_0^2)^2b^TP_\rho b}\right\} \tag{27}$$

and we have

$$X_\rho=\frac{1+\rho_0}{2}\hat{X}_\rho \tag{28}$$

Although $X_\rho<\hat{X}_\rho$, no performance grantee can be delivered by the controller in (26). To make the comparison fair, we take $\rho\to\infty$ and note that

$$X_\infty=\hat{X}_\infty \tag{29}$$

This gives a somewhat surprising result:

Corollary 1. *The largest invariant set given by the controller in (19) is the same as the largest invariant set given by an unsaturated controller (26).*

One implication of the results above is that the saturated controller can bring a good benefit when ρ is not close to 0 and not too large.

Next, we study the nesting property of X_ρ and monotonicity of P_ρ. To this end, define

$$S_\rho=(1-\rho_0)P_\rho \tag{30}$$

We then rewrite Ricatti equation in (19) as

$$A^T S_\rho + S_\rho A + (1-\rho_0)Q - (1+\rho_0)r^{-1} S_\rho bb^T S_\rho = 0 \tag{31}$$

and the invariant set X_ρ as

$$X_\rho = \left\{ x : x^T S_\rho x \le \frac{r^2}{b^T S_\rho b} \right\} \tag{32}$$

Lemma 2. *The solution S_ρ to (31) is monotonically decreasing, i.e., $S_{\rho+\epsilon} < S_\rho$ if $0 \le \rho < \rho + \epsilon$. Consequently, X_ρ are nested in the following sense:*

$$X_\rho \subset X_{\rho+\epsilon}, \quad \forall 0 \le \rho < \rho + \epsilon \tag{33}$$

Further, the solution P_ρ to the Ricatti equation in (19) is monotonically increasing, i.e., $P_{\rho+\epsilon} > P_\rho$ if $0 \le \rho < \rho + \epsilon$.

Proof. The monotonicity of S_ρ is a basic property of the Ricatti equation (31). We only need to show this for sufficiently small $\epsilon > 0$. Denote $E = S_\rho - S_{\rho+\epsilon}$ and Ω_ρ to be the left hand side of (31). Also define

$$\epsilon_0 = \frac{\rho+\epsilon}{2+\rho+\epsilon} - \frac{\rho}{2+\rho} > 0$$

Then,

$$\begin{aligned} 0 &= \Omega_\rho - \Omega_{\rho+\epsilon} \\ &= EA^T + AE + \epsilon_0 Q - (1+\rho_0)S_\rho bb^T S_\rho \\ &\quad + (1+\rho_0+\epsilon_0)(S_\rho - E)bb^T(S_\rho - E) \\ &= E(A - (1+\rho_0+\epsilon_0)S_\rho bb^T)^T + (A - (1+\rho_0+\epsilon_0)S_\rho bb^T)E \\ &\quad + \epsilon_0 Q + (1+\rho_0+\epsilon_0)Ebb^T E + \epsilon_0 S_\rho bb^T S_\rho \end{aligned}$$

From (31), we know that $A - (1+\rho_0)bb^T S_\rho$ is Hurwitz. Therefore $A - (1 + \rho_0 + \epsilon_0)bb^T S_\rho$ is also Hurwitz when ϵ_0 (or equivalently, ϵ) is sufficiently small. Hence, the equation above implies that $E > 0$. Therefore, the monotonicity of S_ρ is established. The nesting property of X_ρ then follows naturally from (32). The monotonicity of P_ρ is proved similarly.

Remark 1. If $\rho = 0$, the control in Theorem 1 recovers the optimal control without saturation. In this case, the invariant set is given by

$$X_0 = \left\{ x : x^T P_0 x \le \frac{r^2}{b^T P_0 b} \right\}$$

Remark 2. The "largest" invariant set, called *region of attraction*, is given by taking $\rho \to \infty$ (or equivalently, $\rho_0 \to 1$) and solving for P_ρ in (19). That is, the region of attraction is given by

$$X_\infty = \left\{ x : x^T P_\rho x < \frac{r^2}{(1-\rho_0)^2 b^T P_\rho b}, \quad \rho_0 \to 1 \right\} \tag{34}$$

Note that the solvability of P_ρ for any $\rho > 0$ is guaranteed by the controllability of (A, b) and positive definiteness of Q.

Remark 3. Suppose A is either Hurwitz or marginally unstable (i.e., the only unstable eigenvalues are the ones with a zero real part). Then, the solution to the Ricatti equation in (19) is such that the directions of P_ρ approach to either a constant (corresponding to stable eigenvalues of A) or $O(\sqrt{1-\rho_0})$ (corresponding to marginal eigenvalues of A). In either case, the limiting invariant set is the whole space, i.e., $X_1 = \mathbf{R}^n$.

4 Switching Control

A common control strategy for combating the control saturation is to start with a small gain when the state is "large" (to avoid or reduce saturation) and then gradually increase the gain when the state is "small" (to improve the performance). This strategy can be easily applied to the controller in the previous section due to the nesting properties of X_ρ and monotonicity of P_ρ.

