PASSIVITY ANALYSIS FOR UNCERTAIN SIGNAL PROCESSING SYSTEMS

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ABSTRACT

The problem of passivity analysis finds important applications in many signal processing systems such as digital quantizers, decision feedback equalizers and digital and analog filters. This paper considers the passivity analysis problem for a large class of systems which involve uncertain parameters, time delays, quantization errors, and unmodeled high order dynamics. By characterizing these and many other types of uncertainty using a general tool called integral quadratic constraints (IQCs), we present a solution to the problem of robust passivity analysis. More specifically, we determine if a given uncertain system is robustly passive. The solution is given in terms of the feasibility of a linear matrix inequality (LMI) which can be solved efficiently.

1. INTRODUCTION

The notion of passivity plays an important role in design and analysis of signal processing systems. For example, it is well-known that suppression of limit cycles of a digital quantizer requires certain dynamic part of the system to be passive [6]. Another example where passivity analysis finds important use is the so-called decision feedback equalization (DFE) problem. It is known [5] that a decision feedback equalizer guarantees finite error recovery if certain passivity condition is satisfied.

Many signal processing systems are feedback systems consisting of both a linear time-invariant (LTI) dynamic part and a nonlinear and/or time-varying part. For example, a differential pulse-code modulation (DPCM) system involves a linear predictor and a quantizer. Time-varying filters are popularly used in multirate signal processing. Nonlinear and time-varying systems also arise in many adaptive filtering problems. Passivity analysis is a major tool for studying stability of such systems, especially for high order systems.

The motivation for our paper stems from the fact that, in many applications, the system (or subsystem) which is required to be passive is not a simple LTI transfer function, rather it involves additional uncertainty. For example, in adaptive DPCM (ADPCM) or adaptive DFE, the filter coefficients are subject to time variations. Even in non-adaptive cases, filter coefficients are also subject to quantization effects. Other types of uncertainty include unknown time-delays in a communication channel, variations in analog components, and unmodeled high order dynamics. Note that if there exists no uncertainty, checking if an LTI dynamic system is passive or not is a simple matter. However, for uncertain systems the problem becomes much more involved.

In this paper, we use the so-called integral quadratic constraints (IQCs) introduced in [7] to describe uncertain components. The IQCs encompass all of the commonly encountered types of uncertainty mentioned earlier. Our main result is a sufficient condition for guaranteeing an uncertain system to be robustly passive. This sufficient condition is expressed in terms of a linear matrix inequality (LMI) which can be solved efficiently. We also study a digital quantizer to demonstrate our result.

2. PASSIVITY ANALYSIS

Definition 2.1 (Passivity) *An operator $\mathcal{H}$: $\ell_2^e \to \ell_2^e$, is called* passive *if there exists β such that*

$$\sum_{t=0}^{T} (\mathcal{H}u(t))' u(t) \geq \beta, \;\; \forall u \in \ell_2^e,\; T > 0 \qquad (2.1)$$

Similarly, $\mathcal{H}$ is called strictly passive *if there exist $\alpha > 0$ and β such that*

$$\sum_{t=0}^{T} (\mathcal{H}u(t))' u(t) \geq \;\beta + \alpha \sum_{t=0}^{T} u(t)' u(t), \;\; \forall u \in \ell_2^e,\; T > 0 \qquad (2.2)$$

When $\mathcal{H}$ is a linear time-invariant (LTI) real operator and it is passive (resp. strictly passive), its transfer function is called positive real *(PR) (resp.* strictly positive real *(SPR)).*

Consider the feedback system depicted in Figure 1, where $\mathcal{H}_1$ is an LTI operator and $\mathcal{H}_2$ is a (possibly) nonlinear operator. The lemma below gives a sufficient condition for ℓ_2-stability (or stability for short).

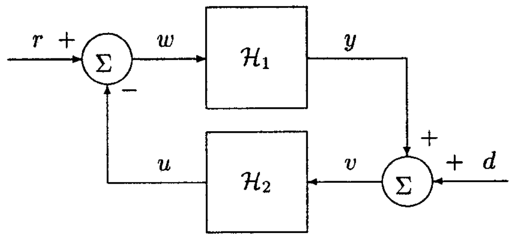

Figure 1: Interconnected Feedback System

Lemma 2.1 *Suppose $\mathcal{H}_1$ is linear and strictly passive and $\mathcal{H}_2$ is passive. Then, the mapping from (r, d) to (w, v) is ℓ_2-stable, i.e., (w, v) has a bounded ℓ_2 norm should (r, d) do.*

