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Approximation Algorithms for Quadratic Programming

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Abstract. We consider the problem of approximating the global minimum of a general quadratic program (QP) with n variables subject to m ellipsoidal constraints. For $m = 1$, we rigorously show that an ϵ-minimizer, where error $\epsilon \in (0, 1)$, can be obtained in polynomial time, meaning that the number of arithmetic operations is a polynomial in n, m, and $\log(1/\epsilon)$. For $m \geq 2$, we present a polynomial-time $(1 - \frac{1}{m^2})$-approximation algorithm as well as a semidefinite programming relaxation for this problem. In addition, we present approximation algorithms for solving QP under the box constraints and the assignment polytope constraints.

Keywords: quadratic programming, global minimizer, polynomial-time approximation algorithm

1. Introduction

Consider the general quadratic programming (QP) problem

$$\text{(QP)}\quad \begin{array}{ll} \text{Minimize} & q(x) := \frac{1}{2}x^T Qx + c^T x \\ \text{Subject to} & x \in \mathcal{F}, \end{array}$$

where $Q \in \Re^{n\times n}$ and $c \in \Re^n$ are given data, and $\mathcal{F}$ is a full-dimensional convex set in $\Re^n$. Throughout this paper, we assume $\mathcal{F}$ is bounded and has nonempty interior. QP is a generic problem in optimization theory and practice. Economic equilibrium, combinatorial optimization, numerical partial differential equation, and general nonlinear programming are all sources of QP problems. In particular, many nonlinear programming methods require solving a sequence of QP subproblems.

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If Q is positive semi-definite, that is, for any given $d \in \Re^n$,

$$d^T Q d \geq 0,$$

then (QP) is a convex optimization problem and it can be solved in polynomial time, e.g., see Vavasis (1991) and references therein. In this paper we consider general nonconvex QP problems for which Q has at least one negative eigenvalue. In this case, (QP) becomes a hard problem—an NP-complete problem (see, Garey and Johnson, 1968; Pardalos and Rosen, 1987; Sahni, 1974; Vavasis, 1991).

If the feasible set $\mathcal{F}$ is nonempty and bounded, then (QP) has a minimizer and a maximizer. Let $\underline{z}$ and $\bar{z}$ denote their minimal and maximal objective values over $\mathcal{F}$, respectively. An ϵ-minimal solution or ϵ-minimizer, $\epsilon \in [0, 1]$, for (QP) is defined as an $x \in \mathcal{F}$ such that

$$\frac{q(x) - \underline{z}}{\bar{z} - \underline{z}} \leq \epsilon.$$

(Vavasis (1993) discussed the importance to have the term $(\bar{z} - \underline{z})$ in this criterion for continuous optimization.) According to this definition, any feasible solution $x \in \mathcal{F}$ is a 1-minimizer.

It turns out that finding an ϵ-minimizer for (QP) is a hard problem for $\epsilon \in (0, 1)$. Indeed, when $\mathcal{F}$ is a polytope, Bellare and Rogaway (1995) showed that there exists a constant, say $\frac{1}{4}$, such that no polynomial-time algorithm exists to compute an $\frac{1}{4}$-minimal solution for (QP), unless $P = NP$. They also showed that there exists a constant $\delta > 0$ such that QP has no polynomial time, $(1 - 2^{-\log^\delta n})$-approximation algorithm, unless $NP \subset \tilde{P}$, where $\tilde{P}$ denotes the class of languages recognizable in quasi-polynomial time.

So far there have been several algorithms available for "solving" general polytope-constrained QP problems; these include the principal pivoting method of Lemke-Cottle-Dantzig (e.g., Cottle and Dantzig, 1968), the active-set method (e.g., Gill et al., 1981), the interior-point algorithm (e.g., Kamath et al., 1992; Ye, 1992), and other special-case methods (e.g., Murty, 1988; Pardalos and Rosen, 1987). These algorithms usually generate a sequence of points that converges to a stationary or Karush-Kuhn-Tucker (KKT) point associated with (QP). To our knowledge, there have been no approximation bounds developed for these methods. Consequently, the solutions delivered by these algorithms are not guaranteed to be a good approximate global minimizer of (QP).

There are also several approximation algorithms developed for QP when the feasible region is a polytope. Pardalos and Rosen (1987) developed a partitioning and linear programming based algorithm with an approximation bound $\epsilon = \sigma(Q, c, \mathcal{F})$, where σ, a function of the QP data, is between 1/4 and 1. Recently, Vavasis (1993) and Ye (1992) developed a polynomial-time algorithm to compute an $(1 - \frac{1}{n^2})$-minimal solution.

The objective of this paper is to develop new interior point polynomial-time approximation algorithms for the class of QPs with $\mathcal{F}$ defined by m ellipsoidal constraints:

$$\mathcal{F} := \left\{ x \in \Re^n : d_i - c_i^T x - \frac{1}{2} x^T Q_i x \geq 0, \quad i = 1, \ldots, m \right\}, \tag{1.1}$$

where each $Q_i \in \Re^{n \times n}$ is a symmetric positive semidefinite matrix, $c_i \in \Re^n$ is a vector and d_i is a scalar. Here and throughout this paper, the term "polynomial time approximation algorithm" means an algorithm whose running time is a polynomial of n, m, $\log(1/\delta)$ and $\log(1/\epsilon)$, where δ is the ratio of the radius of the largest inscribing sphere over that of the smallest circumscribing sphere of $\mathcal{F}$. The quantity δ can be thought of as a condition number of the problem. Our definition of polynomial time approximation algorithm is consistent with the traditional notion of polynomial time algorithm for linear programming. This is because any polynomial approximation algorithm for linear programming can be turned into a polynomial time algorithm by setting $\epsilon = 2^{-L}$ and noting $\delta = O(2^{-L})$, where L is the data length in binary encoding.

Our interest in the approximation algorithms for the QP problem with multiple ellipsoid constraints is two fold. First, Celis et al. (1984) and Powell-Yuan (1991) proposed trust-region algorithms for equality constrained nonlinear programming problem whereby QP problems with two ellipsoid constraints (i.e., $m = 2$ in (1.1)) are solved at each iteration. As a solver for the subproblems, any effective method for solving the above QP problem will yield a fast trust region algorithm for the general nonlinear programming problem. Second, for the single ball constrained case (i.e., $m = 1$), we know from Section 2 that the QP problem is polynomial time solvable. However, for the case $m = 2$, neither is there a known polynomial time algorithm for computing the global minimum of the above QP problem, nor do we know if the problem is NP-hard. (Indeed, the complexity status of the two-sphere constrained QP problem remains open.) This leads us to design approximation algorithms for the two ellipsoid constrained QP problem. In general, if m is arbitrary, the QP problem with multiple ellipsoid constraints includes the polytope constrained QP as a special case since a polytope is a degenerate form of (1.1) with $Q_i = 0$. Thus, finding the global minimum for the case with general m is NP-hard. In fact, even the problem of finding an ϵ-minimizer (for $\epsilon \in (0, 1)$) with general m is hard since a subclass of the problem, namely the polytope constrained QPs, is known to be hard to approximate.

The main contributions of this paper are as follows. When $m = 1$, we rigorously show that an ϵ-minimizer of the QP can be obtained in polynomial time $-O(n^3 \log(1/\epsilon))$ arithmetic operations. We show this by a binary section type method (Section 2) and by converting the problem to a semidefinite program which is known to be polynomial time solvable (see Section 6). For $m \geq 2$, we present a semidefinite programming relaxation of this problem (Section 6). We also present $(1 - \frac{1}{m^2})$-approximation algorithms for this problem (Section 3). This algorithm is based on inscribing the feasible region $\mathcal{F}$ by a single ellipsoid and computing a global minimum $q(\bar{x})$ of the objective $q(x)$ over this ellipsoid. We show that if the inscribing ellipsoid is sufficiently centered in $\mathcal{F}$, then $q(\bar{x})$ is a good approximation of the global minimum of $q(x)$ over $\mathcal{F}$. Using the same technique of inscribing ellipsoid, we further develop several approximation algorithms for solving QP under the box constraints and the assignment polytope constraints (see Sections 4 and 5).

