THE SECTOR BOUND APPROACH TO QUANTIZED FEEDBACK CONTROL

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Abstract—This paper studies a number of quantized feedback design problems for linear systems. We consider the case where quantizers are static (memoryless). The common aim of these design problems is to stabilize the given system or to achieve certain performance with the coarsest quantization density. Our main discovery is that the classical sector bound approach is non-conservative for studying these design problems. Consequently, we are able to convert many quantized feedback design problems to well-known robust control problems with sector bound uncertainties. In particular, we derive the coarsest quantization densities for stabilization for multi-input, multi-output systems in both state feedback and output feedback cases; and we have also derived conditions for quantized feedback control for quadratic cost and \(H_\infty\) performances.

I. INTRODUCTION

Control using quantized feedback has been an important research area for a long time. Even as early as in 1956, Kalman [1] studied the effect of quantization in a sampled data system and pointed out that if a stabilizing controller is quantized using a finite-alphabet quantizer, the feedback system would exhibit limit cycles and chaotic behavior. Most of the work on quantized feedback control concentrates on understanding and mitigation of quantization effects; see, e.g., [2], [3], [4].

A simple classical approach to analysis and mitigation of quantization effects is to treat the quantization error as uncertainty or nonlinearity and bound it using a sector bound. By doing so, robustness analysis tools, such as absolute stability theory (see [5], [6]), can be applied to study the quantization effect. Further, control parameters can be optimized to minimize the quantization effect. We will call this the sector bound method.

There is a new line of research on quantized feedback control where an quantizer is regarded as an information coder. The fundamental question of interest is how much information needs to be communicated by the quantizer in order to achieve a certain control objective. Noticeable works include [7], [8], [9], [10], [11]. In [11], the problem of quadratic stabilization of discrete-time single-input-single-output (SISO) linear time-invariant (LTI) systems using quantized feedback is studied. The quantizer is assumed to be static and time-invariant (i.e. memoryless and with fixed quantization levels). It is proved in [11] that for a quadratically stabilizable system, the quantizer is the so-called logarithmic (i.e., the quantization levels are linear in logarithmic scale). Further, the coarsest quantization density is given explicitly in terms of the system’s unstable poles. Note that the required quantizer has an infinite number of quantization levels because of its time-invariance nature. When a finite number of quantization levels are available, the so-called practical stability is obtained where there is a region of attraction in the state and the steady state converges to a small limit cycle.

When the quantizer is allowed to be dynamic and time-varying, it is obviously advantageous to scale the quantization levels dynamically so that the region of attraction is increased and the steady-state limit cycle is reduced. This is indeed the basic idea behind [9], [10]. In fact, it is shown in [10] that stabilization of a SISO LTI system (in some stochastic sense) can be achieved using only a finite number of quantization levels, and the minimum number of quantization levels is explicitly related to the unstable poles of the system. There is no doubt that the results in [11] and [10] are important because they give some sort of fundamental limitations of quantization.

The most pertinent work to this paper is [11]. In fact, this paper stems from the following motivations. First, the results in [11] (also those in [10]) are for SISO systems and for stabilization only. We want to know how to generalize their results to multi-input-multi-output (MIMO) systems and to control design for performances. Secondly, the technique used in [11], although being novel, does not seem to have a simple interpretation. This is perhaps why the generalization of their results appears to be difficult.

In this paper, we first review the key result in [11] which is on quadratic stabilization of SISO linear systems using quantized state feedback. We show that coarsest quantization density can be simply obtained using the sector bound method. This not only gives a simpler interpretation of the result, but also provides the basis for generalization of the result. Further, the coarsest quantization density is directly related to a \(H_\infty\) optimization problem, which is better than relating it to an “expensive” control problem as done in [11]. Secondly, we study the output feedback stabilization of SISO systems. Two cases are considered: observer-based quantized state feedback and dynamic feedback using quantized output. We show that the coarsest quantization density in the former case is the same as in quantized state feedback, whereas the latter case is related to a different \(H_\infty\) optimization problem and in general requires a finer quantization density. Thirdly, we generalize the quadratic stabilization problem to MIMO systems and show that quadratic stabilization with a set of logarithmic quantizers is the same as quadratic stabilization for an associated system with sector-bounded uncertainty. Because the latter problem has been well studied, the technical difficulty for the first problem is clearly revealed. A sufficient condition is then given, in terms of an \(H_\infty\) optimization problem, for the quantizers to render a quadratic stabilizer. As in the SISO case, both state feedback and output feedback are considered. Finally, we generalize the results to performance control problems. Both linear quadratic performance and \(H_\infty\) performance problems are studied and conditions are given for a set of quantizers to render a given performance level.
II. STABILIZATION USING QUANTIZED STATE FEEDBACK

In this section, we revisit the work of Elia and Mitter [11] on stabilization using quantized state feedback and reinterpret their result using the sector bound method.