Proof: See [3, p. 182] and [5]. ■

Consider the following uncertain system:

$$\begin{aligned} x(t+1) &= Ax(t) + Bw(t) + \sum_{i=1}^{p} F_{1i}\xi_i(t) \quad (2.3) \\ y(t) &= Cx(t) + Dw(t) + \sum_{i=1}^{p} F_{2i}\xi_i(t) \quad (2.4) \\ z_i(t) &= E_{1i}x(t) + E_{2i}w(t) + E_{3i}\xi(t), \quad (2.5) \\ & \quad i = 1, 2, \ldots, p \end{aligned}$$

where $x(t) \in \mathbf{R}^n$ is the state, $w(t) \in \mathbf{R}^q$ is the exogenous input, $y(t) \in \mathbf{R}^q$ is the output, $z_i(t) \in \mathbf{R}^{k_i}$, $i = 1, 2, \ldots, p$, are fictitious outputs, and $\xi_i(t) \in \mathbf{R}^{k_i}$, $i = 1, 2, \ldots, p$, denote uncertain variables which satisfy the following integral quadratic constraints (IQCs):

$$\lim_{T\to\infty} \sum_{t=0}^{T} (\|\xi_i(t)\|^2 - \|z_i(t)\|^2) \le 0, \; i = 1, 2, \ldots, p. \quad (2.6)$$

In the above, $A, B, C, D, F_{1i}, F_{2i}, E_{1i}, E_{2i}$ and E_{3i} are constant matrices of appropriate dimensions. Note that a sum is used in (2.6), but the term IQC was originated in continuous-time systems where an integral is used rather.

Remark 2.1 *The uncertainty represented by the IQCs (2.6) is very general. It includes time-delays, quantization errors, uncertain parameters, unmodeled dynamics, and many nonlinear and/or time-varying components; see [4].*

Definition 2.2 *The uncertain system (2.3)-(2.6) is called robustly passive (resp. robustly strictly passive) if it is passive (resp. strictly passive) for all admissible uncertainty.*

Our objective is to analyze the robust strict passivity of the uncertain system (2.3)-(2.6). Before proceeding further, we introduce the following short-hand notation:

$$\begin{aligned} F_1 &= [F_{11} \; \ldots \; F_{1p}]; \; F_2 = [F_{21} \; \ldots \; F_{2p}] \\ E_1 &= [E_{11}' \; \ldots \; E_{1p}']'; \; E_2 = [E_{21}' \; \ldots \; E_{2p}']'; \\ E_3 &= [E_{31}' \; \ldots \; E_{3p}']' \\ \tau &= (\tau_1, \ldots, \tau_p), \; \tau_i \in \mathbf{R}; \\ J &= \mathrm{diag}\{\tau_1 I_{k_1}, \ldots, \tau_p I_{k_p}\} \quad (2.7) \end{aligned}$$

Lemma 2.2 *The uncertain system of (2.3)-(2.6) is robustly strictly passive if there exist a symmetric positive definite matrix $P \in \mathbf{R}^{n\times n}$ and scaling parameters $\tau_1, \ldots, \tau_p > 0$ such that the following condition holds for some $\alpha > 0$:*

$$\begin{aligned} &(Ax + Bw + \sum_{i=1}^{p} F_{1i}\xi_i)' P (Ax + Bw + \sum_{i=1}^{p} F_{1i}\xi_i) \\ &-x'Px - 2w'y + 2\alpha w'w + \\ &\sum_{i=1}^{p} \tau_i (\|E_{1i}x + E_{2i}w + E_{3i}\xi\|^2 - \|\xi_i\|^2) < 0 \quad (2.8) \end{aligned}$$

for all $x \in \mathbf{R}^n$, $w \in \mathbf{R}^q$ and $\xi_i \in \mathbf{R}^{k_i}$, $i = 1, 2, \ldots, p$, such that $[x', w', \xi_1', \ldots, \xi_p'] \neq 0$.

Proof: Let $V(x) = x'Px$ and sum the inequality of (2.8) from 0 to T along any trajectory of (2.3). Then, we have

$$\begin{aligned} &V[x(T+1)] - V[x(0)] + 2\alpha \sum_{t=0}^{T} w'(t)w(t) \\ &+ \sum_{i=1}^{p} \tau_i \left\{ \sum_{t=0}^{T} \|E_{1i}x(t) + E_{2i}w(t) + E_{3i}\xi(t)\|^2 \right. \\ &\left. - \sum_{t=0}^{T} \|\xi_i(t)\|^2 \right\} - 2\sum_{t=0}^{T} w'(t)y(t) \le 0 \end{aligned}$$