2. The single-ellipsoid constrained QP problem

In this section we consider the ball-constrained QP problem, BQP(r), where

$$\mathcal{F} = \mathcal{B}(r) := \{x \in \Re^n : \|x\| \leq r\},$$

radius r is a given positive number, and $\|\cdot\|$ denotes L_2 norm. By an affine transformation, the single-ball constrained QP can be used to solve a single-ellipsoid constrained QP.

We begin with a brief history of this problem. There is a class of nonlinear programming algorithms called model trust region methods. In these algorithms, a quadratic function is used as an approximate model of the true objective function around the current iterate. Then the main step is to minimize the model function. In general, however, the model is expected to be accurate or trusted only in a neighborhood of the current iterate. Accordingly, the quadratic model is minimized in a L_2-norm neighborhood, which is a ball, around the current iterate.

The model-trust region method is due to Levenberg (1963) and Marquardt (1963). These authors considered only the case where Q is positive definite. Moré (1977) proposed an algorithm with a convergence proof for this case. Gay (1981) and Sorenson (1982) proposed algorithms for the general case, also see Dennis and Schnabel (1983). These algorithms work very well in practice, but no complexity result was established for this problem then.

It is well known (Gay, 1981; Sorenson, 1982) that the solution x of problem BQP(r) satisfies the following necessary and sufficient conditions:

$$\begin{aligned} (Q + \mu I)x &= -c \\ \mu &\geq \max\{0, -\underline{\lambda}\} \end{aligned} \tag{2.1}$$

and

$$\|x\| = r,$$

where $\underline{\lambda}$ denotes the least eigenvalue of matrix Q. Since Q is not positive semi-definite, we must have $\underline{\lambda} < 0$.

Let μ^* and x^* satisfy conditions (2.1). It has been shown that μ^* is unique and

$$\mu^* \leq |\underline{\lambda}| + \frac{\|c\|}{r}. \tag{2.2}$$

It is also known that

$$|\underline{\lambda}| \leq n \max\{|q_{ij}|\},$$

where q_{ij} is the (i, j)th component of matrix Q. Thus, we have

$$0 \leq \mu^* \leq \mu^0 := n \max\{|q_{ij}|\} + \frac{\|c\|}{r}, \tag{2.3}$$

where μ^0 is a computable upper bound. It is further proved that Ye (1992)

$$\frac{1}{2} r^2 |\underline{\lambda}| \leq \frac{1}{2} r^2 \mu^* \leq q(0) - q(x^*) \leq \frac{1}{2} r^2 |\underline{\lambda}| + r\|c\|. \tag{2.4}$$

We now analyze the complexity of solving BQP(r). A simple bisection method was proposed by Vavasis and Zippel (1990) and Ye (1992). For any given μ, denote solutions of the top linear equations by x_μ in conditions (2.1), i.e.,

$$x_\mu := -(Q + \mu I)^{-1} c, \quad \forall \mu > |\underline{\lambda}|. \tag{2.5}$$

For any given μ we can check to see if $\mu \geq |\underline{\lambda}|$ by checking the positive definiteness of matrix $Q + \mu I$, which can be solved as a LDL^T decomposition. These facts lead to a bisection method to search for the root of $\|x_\mu\| = r$ over the interval $\mu \in [|\underline{\lambda}|, \mu^0] \subset [0, \mu^0]$. Obviously, for a given $\epsilon' \in (0, 1)$, a μ such that, say $0 \le \mu - \mu^* \le \epsilon'\mu^*/8$, can be obtained in $O(\log(\mu^0/\mu^*) + \log(1/\epsilon'))$ bisection steps, and the cost of each step is $O(n^3)$ arithmetic operations (for performing LDL^T decomposition).

The remaining question is what ϵ' would be sufficient to generate an ϵ-minimizer of $q(x)$ over the ball $\mathcal{B}(r)$. Let μ denote the right end point of the interval generated by the bisection search. Then, $\mu \ge \mu^*$. If $\mu = \mu^*$, then we get an exact solution. Thus, we assume $\mu > \mu^* \ge |\underline{\lambda}|$. By the positive semi-definiteness of $Q + \mu^* I$, we have

$$\|x_\mu\| < \|x_{\mu^*}\| = r.$$

We consider two cases.

Case I. In the first case we assume

$$\left(1 - \frac{\epsilon}{8\sqrt{n}}\right)\mu^* \ge |\underline{\lambda}| \quad \text{or} \quad \mu^* \ge |\underline{\lambda}| + \frac{\epsilon}{8\sqrt{n}}\mu^*.$$

Using the relation (2.5) and simplifying, we obtain

$$\begin{aligned}\|x_{\mu^*}\|^2 - \|x_\mu\|^2 &= x^T(\mu^*)(I - (Q + \mu^* I)(Q + \mu I)^{-2}(Q + \mu^* I))x_{\mu^*}\\ &= x^T(\mu^*)(2(\mu - \mu^*)(Q + \mu I)^{-1} - (\mu - \mu^*)^2(Q + \mu I)^{-2})x_{\mu^*}.\end{aligned}$$

Next we consider a diagonalization of Q and bound the resulting diagonal entries of the above expression by using the smallest eigenvalue $\underline{\lambda}$. This gives

$$\begin{aligned}\|x_{\mu^*}\|^2 - \|x_\mu\|^2 &\le \left(\frac{2(\mu - \mu^*)}{(\mu - |\underline{\lambda}|)} - \frac{(\mu - \mu^*)^2}{((\mu - |\underline{\lambda}|))^2}\right)\|x_{\mu^*}\|^2\\ &= \left(\frac{2(\mu - \mu^*)}{(\mu - \mu^*) + (\mu^* - |\underline{\lambda}|)} - \frac{(\mu - \mu^*)^2}{((\mu - \mu^*) + (\mu^* - |\underline{\lambda}|))^2}\right)\|x_{\mu^*}\|^2\\ &= \frac{(\mu - \mu^*)^2 + 2(\mu - \mu^*)(\mu^* - |\underline{\lambda}|)}{((\mu - \mu^*) + (\mu^* - |\underline{\lambda}|))^2} r^2\\ &= \left(1 - \frac{(\mu^* - |\underline{\lambda}|)^2}{((\mu - \mu^*) + (\mu^* - |\underline{\lambda}|))^2}\right) r^2\\ &\le \left(1 - \frac{(\epsilon\mu^*/8\sqrt{n})^2}{((\mu - \mu^*) + \epsilon\mu^*/8\sqrt{n})^2}\right) r^2,\end{aligned}$$

where the in last step we used the assumption $\mu^* \ge |\underline{\lambda}| + \frac{\epsilon}{8\sqrt{n}}\mu^*$. Therefore, if we have $\mu - \mu^* \le \epsilon'\mu^*/8$, then

$$\|x_{\mu^*}\|^2 - \|x_\mu\|^2 \le \frac{(2\sqrt{n}\epsilon'/\epsilon) + (\sqrt{n}\epsilon'/\epsilon)^2}{(1 + (\sqrt{n}\epsilon'/\epsilon))^2} r^2 \le \frac{2\sqrt{n}\epsilon'}{\epsilon} r^2. \tag{2.6}$$

