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Piecewise Lyapunov functions for robust stability of linear time-varying systems[1]

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Abstract

In this paper, we investigate the use of two-term piecewise quadratic Lyapunov functions for robust stability of linear time-varying systems. By using the so-called S-procedure and a special variable reduction method, we provide numerically efficient conditions for the robust asymptotic stability of the linear time-varying systems involving the convex combinations of two matrices. An example is included to demonstrate the usefulness of our results. © 1997 Elsevier Science B.V.

Keywords: Robust stability; Piecewise Lyapunov function; S-procedure; Linear matrix inequality

1. Introduction

The quadratic stability approach is popularly used for robust stability analysis of time-varying uncertain systems. This approach, however, may lead to very conservative results. Alternatively, non-quadratic Lyapunov functions have been used to improve the estimate of robust stability (see [2, 1, 7–10]). The difficulty with non-quadratic Lyapunov functions is that the resulting optimization problem is typically non-convex. In this short paper, we investigate the use of two-term piecewise Lyapunov functions on time-varying linear systems involving the convex combination of two matrices. These Lyapunov functions are either the maximum or the minimum of two quadratic terms. By using the so-called S-procedure [11] and a variable reduction technique, we obtain necessary and sufficient conditions for establishing robust asymptotic stability of the uncertain system using such a piecewise Lyapunov function. The resulting optimization problem involves a set of linear matrix inequalities (LMIs) with two scaling parameters which can be numerically searched.

We show via an example that good improvement on the estimate of robust stability margin can be obtained by using these piecewise Lyapunov functions when compared with the quadratic stability technique.

2. Problem formulation

Consider the linear time-varying system

$$\dot{x} = A(t)x, \qquad A(t) \in \mathscr{A} = \mathrm{Co}\{A_1, A_2\}, \tag{2.1}$$

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where $\mathrm{Co}\{A_1,A_2\}$ stands for the convex hull of A_1 and A_2. Our aim is to produce a test which is less conservative than quadratic stability result with reasonable computational cost. In particular, we use two kinds of piecewise Lyapunov functions as follows:

$$V(x) = \max\{x'P_1x, x'P_2x\}, \quad P_1 > 0,\ P_2 > 0 \tag{2.2}$$

and

$$V(x) = \min\{x'P_1x, x'P_2x\}, \quad P_1 > 0,\ P_2 > 0. \tag{2.3}$$

To check whether (2.2) proves the stability of (2.1) or not, we only need to check that

$$\mathrm{d}V(x(t))/\mathrm{d}t < 0 \tag{2.4}$$

for all $x(t) \neq 0$ along the trajectory of system (2.1). Note that the derivative above is given by

$$\mathrm{d}V(x(t))/\mathrm{d}t = \begin{cases} x'(t)(A'(t)P_1 + P_1A(t))x(t) & \text{when } x'P_1x \geqslant x'P_2x, \\ x'(t)(A'(t)P_2 + P_2A(t))x(t) & \text{when } x'P_2x \geqslant x'P_1x. \end{cases} \tag{2.5}$$

For the Lyapunov function in (2.3), the derivative in (2.5) should be replaced with

$$\mathrm{d}V(x(t))/\mathrm{d}t = \begin{cases} x'(t)(A'(t)P_1 + P_1A(t))x(t) & \text{when } x'P_1x \leqslant x'P_2x, \\ x'(t)(A'(t)P_2 + P_2A(t))x(t) & \text{when } x'P_2x \leqslant x'P_1x. \end{cases} \tag{2.6}$$

In order to derive conditions for robust stability of (2.1) with the Lyapunov function (2.2) or (2.3), we need the lemma below.

Lemma 2.1 (S-procedure lemma [11]). *Let $F_0(x)$ and $F_1(x)$ be two arbitrary quadratic forms over $\mathbb{R}^n$. Then $F_0(x) < 0$ for all $x \in \mathbb{R}^n$ satisfying $F_1(x) \leqslant 0$ if and only if there exist $\tau \geqslant 0$ such that*

$$F_0(x) - \tau F_1(x) \leqslant 0, \quad \forall x \in \mathbb{R}^n.$$

From the lemma above, the robust stability conditions (2.2)–(2.4) can be rewritten as in the following lemma.