It follows that

$$\begin{aligned} &\sum_{t=0}^{T} w'(t)y(t) \ge -\frac{1}{2}V[x(0)] + \alpha \sum_{t=0}^{T} w'(t)w(t) \\ &+ \frac{1}{2}\sum_{i=1}^{p} \tau_i \left\{ \sum_{t=0}^{T} \|E_{1i}x(t) + E_{2i}w(t) + E_{3i}\xi(t)\|^2 \right. \\ &\left. - \sum_{t=0}^{T} \|\xi_i(t)\|^2 \right\} \end{aligned}$$

for all $T \ge 0$. Now take $T \to \infty$. By considering (2.6) and noting the fact that $\tau_1, \ldots, \tau_p > 0$, we have

$$\sum_{t=0}^{\infty} w'(t)y(t) \ge -\frac{1}{2}V[x(0)] + \alpha \sum_{t=0}^{\infty} w'(t)w(t)$$

That is, the system (2.3)-(2.6) is robustly strictly passive. ■

Now we present the main result of this paper.

Theorem 2.1 *Consider the uncertain system of (2.3)-(2.6). The following conditions, all guaranteeing the system to be robustly strictly passive, are equivalent:*

(a) There exists $P = P' > 0$ such that (2.8) holds;

(b) For some $J > 0$ of (2.7), there exists $P = P' > 0$ such that

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12}' & L_{22} & L_{23} \\ L_{13}' & L_{23}' & L_{33} \end{bmatrix} < 0 \tag{2.9}$$

where

$$\begin{aligned} L_{11} &= A'PA - P + E_1'JE_1 \\ L_{12} &= A'PB - C' + E_1'JE_2 \\ L_{13} &= A'PF_1 + E_1'JE_3 \\ L_{22} &= -(D + D' - B'PB - E_2'JE_2) \\ L_{23} &= B'PF_1 - F_2 + E_2'JE_3 \\ L_{33} &= F_1'PF_1 + E_3'JE_3 - J \end{aligned}$$

(c) A is stable and for some $J > 0$, the following auxiliary system is strictly positive real:

$$x_a(t+1) = Ax_a(t) + [B \ \ F_1 \ \ 0]w_a(t) \tag{2.10}$$

$$y_a(t) = \begin{bmatrix} C \\ 0 \\ -JE_1 \end{bmatrix} x_a(t) + \begin{bmatrix} D & F_2 & 0 \\ 0 & \frac{1}{2}J & 0 \\ -JE_2 & -JE_3 & \frac{1}{2}J \end{bmatrix} w_a(t) \tag{2.11}$$

Moreover, the set of all J satisfying (c) is convex, where J is given in (2.7).

Proof: The proof uses Lemma 2.2 and some matrix inequality manipulations. The details are omitted. ■

Remark 2.2 *Theorem 2.1 shows that the robust strict passivity of system (2.3)-(2.6) is guaranteed if the auxiliary system (2.10)-(2.11) is strictly positive real for some $J > 0$. It can be observed that the inequality in (2.9) is jointly linear in P and J. Note that very efficient numerical algorithms exist for solving LMIs, owing to the recent advancement in interior point algorithms for convex optimization [2].*

3. ILLUSTRATIVE EXAMPLE

Consider the overflow limit cycle problem associated with the digital quantizer in Figure 2. Let $G(z)$ be of the form:

$$G(z) = \frac{0.0375(z^2 + 0.6875z + 1)}{z^3 - 0.875z^2 + (0.75 + \delta a)z - 0.625 - \delta b}$$

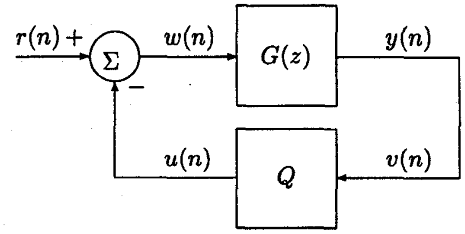

Figure 2: Digital Quantizer

where δa and δb represent quantization errors after the corresponding coefficients are coded with 4 bits. It is known that

$$|\delta a| \le 2^{-4}, \quad |\delta b| \le 2^{-4}$$

It can be easily checked that the nominal transfer function $G_0(z)$ of $G(z)$ (setting $\delta a \equiv 0$ and $\delta b \equiv 0$) is stable but not SPR. Although the quantizer is passive, we are unable to conclude even if the nominal quantizer is void of overflow limit cycles.

To reduce the conservatism, we consider the transformed system in Figure 3, where $0 < \alpha < 1$ is a tuning parameter and $H(z)$ is any stable function with L_1 norm less than or equal to 1, i.e.,

$$\sum_{t=0}^{\infty} |h(t)| \le 1 \tag{3.12}$$

where $h(t)$ is the impulse response corresponding to $H(z)$. In addition, it is required that $1 + H(z)$ is invertible.