On the other hand, note that

$$\begin{aligned} q(x_\mu) - q(x_{\mu^*}) &= \frac{1}{2}x_\mu^T Q x_\mu + c^T x_\mu - \frac{1}{2}x_{\mu^*}^T Q x_{\mu^*} - c^T x_{\mu^*} \\ &= \frac{1}{2}(Qx_\mu + c)^T(x_\mu - x_{\mu^*}) + \frac{1}{2}(Qx_{\mu^*} + c)^T(x_\mu - x_{\mu^*}) \\ &= -\frac{1}{2}\mu x_\mu^T(x_\mu - x_{\mu^*}) - \frac{1}{2}\mu^* x_{\mu^*}^T(x_\mu - x_{\mu^*}) \\ &= -\frac{1}{2}(\mu - \mu^*)x_\mu^T(x_\mu - x_{\mu^*}) - \frac{1}{2}\mu^*(\|x_\mu\|^2 - \|x_{\mu^*}\|^2). \end{aligned} \tag{2.7}$$

Now we use the bound (2.6), the assumption $\mu - \mu^* \le \epsilon'\mu^*/8$ and the fact $\|x_\mu\| \le \|x_{\mu^*}\| = r$ to obtain:

$$\begin{aligned} q(x_\mu) - q(x_{\mu^*}) &\le \mu^* r^2 \epsilon'/8 + r^2\mu^* \frac{\sqrt{n}\epsilon'}{\epsilon} \\ &= \left(\epsilon'/4 + \frac{2\sqrt{n}\epsilon'}{\epsilon}\right)\mu^* r^2/2 \\ &\le \left(\epsilon'/4 + \frac{2\sqrt{n}\epsilon'}{\epsilon}\right)(q(0) - q(x^*)), \end{aligned}$$

where the last step is due to (2.4). Thus, if we select

$$\epsilon' \le \frac{\epsilon^2}{2\sqrt{n} + 1/4},$$

then x_μ is feasible for BQP(r) and

$$q(x_\mu) - q(x_{\mu^*}) \le \epsilon(q(0) - q(x^*)),$$

i.e., x_μ is an ϵ-minimizer to x^*.

Case II. In this case, we have

$$\left(1 - \frac{\epsilon}{8\sqrt{n}}\right)\mu^* < |\underline{\lambda}| \quad \text{or} \quad \mu^* < |\underline{\lambda}| + \mu^* \frac{\epsilon}{8\sqrt{n}}.$$

Again, if we have $\mu - \mu^* < \epsilon'\mu^*/8$, then $\mu - |\underline{\lambda}| < \epsilon'\mu^*/8 + \mu^*\epsilon/8\sqrt{n}$. However, unlike Case I, we find $\|x_\mu\|$ is not sufficiently close to r. When we observe this fact, we do the following computation, essentially due to Vavasis and Zippel (1990), to enhance x_μ.

Let $\underline{q}$, $\|\underline{q}\| = 1$, be an eigenvector associated with the eigenvalue $\underline{\lambda}$. Then, one of the unit vector e_j, $j = 1, \ldots, n - m$, must have $|e_j^T \underline{q}| \ge 1/\sqrt{n}$. (In fact, we can use any unit vector q to replace e_j as long as $q^T\underline{q} \ge 1/\sqrt{n}$. A randomly generated q will do it with high probability.) Now we solve for y from

$$(Q + \mu I)y = e_j$$

and let

$$x = x_\mu + \alpha y,$$

where α is chosen such that $\|x\| = r$. Note we have

$$(Q + \mu I)x = -c + \alpha e_j,$$

and in the computation of x_μ and y, matrix $Q + \mu I$ needs to be factorized only once.

It is easy to show that

$$\|y\| \geq \frac{1}{\sqrt{n}(\mu - |\underline{\lambda}|)}$$

and

$$|\alpha| \leq 2r(\mu - |\underline{\lambda}|)\sqrt{n} \leq 2r(\epsilon'\mu^*/8 + \epsilon\mu^*/8\sqrt{n})\sqrt{n}.$$

Then, we have from (2.7)

$$\begin{aligned}
&q(x) - q(x_{\mu^*}) \\
&\quad = \frac{1}{2}(Qx + c)^T(x - x_{\mu^*}) + \frac{1}{2}(Qx_{\mu^*} + c)^T(x - x_{\mu^*}) \\
&\quad = \frac{1}{2}(Qx + c - \alpha e_j)^T(x - x_{\mu^*}) + \frac{1}{2}\alpha e_j^T(x - x_{\mu^*}) - \frac{1}{2}\mu^* x_{\mu^*}^T(x - x_{\mu^*}) \\
&\quad = -\frac{1}{2}\mu x^T(x - x_{\mu^*}) + \frac{1}{2}\alpha e_j^T(x - x_{\mu^*}) - \frac{1}{2}\mu^* x_{\mu^*}^T(x - x_{\mu^*}) \\
&\quad = -\frac{1}{2}(\mu x + \mu^* x_{\mu^*})^T(x - x_{\mu^*}) + \frac{1}{2}\alpha e_j^T(x - x_{\mu^*}) \\
&\quad = -\frac{1}{2}(\mu - \mu^*)x^T(x - x_{\mu^*}) + \frac{1}{2}\alpha e_j^T(x - x_{\mu^*}),
\end{aligned}$$

where the last step follows from $\|x\| = \|x_{\mu^*}\| = r$. Now we use $\mu - \mu^* < \epsilon'\mu^*/8$ and the preceding upper bound on α to estimate the right hand side:

$$\begin{aligned}
q(x) - q(x_{\mu^*}) &\leq r^2\mu^*\epsilon'/8 + 2(\epsilon'\mu^*/8 + \epsilon\mu^*/8\sqrt{n})r^2\sqrt{n} \\
&= (\epsilon'/4 + \sqrt{n}\epsilon'/2 + \epsilon/2)\mu^* r^2/2 \\
&\leq (\epsilon'/4 + \sqrt{n}\epsilon'/2 + \epsilon/2)(q(0) - q(x^*)),
\end{aligned}$$

where the last step is due to (2.4). Thus, if we choose

$$\epsilon' \leq \frac{\epsilon}{\sqrt{n} + 1/2},$$

then x is feasible for BQP(r) and

$$q(x) - q(x_{\mu^*}) \le \epsilon(q(0) - q(x^*)) \le \epsilon(\bar{z} - \underline{z}),$$

i.e., x is an ϵ-minimizer of $q(x)$ over $\mathcal{B}(r)$.

Hence, the bisection method will terminate with an ϵ-minimizer of BQP(r) in at most

$$O(\log(\mu^0/\mu^*) + \log(1/\epsilon) + \log n)$$

steps, or in a total of $O(n^3(\log(\mu^0/\mu^*) + \log(1/\epsilon) + \log n))$ arithmetic operations.

Theorem 1. *The total running time of the bisection algorithm for generating an ϵ-minimal solution to the ball-constrained QP is bounded by $O(n^3(\log(\mu^0/\mu^*) + \log(1/\epsilon) + \log n))$ arithmetic operations.*

The polynomial complexity in Theorem 1 can be further improved. In particular, we (Ye, 1994) developed a Newton-type method for solving (BQP(r)) and established an arithmetic operation bound $O(n^3 \log(\log(\mu^0/\mu^*) + \log(1/\epsilon')))$ to yield a μ such that $0 \le \mu - \mu^* \le \epsilon'$. To compute an ϵ-minimizer of BQP(r), we first find an approximate $\underline{\mu}$ to the absolute value of the least eigenvalue $|\underline{\lambda}|$ and an approximate eigenvector q to the true $\underline{q}$, such that $0 \le \underline{\mu} - |\underline{\lambda}| \le \epsilon'$ and $q^T \underline{q} \ge 1 - \epsilon'$. This approximation can be done in $O(n^3 \log(\log(1/\epsilon')))$ arithmetic operations. Then, we will use q to replace e_j in Case II (i.e., $\|x_{\underline{\mu}}\| < r$) to enhance $x(\underline{\mu})$ and generate a desired approximation. Otherwise, we know $\mu^* > \underline{\mu}$ and, using the method in (Ye, 1994), we will generate a $\mu \in (\underline{\mu}, \mu^0)$ such that $|\mu - \mu^*| \le \epsilon' \mu^*/8$ in $O(n^3 \log(\log(\mu^0/\mu^*) + \log(1/\epsilon')))$ arithmetic operations.