Lemma 2.2 (Boyd [3]). *System* (2.1) *is robustly stable for all $A(t) \in \mathscr{A}$ with the Lyapunov function* (2.2) *if and only if there exist solutions to the following set of inequalities*:

$$A_1'P_1 + P_1A_1 - \lambda_1(P_2 - P_1) < 0, \tag{2.7}$$

$$A_2'P_1 + P_1A_2 - \lambda_2(P_2 - P_1) < 0, \tag{2.8}$$

$$A_1'P_2 + P_2A_1 + \lambda_3(P_2 - P_1) < 0, \tag{2.9}$$

$$A_2'P_2 + P_2A_2 + \lambda_4(P_2 - P_1) < 0, \tag{2.10}$$

$$P_1 > 0, \quad P_2 > 0, \tag{2.11}$$

$$\lambda_1 \geqslant 0, \quad \lambda_2 \geqslant 0, \quad \lambda_3 \geqslant 0, \quad \lambda_4 \geqslant 0. \tag{2.12}$$

Similarly, we can have a result for the Lyapunov function in (2.3).

Lemma 2.3. *System* (2.1) *is robustly stable for all $A(t) \in \mathscr{A}$ with the Lyapunov function* (2.3) *if and only if there exist solutions to the following set of inequalities*:

$$A_1'P_1 + P_1A_1 + \lambda_1(P_2 - P_1) < 0, \tag{2.13}$$

$$A_2'P_1 + P_1A_2 + \lambda_2(P_2 - P_1) < 0, \tag{2.14}$$

$$A_1'P_2 + P_2A_1 - \lambda_3(P_2 - P_1) < 0, \tag{2.15}$$

$$A_2'P_2 + P_2A_2 - \lambda_4(P_2 - P_1) < 0, \tag{2.16}$$

$$P_1 > 0, \quad P_2 > 0, \tag{2.17}$$

$$\lambda_1 \geqslant 0, \quad \lambda_2 \geqslant 0, \quad \lambda_3 \geqslant 0, \quad \lambda_4 \geqslant 0. \tag{2.18}$$

Remark 2.1. The use of Lyapunov functions (2.2)–(2.3) and the S-procedure is not new [5, 3]. In fact, it is shown [7, 8] that the system (2.1) is robustly asymptotically stable if and only if there exists a piecewise quadratic Lyapunov function of the following form:

$$V(x) = \max_{1 \leqslant i \leqslant p} \{x'P_ix\}, \; P_i > 0, \; i = 1, \ldots, p$$

for some p. But the stumbling block is that the subsequent optimization problem is non-convex. This is true even for (2.7)–(2.12) or (2.13)–(2.18). If we fix P_1 and P_2, then (2.7)–(2.12) and (2.13)–(2.18) are convex problems. This is also the case if we fix λ_i, $i = 1, \ldots, 4$ (refer to [3] for example). However, (2.7)–(2.12) and (2.13)–(2.18) are not jointly convex in P_1, P_2 and λ_i.

3. Main result

In this section, we are going to provide a variable reduction procedure which can reduce the number of variables λ_i in (2.7)–(2.18) from four to two. Also, the range of the new variables will be set to [0, 1]. The resulting problem involves a set of matrix inequalities which are linear in all variables except for two. Numerical searching of these two variables becomes a moderate problem. In particular, we can use a simple gridding scheme to find an estimate of them. The estimate can then be refined by using either further gridding or local searching methods such as Newton's method.