The simplest and most fundamental case considered in [11] is the problem of quadratic stabilization for the following system:

\[ x(k + 1) = Ax(k) + Bu(k) \]  

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times 1} \), \( x \) is the state and \( u \) is a quantized state feedback in the following form:

\[ u(k) = f(v(k)) \]  

\[ v(k) = Kx(k) \]

In the above, \( K \in \mathbb{R}^{1 \times n} \) is the feedback gain, and \( f(\cdot) \) is a quantizer which is assumed to be symmetric, i.e., \( f(-v) = -f(v) \). Note that the quantizer is static and time-invariant.

The set of quantized levels is described by

\[ U = \{ \pm u_i, i = 0, \pm 1, \pm 2, \cdots \} \cup \{ 0 \} \]

Denote by \( \# g[\epsilon] \) the number of quantization levels in the interval \( [\epsilon, 1/\epsilon] \). The density of the quantizer \( f(\cdot) \) is defined as follows:

\[ \eta_f = \lim_{\epsilon \to 0} \sup \frac{\# g[\epsilon]}{-\ln \epsilon} \]

With this definition, the number of quantization levels of a quantizer with a nonzero, finite quantization density grows logarithmically as the interval \( [\epsilon, 1/\epsilon] \) increases. A small \( \eta_f \) corresponds to a coarse quantizer. A finite quantizer (i.e., a quantizer with a finite number of quantization levels) has \( \eta_f = 0 \), and a linear quantizer has \( \eta_f = \infty \).

A quantizer is called logarithmic if it has the form:

\[ U = \{ \pm u_i : v^{(i)} = \rho^i u^{(0)}, i = \pm 1, \pm 2, \cdots \} \cup \{ 0 \}, \quad 0 < \rho < 1, u^{(0)} > 0 \]

It is easily verified that \( \eta_f = 2/\ln(1/\rho) \) for the logarithmic quantizer. This means that the smaller the \( \rho \), the smaller the \( \eta_f \). For this reason, we will call \( \rho \) the quantization density as well. The logarithmic quantizer is illustrated in Figure 1. In contrast, a non-logarithmic quantizer is illustrated in Figure 2.

For the quadratic stabilization problem, a quadratic Lyapunov function \( V(x) = x^T P x, P = P^T > 0 \), is used to assess the stability of the feedback system. That is, the quantizer must satisfy

\[ \nabla V(x) = V(Ax + B f(Kx)) - V(x) < 0, \quad \forall x \neq 0 \]

\[ x(k + 1) = Ax(k) + B(1 + \Delta) v(k), \quad \Delta \in [\delta^-, \delta^+] \]

The coarsest quantizer is the one which minimizes \( \eta_f \) subject to (7). But the coarsest quantizer is in general not attainable because the constraint in (7) is a strict inequality.

The required density of the quantizer depends on \( V(x) \) (or \( P \)) and \( K \). This raises the key question: What is the coarsest density, \( \rho_{inf} \), among all possible \( P \) and \( K \) ? In [11], under the assumption that

\[ K = K_{GD} = -\frac{B^T P A}{B^T P B} \]

the answer for \( \rho_{inf} \) is given as

\[ \rho_{inf} = \prod_i \left| \lambda_i^* \right| - 1 \]

\[ \prod_i \left| \lambda_i^* \right| + 1 \]

where \( \lambda_i^* \) are the unstable eigenvalues of \( A \).

We see from Figures 1-2 that a quantizer can be bounded by a sector. For a logarithmic quantizer, the sector bound is described by a single parameter \( \delta \) which is related to the quantization density by

\[ \delta = \frac{1 - \rho}{1 + \rho} \]

whereas for a non-logarithmic quantizer, two parameters, \( \delta^- \) and \( \delta^+ \), are needed to describe the sector in general. For both finite quantizers and linear quantizers, a default output value, \( u_0 \), is needed when the input is smaller than some minimal threshold (in magnitude). If \( u_0 = 0 \), then \( \delta^- = -1 \); otherwise, \( \delta^+ = \infty \).