3. The multiple-ellipsoid constrained QP

In this section we consider the QP problem

$$\begin{array}{ll} \text{Minimize} & q(x) := \frac{1}{2} x^T Q x + c^T x \\ \text{Subject to} & x \in \mathcal{F}, \end{array} \tag{3.1}$$

whose feasible region $\mathcal{F}$ is defined by the intersection of multiple ellipsoids, namely,

$$\mathcal{F} := \left\{ x \in \Re^n : d_i - c_i^T x - \frac{1}{2} x^T Q_i x \ge 0, \quad i = 1, \ldots, m \right\}, \tag{3.2}$$

where each $Q_i \in \Re^{n \times n}$ is a symmetric positive semidefinite matrix, $c_i \in \Re^n$ is a vector and d_i is a scalar. We continue to assume that $\mathcal{F}$ contains an interior point and is bounded. To simplify notations, we define

$$g_i(x) := d_i - c_i^T x - \frac{1}{2} x^T Q_i x, \quad i = 1, \ldots, m.$$

Also, since $\mathcal{F}$ is bounded, we may assume without loss of generality that $\mathcal{F}$ is contained in the unit ball and that $g_1 = 1 - x^T x$.

The approximation algorithms developed below and in the next two sections are all based on the following idea: we approximate the feasible region $\mathcal{F}$ by an inscribing ellipsoid with radius r, and minimize the objective function $q(x)$ over this ellipsoid to obtain a global minimizer $x^*(r)$. We then use $x^*(r)$ as an approximate global minimizer for $q(x)$ over $\mathcal{F}$. If the inscribing ellipsoid has the property that when its radius is enlarged to R the ellipsoid contains $\mathcal{F}$, then we can use the following theorem to show that $x^*(r)$ is a $(1-(r/R)^2)$-minimizer of $q(x)$ over $\mathcal{F}$. This theorem first appeared in (Ye, 1992) and will be used frequently in the remainder of this paper.

Theorem 2. *Let $x^*(r)$ and $x^*(R)$ be minimizers of BQP(r) and BQP(R), where $R > r > 0$, respectively. Then,*

$$q(0) - q(x^*(r)) \geq (r/R)^2 (q(0) - q(x^*(R)))$$

or

$$q(x^*(r)) - q(x^*(R)) \leq (1 - (r/R)^2)(q(0) - q(x^*(R))).$$

We also derive the following corollary.

Corollary 1. *Let $x^*(r)$ and $x^*(R)$ be minimizers of BQP(r) and BQP(R), where $R > r > 0$, respectively. Moreover, let $\bar{x}$ be a ϵ-minimizer of BQP(r) and*

$$q(\bar{x}) - q(x^*(r)) \leq \epsilon (q(0) - q(x^*(r))).$$

Then, for any $\underline{z}$ such that $q(x^*(R)) \leq \underline{z} \leq q(x^*(r))$

$$q(\bar{x}) - \underline{z} \leq (1 - (1-\epsilon)(r/R)^2)(q(0) - \underline{z}).$$

Proof: We have

$$\begin{aligned} q(\bar{x}) - \underline{z} &= q(\bar{x}) - q(x^*(r)) + q(x^*(r)) - \underline{z} \\ &\leq \epsilon(q(0) - q(x^*(r))) + q(x^*(r)) - \underline{z} \\ &= -(1-\epsilon)(q(0) - q(x^*(r)) + q(0) - \underline{z} \\ &= -(1-\epsilon)\,\alpha\,(q(0) - q(x^*(R))) + (q(0) - \underline{z}) \\ &\leq -(1-\epsilon)\,\alpha\,(q(0) - \underline{z}) + (q(0) - \underline{z}) \\ &= (1 - (1-\epsilon)\alpha)(q(0) - \underline{z})), \end{aligned}$$

where

$$\alpha := \frac{q(0) - q(x^*(r))}{q(0) - q(x^*(R))}.$$

According to Theorem 2, we have $\alpha \geq (r/R)^2$. □

Theorem 2 shows that the larger is the ratio r/R, the closer is the approximate minimizer $x^*(r)$ to the global minimum of $q(x)$ over $\mathcal{F}$. This suggests that we should find two ellipsoids of the same orientation and center, one containing $\mathcal{F}$ with radius R while the other contained in $\mathcal{F}$ with radius r, so that the ratio r/R is maximized. In what follows, we shall use the *analytic center* based inscribing ellipsoid for $\mathcal{F}$ and show that when we increase the radius r to $R = mr$, then the inscribing ellipsoid becomes a containing ellipsoid of $\mathcal{F}$, where m is the number of quadratic inequalities defining $\mathcal{F}$. This result should be contrasted with the result of Lowner-John (Schrijver, 1986, p. 205) which says that for each full dimensional compact convex body in $\Re^n$ there exists a pair of inscribing and circumscribing ellipsoids which are concentric and are n-dilation of each other. Thus, when $m < n$, it is better to use the analytic center based ellipsoids.

We introduce some notations. Let $L(x)$ denote the following logarithmic barrier function for $\mathcal{F}$:

$$L(x) := -\sum_{i=1}^{m} \log g_i(x) = -\sum_{i=1}^{m} \log\left(d_i - c_i^T x - \frac{1}{2}x^T Q_i x\right).$$

It can be seen that $L(x)$ is strictly convex and approaches $+\infty$ near the boundary of $\mathcal{F}$. It is in fact *self-concordant* in the terminology of Nesterov and Nemirovskii (1993). The gradient and the Hessian of $L(x)$ are given by

$$\nabla L(x) = \sum_{i=1}^{m} \frac{c_i + Q_i x}{g_i(x)} \tag{3.3}$$

$$\nabla^2 L(x) = \sum_{i=1}^{m} \left(\frac{(c_i + Q_i x)(c_i + Q_i x)^T}{g_i^2(x)} + \frac{Q_i}{g_i(x)}\right). \tag{3.4}$$

For each $r \geq 0$ and each interior point z in $\mathcal{F}$, let us define an ellipsoid centered at z

$$E(z; r) := \{x \in \Re^n : (x - z)^T \nabla^2 L(z)(x - z) \leq r\}.$$

$E(z; 1)$ is the so called Dikin ellipsoid at z and is known that $E(z; 1) \subset \mathcal{F}$ for all z in the interior of $\mathcal{F}$ (see Nesterov and Nemirovskii, 1993). The analytic center of $\mathcal{F}$ is defined to be the unique minimizer $\bar{x} \in \mathcal{F}$ of the logarithmic barrier function $L(x)$. The Dikin ellipsoid at $\bar{x}$ is denoted by $E(\bar{x}; 1)$. Next we show that $\mathcal{F} \subset E(\bar{x}; m)$.

Theorem 3. *There holds*

$$E(\bar{x}; 1) \subset \mathcal{F} \subset E(\bar{x}; m).$$

Proof: The containment $E(\bar{x}; 1) \subset \mathcal{F}$ has been established in (Nesterov and Nemirovskii, 1993). It remains to prove the second containment. Fix any $x \in \mathcal{F}$. To simplify notations, we drop x and $\bar{x}$ in $g_i(x)$, $g_i(\bar{x})$, $\nabla L(\bar{x})$, etc., and denote them by g_i, $\bar{g}_i$, $\nabla \bar{L}$ respectively.