Theorem 3.1. *The system* (2.1) *is robustly stable with the Lyapunov function* (2.2) *if and only if there exist* $\delta_1, \delta_2 \in [0, 1]$ *such that the following set of LMIs have a solution for* H_1 *and* H_2:

$$\begin{aligned} &A_1'H_1 + H_1A_1 < 0, \qquad A_2'H_2 + H_2A_2 < 0, \\ &(1 - \delta_2)(A_1'H_2 + H_2A_1) + \delta_2(H_2 - H_1) < 0, \\ &(1 - \delta_1)(A_2'H_1 + H_1A_2) - \delta_1(H_2 - H_1) < 0, \\ &0 < H_1 < I, \quad 0 < H_2 < I. \end{aligned} \tag{3.19}$$

If this is the case, then $V(x) = \max\{x'H_1x, x'H_2x\}$ *is also a valid Lyapunov function for establishing the robust asymptotic stability of the system* (2.1).

Theorem 3.2. *The system* (2.1) *is robustly stable with the Lyapunov function* (2.3) *if and only if there exist* $\delta_1, \delta_2 \in [0, 1]$ *such that the following set of LMIs have a solution for* H_1 *and* H_2:

$$\begin{aligned} &A_1'H_1 + H_1A_1 < 0, \qquad A_2'H_2 + H_2A_2 < 0, \\ &(1 - \delta_2)(A_1'H_2 + H_2A_1) - \delta_2(H_2 - H_1) < 0, \\ &(1 - \delta_1)(A_2'H_1 + H_1A_2) + \delta_1(H_2 - H_1) < 0, \\ &0 < H_1 < I, \quad 0 < H_2 < I. \end{aligned} \tag{3.20}$$

If this is the case, then $V(x) = \min\{x'H_1x, x'H_2x\}$ *is also a valid Lyapunov function for establishing the robust asymptotic stability of the system* (2.1).

Remark 3.1. To see the improvement given by (3.19) and (3.20) in comparison with (2.7)–(2.12) and (2.13)–(2.18), we look at the gridding method for finding δ_i and λ_i. If the number of grid points for each variable is N, then N^2 grid points are required for (δ_1, δ_2) while N^4 points are needed for $(\lambda_1, \ldots, \lambda_4)$. For $N = 10$, the former is 100 while the latter is 10 000!

Proof of Theorem 3.1 (*Necessity*). Without loss of generality, we assume that $\lambda_2\lambda_3 \neq \lambda_1\lambda_4$. If $\lambda_2\lambda_3 = \lambda_1\lambda_4$, we can perturb one of these parameters slightly without violating (2.7)–(2.12). Define the transformation

$$H_1 = \mu_1(\lambda_1 P_2 + \lambda_3 P_1), \qquad H_2 = \mu_2(\lambda_2 P_2 + \lambda_4 P_1), \tag{3.21}$$

where $\mu_1 = \lambda_2 + \lambda_4$ and $\mu_2 = \lambda_1 + \lambda_3$. Then (2.7)–(2.12) lead to the following:

$$H_1 > 0,\ H_2 > 0, \tag{3.22}$$

$$A_1'H_1 + H_1A_1 = \mu_1\lambda_3(A_1'P_1 + P_1A_1 - \lambda_1(P_2 - P_1)) + \mu_1\lambda_1(A_1'P_2 + P_2A_1 + \lambda_3(P_2 - P_1)) < 0, \tag{3.23}$$

$$A_2'H_2 + H_2A_2 = \mu_2\lambda_4(A_2'P_1 + P_1A_2 - \lambda_2(P_2 - P_1)) + \mu_2\lambda_2(A_2'P_2 + P_2A_2 + \lambda_4(P_2 - P_1)) < 0, \tag{3.24}$$

$$\begin{aligned} &A_1'H_2 + H_2A_1 + \mu_2(H_2 - H_1) \\ &\quad = \mu_2\lambda_4(A_1'P_1 + P_1A_1 - \lambda_1(P_2 - P_1)) + \mu_2\lambda_2(A_1'P_2 + P_2A_1 + \lambda_3(P_2 - P_1)) < 0, \end{aligned} \tag{3.25}$$

$$\begin{aligned} &A_2'H_1 + H_1A_2 - \mu_1(H_2 - H_1) \\ &\quad = \mu_1\lambda_3(A_2'P_1 + P_1A_2 - \lambda_2(P_2 - P_1)) + \mu_1\lambda_1(A_2'P_2 + P_2A_2 + \lambda_4(P_2 - P_1)) < 0. \end{aligned} \tag{3.26}$$

