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Piecewise Lyapunov functions for robust stability of linear time-varying systems *

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Abstract In this paper, we investigate the use of two-term piecewise quadratic Lyapunov functions for robust stability of linear time-varying systems. By using the so-called S-procedure and a special variable reduction method, we provide numerically efficient conditions for the robust asymptotic stability of the linear time-varying systems involving the convex combinations of two matrices. An example is included to demonstrate the usefulness of our results.

Key words: Robust stability, Piecewise Lyapunov function, S-procedure, Linear matrix inequality.

1 Introduction

The quadratic stability approach is popularly used for robust stability analysis of time-varying uncertain systems. This approach, however, may lead to very conservative results. Alternatively, non-quadratic Lyapunov functions have been used to improve the estimate of robust stability (see [1]). The general difficulty with non-quadratic Lyapunov functions is that the resulting optimization problem is non-convex. In this short paper, we investigate the use of two-term piecewise Lyapunov functions on time-varying linear systems involving the convex combination of two matrices. These Lyapunov functions are either the maximum or the minimum of two quadratic terms. By using the so-called S-procedure [2] and a variable reduction technique, we obtain necessary and sufficient conditions for establishing robust asymptotic stability of the uncertain system using such a piecewise Lyapunov function. The resulting optimization problem involves a set of linear matrix inequalities(LMIs) with two scaling parameters which can be numerically searched.

We show via an example that good improvement on the estimate of robust stability margin can be obtained by using these piecewise Lyapunov functions when compared with the quadratic stability technique.

2 Problem formulation

Consider the linear time-varying system:

$$\dot{x} = A(t)x, \quad A(t) \in \mathcal{A} \triangleq \text{Co}\{A_1, A_2\} \tag{2.1}$$

Our aim is to produce a test which is less conservative than quadratic stability result with reasonable computational cost. In particular, we use two kinds of piecewise Lyapunov functions as follows:

$$V(x) = \max\{x'P_1x, x'P_2x\}, \quad P_1 > 0, P_2 > 0 \tag{2.2}$$

and

$$V(x) = \min\{x'P_1x, x'P_2x\}, \quad P_1 > 0, P_2 > 0 \tag{2.3}$$

In order to derive conditions for robust asymptotic stability of (2.1) with the Lyapunov function (2.2) or (2.3), we need the lemma below.

Lemma 2.1 (**S-procedure lemma** *[2]) Let $F_0(x)$ and $F_1(x)$ be two arbitrary quadratic forms over $\Re^n$. Then $F_0(x) < 0$ for all $x \in \Re^n$ satisfying $F_1(x) \le 0$ if and only if there exist $\tau \ge 0$ such that*

$$F_0(x) - \tau F_1(x) \le 0, \quad \forall x \in \Re^n.$$

Lemma 2.2 *[7] system (2.1) is robustly asymptotically stable for all $A(t) \in \mathcal{A}$ with the Lyapunov function (2.2) if and only if there exist solutions to the following set of inequalities:*

$$A_1'P_1 + P_1A_1 - \lambda_1(P_2 - P_1) < 0 \tag{2.4}$$
$$A_2'P_1 + P_1A_2 - \lambda_2(P_2 - P_1) < 0 \tag{2.5}$$
$$A_1'P_2 + P_2A_1 + \lambda_3(P_2 - P_1) < 0 \tag{2.6}$$
$$A_2'P_2 + P_2A_2 + \lambda_4(P_2 - P_1) < 0 \tag{2.7}$$
$$P_1 > 0, P_2 > 0 \tag{2.8}$$
$$\lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0, \lambda_4 \ge 0 \tag{2.9}$$

Similarly, we can have a result for the Lyapunov function in (2.3):

Lemma 2.3 *system (2.1) is robustly stable for all $A(t) \in \mathcal{A}$ with the Lyapunov function (2.3) if and only if there exist solutions to the following set of inequalities:*

$$A_1'P_1 + P_1A_1 + \lambda_1(P_2 - P_1) < 0 \tag{2.10}$$
$$A_2'P_1 + P_1A_2 + \lambda_2(P_2 - P_1) < 0 \tag{2.11}$$
$$A_1'P_2 + P_2A_1 - \lambda_3(P_2 - P_1) < 0 \tag{2.12}$$
$$A_2'P_2 + P_2A_2 - \lambda_4(P_2 - P_1) < 0 \tag{2.13}$$
$$P_1 > 0, P_2 > 0 \tag{2.14}$$
$$\lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0, \lambda_4 \ge 0 \tag{2.15}$$

Remark 2.1 *The use of Lyapunov functions (2.2)- (2.3) and the S-procedure is not new [6, 7]. In fact, it is shown [1] that the system (2.1) is robustly asymptotically stable if and only if there exists a piecewise quadratic Lyapunov function of the following form:*

$$V(x) = \max_{1 \le i \le p}\{x'P_ix\}, \ P_i > 0, \ i = 1, ..., p.$$

for some p. But the stumbling block is that the subsequent optimization problem is non-convex. This is true to some degree even for (2.4)- (2.9) or (2.10)- (2.15). If we fix P_1 and P_2, then (2.4)- (2.9) and (2.10)- (2.15) are convex problems. This also the case if we fix $\lambda_i, i = 1, ..., 4$ (refer to [7] for example). However, (2.4)- (2.9) and (2.10)- (2.15) are not jointly convex in P_1, P_2 and λ_i.