We also denote $\Delta_i = \frac{1}{2}(x-\bar{x})^T Q_i (x-\bar{x})$. Notice that

$$0 = \nabla \bar{L} = \sum_{i=1}^{m} \frac{c_i + Q_i \bar{x}}{\bar{g}_i}.$$

Multiplying both sides by $(x-\bar{x})$ yields

$$\sum_{i=1}^{m} \frac{(c_i + Q_i x)^T (x-\bar{x})}{\bar{g}_i} = 0.$$

By Taylor expansion, we have

$$\begin{aligned} -(c_i + Q_i \bar{x})^T (x-\bar{x}) &= \nabla \bar{g}_i^T (x-\bar{x}) \\ &= g_i - \bar{g}_i + \frac{1}{2}(x-\bar{x})^T Q_i (x-\bar{x}) \\ &= g_i - \bar{g}_i + \Delta_i. \end{aligned}$$

Substituting this into the previous relation we obtain

$$0 = \sum_{i=1}^{m} \frac{g_i - \bar{g}_i + \Delta_i}{\bar{g}_i},$$

which implies

$$m = \sum_{i=1}^{m} \frac{g_i + \Delta_i}{\bar{g}_i}. \tag{3.5}$$

Squaring both sides and noting $g_i \geq 0$ (since $x \in \mathcal{F}$) gives

$$\begin{aligned} m^2 &= \sum_{i=1}^{m} \left(\frac{g_i + \Delta_i}{\bar{g}_i}\right)^2 + 2\sum_{i \neq j} \left(\frac{g_i + \Delta_i}{\bar{g}_i}\right)\left(\frac{g_j + \Delta_j}{\bar{g}_j}\right) \\ &\geq \sum_{i=1}^{m} \left(\frac{g_i + \Delta_i}{\bar{g}_i}\right)^2. \end{aligned}$$

This further implies

$$\begin{aligned} \sum_{i=1}^{m} \left(\frac{g_i + \Delta_i}{\bar{g}_i} - 1\right)^2 &= \sum_{i=1}^{m} \left(\frac{g_i + \Delta_i}{\bar{g}_i}\right)^2 - 2\sum_{i=1}^{m} \left(\frac{g_i + \Delta_i}{\bar{g}_i}\right) + m \\ &\leq m^2 - 2\sum_{i=1}^{m} \left(\frac{g_i + \Delta_i}{\bar{g}_i}\right) + m \\ &= m^2 - 2m + m \\ &= m^2 - m, \end{aligned} \tag{3.6}$$

where the second equality follows from (3.5). Furthermore, the equality (3.5) implies

$$\sum_{i=1}^{m} \frac{\Delta_i}{\bar{g}_i} = m - \frac{g_i}{\bar{g}_i} \leq m, \tag{3.7}$$

where the last step is due to $g_i \geq 0$, for all i. Combining the relations (3.6) and (3.7) and noting (3.4) yields

$$\begin{aligned}(x-\bar{x})^T \nabla^2 L(\bar{x})(x-\bar{x}) &= \sum_{i=1}^{m} \left(\frac{((c_i + Q_i\bar{x})^T(x-\bar{x}))^2}{g_i^2(\bar{x})} + \frac{(x-\bar{x})^T Q_i (x-\bar{x})}{g_i(\bar{x})} \right) \\ &= \sum_{i=1}^{m} \left(\frac{g_i + \Delta_i}{\bar{g}_i} - 1 \right)^2 + \sum_{i=1}^{m} \frac{\Delta_i}{\bar{g}_i} \\ &\leq m^2 - m + m = m^2.\end{aligned}$$

This shows that $x \in E(\bar{x}; m)$ as desired. □

Theorem 3 shows that the Dikin ellipsoid at the analytic center $\bar{x}$ has the property that when its radius is enlarged m times it contains the feasible region $\mathcal{F}$. Thus, Theorem 2 and Corollary 1 imply that the minimizer of $q(x)$ over the Dikin ellipsoid $E(\bar{x}; 1)$ at the analytic center $\bar{x}$ is a $(1 - 1/m^2)$-minimizer of $q(x)$ over $\mathcal{F}$. The remaining issue is how to find the analytic center of $\mathcal{F}$.

The following is a column generation algorithm for computing an ϵ-approximate analytic center of $\mathcal{F}$. This algorithm is adopted from (Luo and Sun, 1995).

The column generation algorithm.

— **Step 1**
Ω^1 is defined by the quadratic inequality $g_1(x) \geq 0$ or equivalently $\|x\|^2 \leq 1$. Formally, let

$$\Omega^1 = \{x \in \Re^n : 1 - \|x\|^2 \geq 0\}$$

and let

$$x^1 = 0 \in \Re^n.$$

Then x^1 is the analytic center of Ω^1.

— **Step k ($2 \leq k \leq m$)**
Let x^{k-1} be an approximate analytic center of the set

$$\Omega^{k-1} := \{x \in \Re^n : g_i(x) \geq 0, \quad i = 1, \ldots, k-1, \}$$

in the sense that

$$(\nabla L_{k-1}(x^{k-1}))^T (\nabla^2 L_{k-1}(x^{k-1}))^{-1} \nabla L_{k-1}(x^{k-1}) \le 0.01$$

where

$$L_{k-1}(x) := -\sum_{i=1}^{k-1} \log\left(d_i - c_i^T x - \frac{1}{2} x^T Q_i x\right).$$

Let $x_0^k := x^{k-1}$ and check if

$$x_0^k \in \Omega^k := \Omega_{k-1} \cap \{x \in \Re^n : g_k(x) \ge 0\}$$

and if it satisfies

$$\left(\nabla L_k\left(x_0^k\right)\right)^T \left(\nabla^2 L_k\left(x_0^k\right)\right)^{-1} \nabla L_k\left(x_0^k\right) \le 0.01$$

where

$$L_k(x) := -\sum_{i=1}^{k} \log\left(d_i - c_i^T x - \frac{1}{2} x^T Q_i x\right).$$

If yes, x_0^k is an approximate analytic center of Ω_k; set $x^k = x_0^k$ and go to step $k+1$. Else do the following.

(Inner loop for step k)

- *Step A*: Let $j = 0$ and

  $$d_k^j := \max\left\{d_k, 16\sqrt{\left(\nabla g_k\left(x_j^k\right)\right)^T \nabla^2 L_{k-1}\left(x_j^k\right) \nabla g_k\left(x_j^k\right)} + d_k - g_k\left(x_j^k\right)\right\}.$$

  Define

  $$\Omega_j^k = \left\{ x \in \Re^n : \begin{array}{ll} d_i - c_i^T x - \frac{1}{2} x^T Q_i x \ge 0, & i = 1, \ldots, k-1, \\ d_k^j - c_k^T x - \frac{1}{2} x^T Q_k x \ge 0. & \end{array} \right\}. \tag{3.8}$$

- *Step B*: Perform the following damped Newton iterations (initialized at x_j^k)

  $$x^{\text{new}} := x - 0.6\left(\nabla^2 L_k^j(x)\right)^{-1} \nabla L_k^j(x) \tag{3.9}$$

  until

  $$\left(\nabla L_k^j(x^{\text{new}})\right)^T \left(\nabla^2 L_k^j(x^{\text{new}})\right)^{-1} \nabla L_k^j(x^{\text{new}}) \le 0.01$$

  where $L_k^j(x)$ denotes the logarithmic barrier function of Ω_j^k. Set

  $$x_j^k := x^{\text{new}}.$$

Check if $d_k^j = d_k$. If yes, then x_j^k is an approximate analytic center of Ω^k; set $x^k = x_j^k$, exit the inner loop and return to Step $k+1$.
Else, define

$$d_k^j := \max\left\{d_k, d_k^j - 0.01\left(d_k^j - c_k^T x_j^k - \frac{1}{2}\left(x_j^k\right)^T Q_k\left(x_j^k\right)\right)\right\}$$

and update Ω_j^k by (3.8); return to Step B.