Now, introducing the variable substitution

$$\mu_1 = \frac{\delta_1}{1 - \delta_1}, \qquad \mu_2 = \frac{\delta_2}{1 - \delta_2},$$

we obtain (3.19). Note that the constraints $H_1 < I$ and $H_2 < I$ are harmless because we can always rescale H_1 and H_2 to satisfy them.

(*Sufficiency*): Assume that (3.19) holds. If we set $P_1 = H_1$, $P_2 = H_2$, $\lambda_1 = \lambda_4 = 0$, $\lambda_2 = \delta_1/(1 - \delta_1)$ and $\lambda_3 = \delta_2/(1 - \delta_2)$, then we find a solution to the system (2.7)–(2.12). □

The proof for Theorem 3.2 is very similar and therefore omitted.

4. Extension to discrete-time systems

The results in Theorems 3.1 and 3.2 are readily extendible to the discrete-time case. More specifically, we consider the following discrete-time system:

$$x(k + 1) = A(k)x(k), \qquad A(k) \in \mathscr{A} = \mathrm{Co}\{A_1, A_2\}. \tag{4.27}$$

The discrete-time counterpart of Theorems 3.1 and 3.2 is given below.

Theorem 4.1. *The system* (4.27) *is robustly stable with the Lyapunov function* (2.2) *if and only if there exist* $\delta_1, \delta_2 \in [0, 1]$ *such that the following set of LMIs have a solution for* H_1 *and* H_2:

$$\begin{aligned} &A_1'H_1A_1 - H_1 < 0, \qquad A_2'H_2A_2 - H_2 < 0, \\ &(1 - \delta_2)(A_1'H_2A_1 - H_2) + \delta_2(H_2 - H_1) < 0, \\ &(1 - \delta_1)(A_2'H_1A_2 - H_1) - \delta_1(H_2 - H_1) < 0, \\ &0 < H_1 < I, \quad 0 < H_2 < I. \end{aligned} \tag{4.28}$$

If this is the case, then $V(x) = \max\{x'H_1x, x'H_2x\}$ *is also a valid Lyapunov function for establishing the robust asymptotic stability of the system* (4.27).

Theorem 4.2. *The system* (4.27) *is robustly stable with the Lyapunov function* (2.3) *if and only if there exist* $\delta_1, \delta_2 \in [0, 1]$ *such that the following set of LMIs have a solution for* H_1 *and* H_2:

$$\begin{aligned} &A_1'H_1A_1 - H_1 < 0, \qquad A_2'H_2A_2 - H_2 < 0, \\ &(1-\delta_2)(A_1'H_2A_1 - H_2) - \delta_2(H_2 - H_1) < 0, \\ &(1-\delta_1)(A_2'H_1A_2 - H_1) + \delta_1(H_2 - H_1) < 0, \\ &0 < H_1 < I, \quad 0 < H_2 < I. \end{aligned} \tag{4.29}$$

If this is the case, then $V(x) = \min\{x'H_1x, x'H_2x\}$ *is also a valid Lyapunov function for establishing the robust asymptotic stability of the system* (4.27).

5. Numerical example

In this section, we shall test our results on an example which has been used in [12]. Two tests will be carried out to show how the Lyapunov functions (2.2) and (2.3) can be used to obtain better estimation of robust stability margins.