More precisely, a switching control strategy is simply formed by choosing a sequence of saturation indices $0 = \rho^{(0)} < \rho^{(1)} < \cdots < \rho^{(N)}$ and solving for the corresponding Lyapunov matrices P_i, invariant sets X_i and control gains k_i. The control law simply selects the control gain k_i when $x \in X_i$ and $x \notin X_{i-1}$ (unless $i = 0$).

Theorem 2. *Suppose $x_0 \in X_N$ and we apply the switching control law above by starting with k_N (or ρ_N equivalently). Denote the switching control law by u_s and the switching time from $\rho^{(i)}$ to $\rho^{(i-1)}$ by T_i. Then the cost of this switching control is bounded by*

$$J(x_0, u_s) \le x_0^T P_N x_0 - \sum_{i=1}^{N} x^T(T_i)(P_i - P_{i-1})x(T_i) < x_0^T P_N x_0, \quad \forall x_0 \in X_N,\ x_0 \neq 0 \tag{35}$$

Proof. Follows directly from Theorem 1 and the monotonicity of P_ρ.

The advantage of the switching law is clearly seen from the theorem above where the negative terms in (35) are a result of the switching. However, it is somewhat difficult to express the cost explicitly in terms of x_0 and $\rho^{(i)}$. More work needs to be done to study this issue and also on how to choose the sequence $\{q^{(i)}\}$ to optimise $J(x_0, u)$.

5 Illustrative Example

To illustrate the design approach presented in this paper, we consider the following simple system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1.25 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(u(t)) \tag{36}$$

The cost function has $r = 1$ and

$$Q = \begin{bmatrix} 1 & 0 \\ \\ 0 & 0 \end{bmatrix}$$

Choose $\rho = 0, 2, 5, 10, 20, 40, 70, 100$. The corresponding invariant sets X_ρ are shown in Figure 1. We take $x_0 = [0.87 \quad 0]'$. Figure 1 also shows the two state trajectories corresponding the switching controller and non-switching controller. Their performances and control inputs are given in Figures 2-3, respectively. It is seen clearly that the switching controller significantly outperforms the non-switching controller.

6 Conclusion

In this paper, we have presented a new approach to designing linear quadratic controllers for systems with input saturation. The key contribution of the paper is of two-fold: 1) We optimise the sector bound which models the mismatch between the unsaturated controller and the saturated one; and 2) We determine the largest invariant set for the given sector bound above and the associated optimal controller. The invariant sets and the corresponding Lyapunov matrices have the nice properties of nesting and monotonicity, respectively. These properties allow a switching controller to be designed easily to yield substantially lower quadratic cost (in comparison to non-switching controllers) while guaranteeing stability.

The sector bound used for control design can be generalised to include integral quadratic constraints. This allows a dynamic relationship between the linear control input and the saturated control input. It is expected that this approach can yield some improvement in the performance at the expense of somewhat more complicated control design. More specifically, the state of the system needs to include the dynamics of the integral quadratic constraints, which implies that the control gain will be dynamic. More work on this topic will be carried out by the author.

Finally, it should be noted that the design approach given in this paper can be easily generalised to discrete-time systems.

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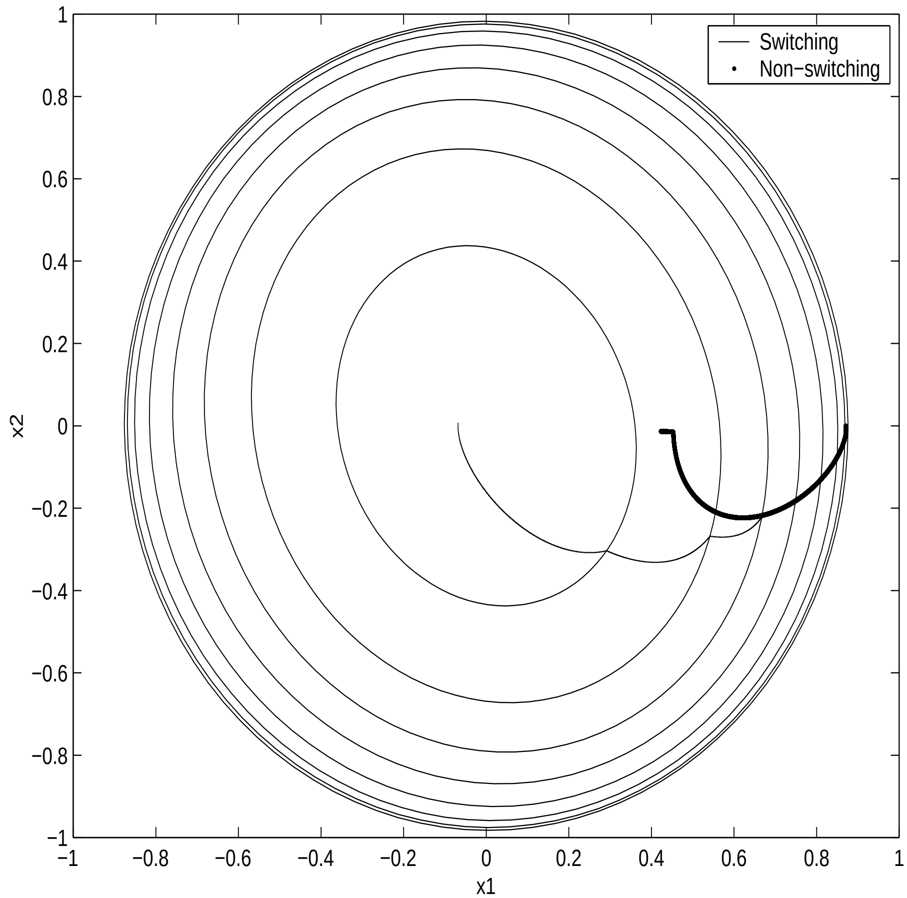