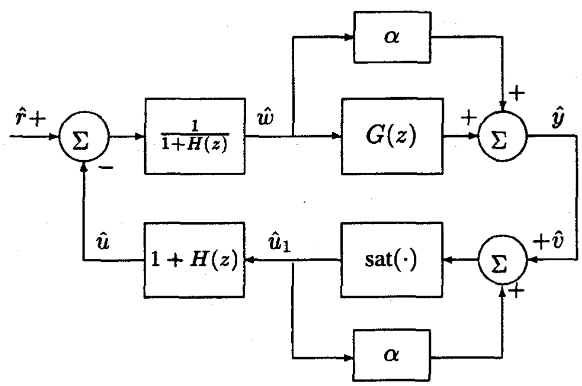

Figure 3: Transformed Quantizer System

It is easy to check that the lower block of Figure 3 reamins passive while the upper block approaches $(1+H(z))^{-1}(1+G(z))$ when $\alpha \to 1$. Therefore, the system in Figure 3 (hence the one in Figure 2) will not observe limit cycles if $(1 + H(z))^{-1}(1 + G(z))$ is SPR. Clearly, this is weaker than requiring $G(z)$ to be SPR because if $G(z)$ is indeed

SPR, one can simply choose $H(z)$ to be zero. The condition above is actually a special case of a more general result studied by Zames and Falb [8] where the feedback block is allowed to be a general monotone and odd function.

Next, let $H(z) = -G_0(z)$. We have $\sum_{n=0}^{\infty} |h(t)| < 1$ and

$$\hat{G}_0(z) = (1 + G_0(z))/(1 + H(z))$$

is SPR. Hence, from the discussion above we conclude that the system does not exhibit overflow limit cycles in the nominal case.

Next, we analyze the effect of the quantization errors. To this end, a state space realization for the transfer function

$$\hat{G}(z) = (1 + G(z))/(1 + H(z))$$

is given by

$$x(k+1) = (A + \Delta A)x(k) + Bw(k) \quad (3.13)$$

$$y(k) = Cx(k) + Dw(k) \quad (3.14)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.6625 & -0.7242 & 0.9125 \end{bmatrix},$$

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\delta a & -\delta b & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = [0.075 \; 0.0516 \; 0.075], \; D = 1$$

Denote

$$F_{11} = F_{12} = \begin{bmatrix} 0 & 0 & -2^{-4} \end{bmatrix}' \quad (3.15)$$

$$E_{11} = [1 \; 0 \; 0], \; E_{12} = [0 \; 1 \; 0] \quad (3.16)$$

$$F_{2i} = 0, \; E_{2i} = E_{3i} = 0, \; i = 1, 2 \quad (3.17)$$

Then, the uncertainty, $\Delta Ax(k)$, in the state equation (3.13) can be represented by

$$\Delta Ax(k) = F_{11}\xi_1 + F_{12}\xi_2$$

where

$$\xi_i(k) = \delta_i z_i(k), \; z_i(k) = E_{1i}x(k)$$

with $|\delta_i| \leq 1, i = 1, 2$. Clearly, ξ_i and z_i satisfy the IQCs:

$$\sum_{t=0}^{T} (||\xi_i||^2 - ||z_i||^2) \leq 0, \; i = 1, 2$$

We now apply Theorem 2.1 to check whether $\hat{G}(z)$ is SPR for any admissible quantization errors δa and δb. Efficient interior-point algorithms are available to solve (2.9); see [2]. We obtain a solution

$$P = \begin{bmatrix} 0.7085 & -0.5375 & 0.2134 \\ -0.5375 & 1.1664 & -0.5843 \\ 0.2134 & -0.5843 & 0.8706 \end{bmatrix} > 0,$$

$$J = \begin{bmatrix} 0.0422 & 0 \\ 0 & 0.0413 \end{bmatrix} > 0$$

Hence, Theorem 2.1 guarantees that no overflow limit cycles exist even when the quantization errors δa and δb are present.

4. CONCLUSION

This paper has studied the problem of robust passivity analysis for a large class of uncertain systems with the uncertainty described by integral quadratic constraints. LMI solutions have been presented. In view of the recent development in convex optimization, especially in solving LMIs (see [2]), our results offer efficient solutions to these problems. Applications of these problems in signal processing systems have been studied. In particular, we note that passivity analysis is an important tool in studying robust stability of signal processing systems involving nonlinear elements.

5. REFERENCES

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