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3 Main Result

In this section, we are going to provide a variable reduction procedure which can reduce the number of non-convex variables λ_i in (2.4)- (2.15) from four to two. Also, the range of the new convex variable will be set to $[0,1]$ for easy numerical implementation. Since the resulting problem involves only two non-convex variables, numerical searching of these parameters becomes a moderate problem.

Theorem 3.1 *The system (2.1) is robustly asymptotically stable with the Lyapunov function (2.2) if and only if there exist $\delta_1, \delta_2 \in [0,1]$ such that the following set of LMIs have a solution for H_1 and H_2:*

$$\begin{cases} A_1'H_1 + H_1A_1 < 0 \\ A_2'H_2 + H_2A_2 < 0 \\ (1-\delta_2)(A_1'H_2 + H_2A_1) + \delta_2(H_2 - H_1) < 0 \\ (1-\delta_1)(A_2'H_1 + H_1A_2) - \delta_1(H_2 - H_1) < 0 \\ H_1 = H_1' > 0, H_2 = H_2' > 0 \end{cases} \quad (3.16)$$

If this is the case, then $V(x) = \max\{x'H_1x, x'H_2x\}$ is also a valid Lyapunov function for establishing the robust asymptotic stability of the system (2.1).

Theorem 3.2 *The system (2.1) is robustly asymptotically stable with the Lyapunov function (2.3) if and only if there exist $\delta_1, \delta_2 \in [0,1]$ such that the following set of LMIs have a solution for H_1 and H_2:*

$$\begin{cases} A_1'H_1 + H_1A_1 < 0 \\ A_2'H_2 + H_2A_2 < 0 \\ (1-\delta_2)(A_1'H_2 + H_2A_1) - \delta_2(H_2 - H_1) < 0 \\ (1-\delta_1)(A_2'H_1 + H_1A_2) + \delta_1(H_2 - H_1) < 0 \\ H_1 = H_1' > 0, H_2 = H_2' > 0 \end{cases} \quad (3.17)$$

If this is the case, then $V(x) = \min\{x'H_1x, x'H_2x\}$ is also a valid Lyapunov function for establishing the robust asymptotic stability of the system (2.1).

Proof of theorem 3.1 See the full paper [9] for details. □

4 Extension to discrete-time systems

The results in theorems 3.1 and 3.2 are readily extendible to the discrete-time case. More specifically, we consider the following discrete-time system

$$x(k+1) = A(k)x(k), \; A(k) \in \mathcal{A} \triangleq \mathrm{Co}\{A_1, A_2\} \quad (4.18)$$

The discrete-time counterpart of theorems 3.1 and 3.2 is given below.

Theorem 4.1 *The system (4.18) is robustly asymptotically stable with the Lyapunov function (2.2) if and only if there exist $\delta_1, \delta_2 \in [0,1]$ such that the following set of LMIs have a solution for H_1 and H_2:*

$$\begin{cases} A_1'H_1A_1 - H_1 < 0 \\ A_2'H_2A_2 - H_2 < 0 \\ (1-\delta_2)(A_1'H_2A_1 - H_2) + \delta_2(H_2 - H_1) < 0 \\ (1-\delta_1)(A_2'H_1A_2 - H_1) - \delta_1(H_2 - H_1) < 0 \\ H_1 = H_1' > 0, H_2 = H_2' > 0 \end{cases} \quad (4.19)$$

If this is the case, then $V(x) = \max\{x'H_1x, x'H_2x\}$ is also a valid Lyapunov function for establishing the robust asymptotic stability of the system (4.18).

Theorem 4.2 *The system (4.18) is robustly asymptotically stable with the Lyapunov function (2.3) if and only if there exist $\delta_1, \delta_2 \in [0,1]$ such that the following set of LMIs have a solution for H_1 and H_2:*

$$\begin{cases} A_1'H_1A_1 - H_1 < 0 \\ A_2'H_2A_2 - H_2 < 0 \\ (1-\delta_2)(A_1'H_2A_1 - H_2) - \delta_2(H_2 - H_1) < 0 \\ (1-\delta_1)(A_2'H_1A_2 - H_1) + \delta_1(H_2 - H_1) < 0 \\ H_1 > 0, H_2 > 0 \end{cases} \quad (4.20)$$

If this is the case, then $V(x) = \min\{x'H_1x, x'H_2x\}$ is also a valid Lyapunov function for establishing the robust asymptotic stability of the system (4.18).

5 Numerical Example

In the full paper [9], an example is given to demonstrate that better robustness bounds can be achieved using the Lyapunov functions (2.2) and (2.3).

6 Conclusions

In this paper, we have investigated the use of piecewise Lyapunov functions for providing better estimation of robust stability. The Lyapunov functions we used are in the form (2.2) and (2.3). We have proposed necessary and sufficient conditions for robust stability of convex combinations of two matrices. These conditions are numerically efficient. We show through an example that our method can produce better estimation than existing methods.

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