Example. Consider the following system:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -2x_1 - x_2 - u(t)x_1, \qquad 0 \leqslant u(t) \leqslant k, \tag{5.30}$$

where $u(t)$ is a time-varying uncertain parameter and k is its bound. The goal is to find the largest k for which the system still remains robustly asymptotically stable. This system is found to be quadratically stable for $k \leqslant 3.828$ only ($k \leqslant 3.82$ in [12]) and admit a fixed Lyapunov function $x'Px$, where

$$P = \begin{pmatrix} 258.4972 & 32.9855 \\ 32.9855 & 66.0536 \end{pmatrix}.$$

In order to estimate the robust stability margin $k_{\max}$, we use a bisection method. That is, we start with some initial k_0 and k_1 as lower and upper bounds for $k_{\max}$, respectively, and then apply bisection to improve the bounds until their gap is sufficiently small.

Test one: Now, we apply Theorems 3.1 and 3.2 to (5.30). We simply use a gridding method to search for δ_1 and δ_2. With the grid size equal to 0.1, we find that (5.30) is robustly asymptotically stable with the Lyapunov function (2.2) for $k \leqslant 4.7$ and the Lyapunov function is given by

$$\max\left\{x'\begin{pmatrix} 0.7685 & 0.1326 \\ 0.1326 & 0.1923 \end{pmatrix}x,\ x'\begin{pmatrix} 0.8199 & 0.0690 \\ 0.0690 & 0.1795 \end{pmatrix}x\right\}. \tag{5.31}$$

The corresponding parameters are $\delta_1 = 0.9$, $\delta_2 = 0.8$, respectively. Also, (5.30) is robustly asymptotically stable for $k \leqslant 4.4$ with the Lyapunov function

$$\min\left\{x'\begin{pmatrix} 0.7344 & 0.0737 \\ 0.0737 & 0.1963 \end{pmatrix}x,\ x'\begin{pmatrix} 0.8252 & 0.1155 \\ 0.1155 & 0.1772 \end{pmatrix}x\right\}, \tag{5.32}$$

corresponding to $\delta_1 = 0.8$ and $\delta_2 = 0.8$.

Test two: To further improve the robust stability margin, we use the following nonlinear transformation technique introduced in [4]:

$$x_l^{[p]} = \sqrt{\left(\frac{p}{p_1}\right)\left(\frac{p-p_1}{p_2}\right)\cdots\left(\frac{p-p_1-\cdots p_{p-1}}{p_p}\right)}\, x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n}, \quad \sum_{i=1}^{n} p_i = p,\ p_i \geqslant 0, \tag{5.33}$$

where the integer p is fixed, taking the value of $2,3,\dots$ and l is the index of the transformed state corresponding to a set $\{p_1,\dots,p_n\}$ satisfying the constraints in (5.33). The transformed system can be expressed as

$$\dot{x}^{[p]} = A_{[p]}x^{[p]} \tag{5.34}$$

which is $\binom{n+p-1}{p}$-dimensional. In our example, we use the transformation

$$y_1 = x_1^2, \qquad y_2 = x_1x_2, \qquad y_3 = x_2^2 \tag{5.35}$$

as done in [12], i.e. $p=2$. This transformation changes (5.30) into

$$\begin{aligned}&\dot{y}_1 = 2y_2, \qquad \dot{y}_2 = -2y_1 - y_2 + y_3 - u(t)y_1,\\ &\dot{y}_3 = -4y_2 - 2y_3 - 2u(t)y_2, \quad 0 \leqslant u(t) \leqslant k.\end{aligned} \tag{5.36}$$

Note that (5.36) is linear in $u(t)$. It is well known that [4] the null solution of the system (5.30) is stable (asymptotically stable) if and only if the null solution of the system (5.36) is stable (asymptotically stable). Moreover, if all solutions of (5.30) are bounded by $|x(t)| \leqslant Me^{-\lambda t}$ then all solutions of (5.36) are bounded by $|y(t)| \leqslant M_1e^{-p\lambda t}$. The stability of the transformed system is usually established using a quadratic Lyapunov function which leads to a $2p$-order Lyapunov function for the original system.