— **Step** $m+1$
Let x^m be an approximate analytic center of the set

$$\Omega^m := \mathcal{F}$$

in the sense that

$$(\nabla L(x^m))^T(\nabla^2 L(x^m))^{-1}\nabla L(x^m) \le 0.01$$

where $L(x)$ is the logarithmic barrier function of $\mathcal{F}$. Perform the following damped Newton iterations (initialized at x^m)

$$x^{\text{new}} := x - 0.6(\nabla^2 L(x))^{-1}\nabla L(x) \tag{3.10}$$

until

$$(\nabla L(x^{\text{new}}))^T(\nabla^2 L(x^{\text{new}}))^{-1}\nabla L(x^{\text{new}}) \le ((2-\sqrt{2})\epsilon)^2. \tag{3.11}$$

Suppose $\mathcal{F}$ contains a δ-ball in its interior. Then, the analysis of (Luo and Sun, 1995) shows that the steps 1 up to m will take at most $O(m\log(1/\delta))$ Newton iterations (3.9), or $O(n^3 m\log(1/\delta))$ arithmetic operations. Also, due to the quadratic convergence of the Newton iterations (3.10), Step $m+1$ of the above Column Generation Algorithm takes at most $O(\log\log(1/\epsilon))$ Newton iterations, or $O(n^3\log\log(1/\epsilon))$ arithmetic operations. Thus, the above Column Generation Algorithm finds an ϵ-approximate analytic center $\bar{x}_\epsilon$ of $\mathcal{F}$ in a total of $O(n^3(m\log(1/\delta)+\log\log(1/\epsilon)))$ arithmetic operations, which is polynomial in m, n and $\log(1/\epsilon)$.

Once we have computed $\bar{x}_\epsilon$, we can construct the Dikin ellipsoid $E(\bar{x}_\epsilon; 1)$ and use the polynomial time algorithm described in Section 2 to minimize $q(x)$ over $E(\bar{x}_\epsilon; 1)$. The minimizer, say x_ϵ, of $q(x)$ over $E(\bar{x}_\epsilon; 1)$ can be shown to be a good approximate minimizer of $q(x)$ over $\mathcal{F}$. The proof is based on the fact that the Dikin ellipsoid $E(\bar{x}_\epsilon; 1)$, when scaled up $m(1+\epsilon)$ times, contains $\mathcal{F}$. This property is similar to the one described in Theorem 3 for the ellipsoid $E(\bar{x}; 1)$.

Theorem 4. *For all $\epsilon \in (0, 1-1/\sqrt{2}]$ and all $m \ge 1$, there exists a polynomial-time approximation algorithm for computing an $(1-\frac{1-\epsilon}{(m(1+\epsilon))^2})$-minimizer of $q(x)$ over $\mathcal{F}$.*

Proof: The approximation algorithm consists of two stages: first we compute an approximate analytic center $\bar{x}_\epsilon$ (in the sense (3.11)) using the Column Generation Algorithm described above; second we apply the bisection algorithm described in Section 2 to minimize $q(x)$ over the Dikin ellipsoid $E(\bar{x}_\epsilon; 1)$. We show below that this algorithm generates an $(1-(1-\epsilon)/(m(1+\epsilon))^2)$-minimizer of $q(x)$ over $\mathcal{F}$ in polynomial time.

Since $\bar{x}_\epsilon$ is an ϵ-approximate center of $\mathcal{F}$, we have from (3.11) that

$$(\nabla L(\bar{x}_\epsilon))^T(\nabla^2 L(\bar{x}_\epsilon))^{-1}\nabla L(\bar{x}_\epsilon) \le ((2-\sqrt{2})\epsilon)^2 < (\sqrt{2}-1)^4.$$

By a lemma of Nesterov (1995), it follows that

$$(\bar{x}-\bar{x}_\epsilon)^T(\nabla^2 L(\bar{x}_\epsilon))^{-1}(\bar{x}-\bar{x}_\epsilon) \le \epsilon^2.$$

This shows that $\bar{x} \in E(\bar{x}_\epsilon; \epsilon)$. Then, by the well known result on the self-similarity of Hessian matrix $\nabla^2 L(x)$ over the Dikin ellipsoids, we obtain

$$(1+\epsilon)^2(\nabla^2 L(\bar{x}_\epsilon))^{-1} \succeq (\nabla^2 L(\bar{x}))^{-1} \succeq (1-\epsilon)^2(\nabla^2 L(\bar{x}_\epsilon))^{-1}.$$

Thus, we have from Theorem 3 that

$$\mathcal{F} \subseteq E(\bar{x}; m) \subseteq (1+\epsilon)E(\bar{x}_\epsilon; m) = E(\bar{x}_\epsilon; m(1+\epsilon)).$$

Since $E(\bar{x}_\epsilon; 1) \subseteq \mathcal{F}$, it follows from Theorem 2 and Corollary 1 that the ϵ-minimizer of $q(x)$ over $E(\bar{x}_\epsilon; 1)$ with

$$q(x) - q(x^*(1)) \le \epsilon(q(\bar{x}_\epsilon) - q(x^*(1)))$$

is an $(1-(1-\epsilon)/(m(1+\epsilon))^2)$-minimizer of $q(x)$ over $\mathcal{F}$.

By the discussion following Column Generation Algorithm, we see the work required to compute $\bar{x}_\epsilon$ is $O(n^3(m\log(1/\delta)+\log\log 1/\epsilon))$. Furthermore, the work required to compute an ϵ-minimizer of $q(x)$ over $E(\bar{x}_\epsilon; 1)$ is equal to $O(n^3(\log(\mu^0/\mu^*)+\log(1/\epsilon)+\log n))$, where μ^0, μ^* are defined in (2.2) and (2.3). This shows that we can compute an $(1-(1-\epsilon)/(m(1+\epsilon))^2)$-minimizer of $q(x)$ over $\mathcal{F}$ in polynomial time. □

For $m = 2$, i.e., for the two-ellipsoid constrained QP problem, Theorem 4 implies that we can compute an $(1-(1-\epsilon)/(2(1+\epsilon))^2)$-minimizer in polynomial time. To the best of our knowledge, this is the first polynomial approximation algorithm for this problem.

4. The box constrained QP problem

In this section, we consider another special QP problem whereby

$$\mathcal{F} = \mathcal{C} := \{x \in \Re^n : \|x\|_\infty \le 1\}.$$

Clearly,

$$\mathcal{B}(1) \subset \mathcal{F} \subset \mathcal{B}(\sqrt{n}).$$

Thus, we have

$$q(x^*(1)) \geq \underline{z} \geq q(x^*(\sqrt{n})),$$

where the notation $x^*(r)$ for any $r > 0$ denotes a global minimizer of BQP(r).

Now, we can apply the algorithms in Section 2 to approximately solve BQP(1) and obtain an x with

$$q(x) - q(x^*(1)) \leq \epsilon(q(0) - q(x^*(1))).$$

According to Corollary 1, if we choose $\epsilon = 1/n$, then we can generate an $(1 - 1/n + 1/n^2)$-minimizer for the cubic-constrained QP problem in $O(n^3 \log(\log(\mu^0/\mu^*) + \log n))$ arithmetic operations.