Fig. 1. Nested Ellipsoids and State Trajectories

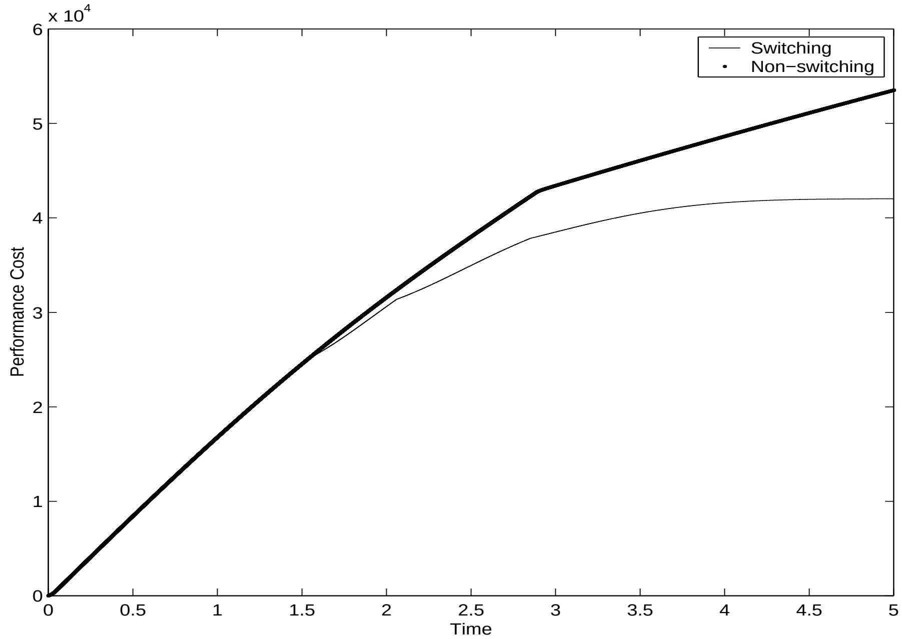

Fig. 2. Performance Costs

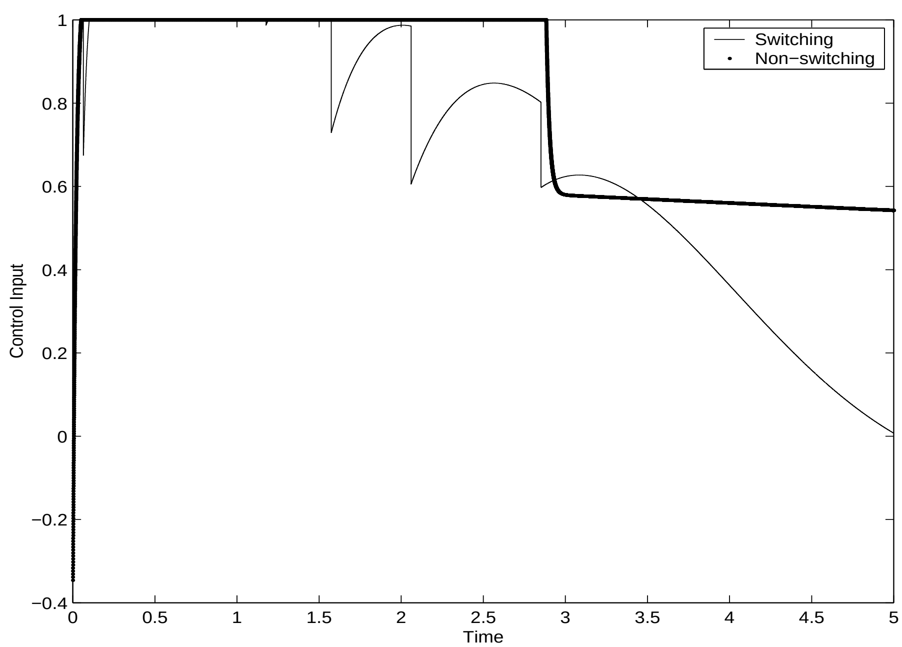

Fig. 3. Control Inputs