The system (5.36) is found to be quadratically stable for only $k \leqslant 5.47$ [12]. A better estimate $k \leqslant 5.73$ is also obtained by adding the quadratic constraint $y_1y_3 = y_2^2$. However, by applying the Lyapunov functions (2.2) and Theorem 3.1 to (5.36), we have found that (5.36) is robustly asymptotically stable for $k \leqslant 5.8$ and admits a Lyapunov function

$$\begin{aligned}&\max\left\{ y'\begin{pmatrix} 0.8454 & 0.1246 & 0.0568\\ 0.1246 & 0.3679 & 0.0627\\ 0.0568 & 0.0627 & 0.0381\end{pmatrix}y,\ y'\begin{pmatrix} 0.8098 & 0.1123 & -0.0077\\ 0.1123 & 0.4552 & 0.0530\\ -0.0077 & 0.0530 & 0.0381\end{pmatrix}y\right\}\\ &\quad= \max\{0.8454x_1^4 + 0.2492x_1^3x_2 + 0.4815x_1^2x_2^2 + 0.1254x_1x_2^3 + 0.0381x_2^4,\\ &\qquad\quad 0.8098x_1^4 + 0.2246x_1^3x_2 + 0.4398x_1^2x_2^2 + 0.1060x_1x_2^3 + 0.0381x_2^4\}\end{aligned} \tag{5.37}$$

with $\delta_1 = 0.9$, $\delta_2 = 0.6$.

Similarly, the use of (2.3) and Theorem 3.2 leads to $k \leqslant 5.8$ with the following Lyapunov function:

$$\begin{aligned}&\min\left\{ y'\begin{pmatrix} 0.7911 & 0.1106 & 0.0111\\ 0.1106 & 0.4285 & 0.0568\\ 0.0111 & 0.0568 & 0.0414\end{pmatrix}y,\ y'\begin{pmatrix} 0.8393 & 0.1638 & 0.0232\\ 0.1638 & 0.4417 & 0.0575\\ 0.0232 & 0.0575 & 0.0393\end{pmatrix}y\right\}\\ &\quad= \min\{0.7911x_1^4 + 0.2212x_1^3x_2 + 0.4507x_1^2x_2^2 + 0.1136x_1x_2^3 + 0.0414x_2^4,\\ &\qquad\quad 0.8393x_1^4 + 0.3276x_1^3x_2 + 0.4881x_1^2x_2^2 + 0.1150x_1x_2^3 + 0.0393x_2^4\}\end{aligned} \tag{5.38}$$

with $\delta_1 = 0.8$, $\delta_2 = 0.8$.

All the above computation is done using the LMI Control Toolbox [6]. The grid size for δ_1 and δ_2 is 0.1. If we use the grid size 0.05, we can obtain the result $k = 5.9$ and $k = 5.8$ for (2.2) and (2.3), respectively.

Remark 5.1. Further improvement can also be achieved by introducing the quadratic constraint

$$y_2^2 = y_1y_2 \tag{5.39}$$

as is done in [12]. This constraint is evident from the transformation (5.35). By using the S-procedure on this constraint, we can obtain $k = 6.2$ and $k = 5.9$ for (2.2) and (2.3), respectively. To explain how this is done for (2.2), we rewrite the constraint (5.39) as

$$y'Qy = y' \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} y = 0. \tag{5.40}$$

Then, (2.2) is modified by adding a term $\tau_i Q$ to the left-hand side of each LMI, where τ_i is a real scalar variable, different for each LMI. Using the S-procedure, a sufficient condition for robust stability of (5.36) is that this modified set of LMIs has a feasible solution. Although this is a sufficient condition, it is a better (weaker) condition than ignoring (5.40) because τ_i are non-zero in general. The increase in computational complexity is very marginal because the resulting LMIs are linear in τ_i. The modification for (2.3) is very similar.

6. Conclusions

In this paper, we have investigated the use of piecewise Lyapunov functions for providing better estimation of robust stability. The Lyapunov functions we used are in the form (2.2) and (2.3). We have proposed necessary and sufficient conditions for robust stability of convex combinations of two matrices. The key contribution of the paper is a variable reduction technique which allows us to gain some computational efficiency. We show through an example that our method can produce better estimation than existing methods.

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