5. The assignment-polytope constrained QP problem

In this section, we consider the assignment-polytope constrained QP whereby

$$\begin{aligned}\mathcal{F} &:= \left\{x \in \Re^{m^2} : \sum_{i=1}^{m} x_{i,j} = 1,\ \sum_{j=1}^{m} x_{i,j} = 1, \quad i, j = 1, \ldots, m,\ x_{i,j} \geq 0\right\} \\ &:= \left\{x \in \Re^{m^2} : x_{i,j} \geq 0\right\} \cap \mathcal{P},\end{aligned}$$

where $\mathcal{P}$ denotes the subset

$$\mathcal{P} := \left\{x \in \Re^{m^2} : \sum_{i=1}^{m} x_{i,j} = 1,\ \sum_{j=1}^{m} x_{i,j} = 1, \quad i, j = 1, \ldots, m\right\}.$$

Obviously, the vector $x = \frac{1}{m}e$, where $e \in \Re^{m^2}$ is the vector of all one's, is a feasible point in $\mathcal{F}$. Moreover, we have

$$\left\{x \in \mathcal{P} : \left\|x - \frac{1}{m}e\right\| \leq \frac{1}{m}\right\} \subset \mathcal{F} \subset \left\{x \in \mathcal{P} : \left\|x - \frac{1}{m}e\right\| \leq \sqrt{2(m-1)}.\right\} \tag{5.1}$$

This is because

$$\begin{aligned}\|x - (1/m)e\|^2 &= \|x\|^2 - 2(1/m)e^T x + (1/m)m^2 \\ &= \|x\|^2 - 2 + m \leq 2m - 2.\end{aligned}$$

Let $d = x - (1/m)e \in \Re^{m^2}$, then we can

$$\begin{aligned} \text{Minimize} \quad & q(d) := \frac{1}{2} d^T Q d + c^T d \\ \text{Subject to} \quad & \sum_{i=1}^{m} d_{i,j} = 0, \quad j = 1, \ldots, m, \\ & \sum_{j=1}^{m} d_{i,j} = 0, \quad i = 1, \ldots, m, \\ & \|d\| \le 1/m. \end{aligned}$$

Let N be an orthonormal basis spanning the null space of the equality constraints, where $N^T N = I$, and let $\hat{Q} = N^T Q N$ and $\hat{c} = N^T c$. Then $d = Ny$ for some y and the above problem can be rewritten as

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2} y^T \hat{Q} x + \hat{c}^T y \\ \text{Subject to} \quad & \|y\| \le 1/m. \end{aligned}$$

This is exactly the single ball-constrained problem discussed earlier in Section 2. Thus, if we choose $\epsilon = 1/m$, then we can generate a $\frac{1}{m}$-minimizer, say $\bar{x}$, of this ball-constrained problem in

$$O(m^6 \log(\log(\mu^0/\mu^*) + \log m))$$

arithmetic operations. By the containment relation (5.1) and using an argument similar to that in Section 5, we can deduce that $\bar{x}$ is an $(1 - 1/m^2(2m-2) + 1/m^3(2m-2))$-minimizer for the assignment polytope-constrained QP problem; the details are omitted.

6. The multiple-ellipsoid constrained QP as a semidefinite program

In this section we revisit the problem of (multiple) ellipsoid constrained QP problem. We present two results. First, we give an alternative formulation of the QP problem with a single ball constraint. This reformulation is in terms of a semidefinite program (SDP) which is similar to that of the recent work by Polijak et al. (1995), Rendl-Wolkowicz (1994) and Fujie-Kojima (1995). However, compared to the proof in this reference, our derivation is simpler and is based on a well-known result in the systems and control literature. Second, we propose a SDP relaxation for the QP problem with multiple ball constraints using a scheme called S-procedure (Yakubovich, 1977). This relaxation provides a polynomial time computable lower bound for the global minimum of a multiple ellipsoid constrained problem.

6.1. The single-ball constrained QP: A SDP formulation

Consider the QP problem with

$$\mathcal{F} = \{x \in \Re^n : (x-a)^T P (x-a) \le 1\} \tag{6.1}$$

where $P = P^T > 0$ is a symmetric and positive-definite matrix in $\Re^{n\times n}$, and $a \in \Re^n$. Denote by $\underline{z}$ the minimum of $q(x)$ subject to $\mathcal{F}$ in (6.1).

Given the QP problem above, let us construct the following SDP problem:

$$\begin{aligned} &\text{Minimize} \quad \rho \\ &\text{Subject to} \quad \begin{pmatrix} Q+\mu P & c-\mu Pa \\ c^T-\mu a^T P & \rho-\mu+\mu a^T Pa \end{pmatrix} \succeq 0 \\ &\qquad\qquad\quad \mu \geq 0 \end{aligned} \tag{6.2}$$

where μ and ρ are variables. Here, for any square matrices M and N the notation $M \succeq N$ means $M - N$ is positive semidefinite. Denote by $\underline{\rho}$ the minimum of ρ obtained above.

Then, we have the following result:

Theorem 5. *The minimum $\underline{z}$ for the QP problem with a single ball constraint $\mathcal{F}$ in* (6.1) *and the minimum $\underline{\rho}$ for the SDP problem in* (6.2) *are related by*

$$\underline{\rho} = -\underline{z} \tag{6.3}$$

Proof: The proof of the result hinges on the following lemma due to Yakubovich (1977) which is well-known in the systems and control area; see also (Boyd et al., 1994, Section 2.6.3).

Lemma 1. *Let $q_0(x)$ and $q_1(x)$ be quadratic functions of the variable $x \in \Re^n$:*

$$q_i(x) = x^T Q_i x + 2c_i^T x + d_i, \quad i = 0, 1 \tag{6.4}$$

where $Q_i = Q_i^T$. Suppose there exists $x_0 \in \Re^n$ such that $q_1(x_0) > 0$. Then, $q_0(x) \geq 0$ for all x satisfying $q_1(x) \geq 0$ if and only if there exists some $\mu \geq 0$ such that

$$q_0(x) - \mu q_1(x) \geq 0, \quad \forall\, x \in \Re^n \tag{6.5}$$

Furthermore, (6.5) *holds if and only if the following holds:*

$$\begin{pmatrix} Q_0 & c_0 \\ c_0^T & d_0 \end{pmatrix} - \mu \begin{pmatrix} Q_1 & c_1 \\ c_1^T & d_1 \end{pmatrix} \succeq 0. \tag{6.6}$$

Now let us return to Theorem 5. Given any ρ, we take

$$q_0(x) = q(x) + \rho, \quad q_1(x) = 1 - (x-a)^T P(x-a).$$

Obviously, $q_1(a) > 0$. Using Lemma 1 above, we know that $q(x) \geq -\rho$ for all x satisfying $(x-a)^T P(x-a) \leq 1$ if and only if there exists some $\mu \geq 0$ such that (6.2) holds. That is, $\underline{z} \geq -\rho$ if and only if there exists some $\mu \geq 0$ such that (6.2) holds. Hence, $\underline{z} \geq -\underline{\rho}$.

On the other hand, suppose $\underline{z} > -\underline{\rho}$. Then, there exists $\epsilon > 0$ such that $\underline{z} - \epsilon \geq -\underline{\rho}$. That is, $q_0(x) - \epsilon + \underline{\rho} \geq 0$ for all x satisfying $q_1(x) \geq 0$. Using Lemma 1 above again, there exists $\mu \geq 0$ such that

$$\begin{pmatrix} Q + \mu P & c - \mu P a \\ c^T - \mu a^T P & (\underline{\rho} - \epsilon) - \mu + \mu a^T P a \end{pmatrix} \succeq 0$$

which violates the definition of $\underline{\rho}$. Hence, we must have $\underline{z} = -\underline{\rho}$. □

When the ellipsoid (6.1) is centered at the origin, i.e., $a = 0$, Theorem 5 specializes to a result by Rendl and Wolkowicz (1994) which states that $\underline{z}$ is given by the minimum of the following SDP problem:

$$\begin{aligned} &\text{Maximize} \quad 2\lambda - t \\ &\text{Subject to} \quad \begin{pmatrix} t & c^T \\ c & Q \end{pmatrix} \succeq \lambda I. \end{aligned} \tag{6.7}$$

where λ and t are variables. This equivalence can be established by simply taking $\lambda = -\mu$ and $t = 2\lambda + \rho$ and permuting the rows of and columns of the constraint matrix in (6.7). This change of variables followed by the permutation of the rows and columns converts (6.7) into (6.2) with $a = 0$. Note that the constraint $\mu \geq 0$ in (6.2) is the same as $\lambda \leq 0$; the latter is implicit in (6.7) because $Q - \lambda I$ must be positive semidefinite and Q has a negative eigenvalue. For additional details of on the relation between the solution of a semidefinite relaxation and of a quadratic program, see Fujie and Kojima (1995).

We should point out that Theorem 5 only implies that the *exact* global minimum $\underline{z}$ of a ball-constrained QP can be obtained via the *exact* global minimum $\underline{\rho}$ of the SDP (6.2). Although the latter can be approximated using interior point methods, it is not clear how to convert an ϵ-minimizer of the SDP into an approximate minimizer of the original ball-constrained QP.

6.2. The multiple-ellipsoid constrained QP: A SDP relaxation

Consider the QP problem with multiple ellipsoid constraints described by

$$\mathcal{F} = \{x \in \Re^n : (x - a_i)^T P_i (x - a_i) \leq 1, \quad i = 1, \ldots, m\} \tag{6.8}$$

where $P_i = P_i^T > 0$ and $a_i \in \Re^n$, $i = 1, \ldots, m$. As always, we shall assume that $\mathcal{F}$ contains an interior point. For any $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$ with

$$\sum_{i=1}^{m} \tau_i = 1, \quad \tau_i \geq 0, \; i = 1, 2, \ldots, m,$$

we define

$$\mathcal{F}(\tau) = \left\{ x : \sum_{i=1}^{m} \tau_i (x - a_i)^T P_i (x - a_i) \leq 1 \right\}. \tag{6.9}$$

Obviously, we have

$$\mathcal{F} \subset \mathcal{F}(\tau) \tag{6.10}$$

and $\mathcal{F}(\tau)$ has non-empty interior. The relaxation method of using $\mathcal{F}(\tau)$ to replace $\mathcal{F}$ is called S-procedure and has been used popularly in the systems and control literature; see (Yakubovich, 1977). If we denote by $\underline{z}(\tau)$ the global minimum of $q(x)$ on $\mathcal{F}(\tau)$ and define

$$\underline{z}^* = \max \left\{ \underline{z}(\tau) : \sum_{i=1}^{m} \tau_i = 1,\ \tau_i \geq 0,\ i = 1, 2, \ldots, m \right\}, \tag{6.11}$$

it follows that $\underline{z}^*$ is a lower bound of $\underline{z}$, i.e.,

$$\underline{z}^* \leq \underline{z}. \tag{6.12}$$

Now we define a related SDP problem as follows:

$$\begin{aligned} &\text{Minimize} \quad \rho \\ &\text{Subject to} \quad \begin{pmatrix} Q + \sum_{i=1}^m \mu_i P_i & c - \sum_{i=1}^m \mu_i P_i a_i \\ c^T - \sum_{i=1}^m \mu_i a_i^T P_i & \rho - \sum_{i=1}^m \mu_i + \sum_{i=1}^m \mu_i a_i^T P_i a_i \end{pmatrix} \succeq 0 \\ &\qquad\qquad\quad \mu_i \geq 0, \quad i = 1, 2, \ldots, m \end{aligned} \tag{6.13}$$

with ρ and $\mu_i,\ i = 1, 2, \ldots, m$ as variables. Also denote by $\underline{\rho}$ the minimum ρ of the above SDP. We then have the following result:

Theorem 6. *Consider the QP problem with multiple constraints described by* (6.8). *The lower bound* $\underline{z}^*$ *as defined in* (6.11) *and the minimum* $\underline{\rho}$ *for the SDP problem in* (6.13) *are related by*

$$\underline{\rho} = -\underline{z}^*. \tag{6.14}$$

Furthermore, $\underline{\rho} = -\underline{z}$ *when* $m = 1$. *In other words, the SDP relaxation* (6.13) *is tight for* $m = 1$.

Proof: The proof is similar to that of Theorem 5. Given any ρ, we take

$$q_0(x) = q(x) + \rho, \quad q_1(x) = 1 - \sum_{i=1}^{m} \tau_i (x - a_i)^T P_i (x - a_i).$$

Note that the set $\{x : q_1(x) \geq 0\} = \mathcal{F}(\tau)$ has non-empty interior. Using Lemma 1, we know that $q(x) \geq -\rho$ for all x satisfying $q_1(x) \geq 0$ if and only if there exists some $\mu \geq 0$ such that

$$\begin{pmatrix} Q + \mu \sum_{i=1}^m \tau_i P_i & c - \mu \sum_{i=1}^m \tau_i P_i a_i \\ c^T - \mu \sum_{i=1}^m \tau_i a_i^T P_i & \rho - \mu + \mu \sum_{i=1}^m \tau_i a_i^T P_i a_i \end{pmatrix} \succeq 0 \tag{6.15}$$

holds. That is, $\underline{z}(\tau) \geq -\rho$ if and only if there exists some $\mu \geq 0$ such that (6.15) holds. Maximizing $\underline{z}(\tau)$ with respect to τ implies that $\underline{z} \geq -\rho$ if there exists some $\mu \geq 0$ such that (6.15) holds. Denote by ρ^1 the minimum ρ subject to (6.15) and

$$\mu \geq 0, \quad \tau_i \geq 0, \quad \sum_{i=1}^m \tau_i = 1. \tag{6.16}$$

We have $\underline{z}^* \geq -\rho^1$.

On the other hand, suppose $\underline{z} > -\rho^1$. Then, there exists $\epsilon > 0$ and τ satisfying (6.16) such that $\underline{z}(\tau) - \epsilon \geq \rho^1$. That is, $q_0(x) - \epsilon + \overline{\rho}^1 \geq 0$ for all x satisfying $q_1(x) \geq 0$. Using Lemma 1 again, there exists some $\mu \geq 0$ such that

$$\begin{pmatrix} Q + \mu \sum_{i=1}^m \tau_i P_i & c - \mu \sum_{i=1}^m \tau_i P_i a_i \\ c^T - \mu \sum_{i=1}^m \tau_i a_i^T P_i & \rho^1 - \epsilon - \mu + \mu \sum_{i=1}^m \tau_i a_i^T P_i a_i \end{pmatrix} \succeq 0$$

which violates the definition of ρ^1. Hence, $\underline{z}^* = -\rho^1$.

Finally, we substitute the variables $\mu\tau_i$ for μ_i, $i = 1, \ldots, m$. Then, the constraints (6.16) are equivalent to

$$\mu \geq 0, \quad \mu_i \geq 0, \quad \sum_{i=1}^m \mu_i = \mu. \tag{6.17}$$

It follows that (6.15) and (6.17) are the same as the constraints in (6.13) once we eliminate τ. Naturally, $\underline{\rho} = \rho^1$.

The last statement in the theorem is identical to Theorem 5. □

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