In the following, we use the sector bound method to show three results: 1) Given any quantizer, the quantized state feedback stabilization problem above is equivalent to a state feedback quadratic stabilization problem with an appropriately defined sector bound uncertainty; 2) The optimal quantizer structure is logarithmic; 3) For a logarithmic quantizer, the quadratic stabilization problem with the sector bound uncertainty has a simple explicit solution which leads to (9). It turns out that the result for \( \rho_{inf} \) remains the same even when \( K \) is without the constraint (8). These results are given below:

**Theorem II.1:** Consider the linear system in (1) and the quantized state feedback (2)-(3). Given a quantizer with a sector bound \( [\delta^-, \delta^+] \), the system (1) is quadratically stabilizable via quantized state feedback if and only if the following uncertain system:

\[ x(k + 1) = Ax(k) + B(1 + \Delta) v(k), \Delta \in [\delta^-, \delta^+] \]

is quadratically stabilizable via (3). If the quantizer is logarithmic with density \( \rho \), then the largest sector bound for (11) to be quadratically stabilizable is given by

\[ \delta_{sup} = \frac{1}{\prod_i |\lambda_i^*|} \]

Consequently, \( \rho_{inf} \) is given by (9). Finally, the logarithmic quantizer with \( \delta_{sup} \) (or \( \rho_{inf} \)) is the coarsest possible, among all quantizers, for quadratic stabilization of the system (1) via quantized state feedback.

**Remark II.1:** For any quantizer, there exists a sector bound for its quantization errors; see Figures 1 and 2. The last part of Theorem II.1 means that this sector bound must be within the largest sector bound, \( \delta_{sup} \), for the quantized feedback system to be quadratically stable. Then, within the maximum sector bound, it is not difficult to see from Figures 1 and 2 that the logarithmic quantizer with \( \rho_{inf} \) has the coarsest quantization density possible.

Two lemmas are needed for the proof of Theorem II.1.
Lemma II.1: Given a constant vector $K \in \mathbb{R}^{1 \times n}$, a constant matrix $\Omega_0 \in \mathbb{R}^{n \times n}$, a vector function $\Omega_1(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n \times 1}$, scalars $\delta^- \leq \delta^+$, and a scalar function $\Omega(\cdot) : \mathbb{R} \rightarrow [\delta^-, \delta^+]$ with the following property: For any $\Delta_0 \in [\delta^-, \delta^+]$, there exists $v_0 \neq 0$ such that $\Delta(v_0) = \Delta_0$. Define the following matrix function:

$$\Omega(\cdot) = \Omega_0 + \Omega_1(\cdot)K + K^T\Omega_1^T(\cdot)$$  \hspace{1cm} (13)

Then,

$$x^T\Omega(\Delta(Kx))x < 0, \forall x \neq 0, x \in \mathbb{R}^n$$  \hspace{1cm} (14)

and we can proceed as before. If $\Omega(\cdot)$ is strictly anti-stable, which implies $G_c(\cdot)$ is stable. Because $\Omega(\cdot)$ is antistable, $|a| < 1$ and $|z^na(\cdot)| < 1$. Hence, $G_c(\cdot)$ is strictly anti-stable, which implies $G_c(\cdot)$ is stable.

Proof. It is obvious that (15) implies (14). To see the converse, we assume (14) holds but (15) fails. Then, there exists some $x_0 \neq 0$ and $\Delta_0 \in [\delta^-, \delta^+]$ such that

$$x_0^T\Omega(\Delta_0)x_0 \geq 0$$  \hspace{1cm} (16)

We claim that $Kx_0 \neq 0$. Indeed, if $Kx_0 = 0$, then

$$x_0^T\Omega(\Delta(Kx_0))x_0 = x_0^T\Omega_0x_0 = x_0^T\Omega(\Delta_0)x_0 \geq 0$$  \hspace{1cm} (16)

by (15) and (16), which contradicts (14). So, $Kx_0 \neq 0$. Because of the property of $\Delta(\cdot)$, there exists a scalar $\alpha \neq 0$ such that $\Delta(\alpha Kx_0) = \Delta_0$. Define $x_1 = \alpha x_0 \neq 0$. Then,

$$x_1^T\Omega(\Delta_0)x_1 = \alpha^2x_0^T\Omega(\Delta_0)x_0 \geq 0$$

which violates (14). Hence, (14) implies (15).

Lemma II.2: Consider the uncertain system in (11). Define

$$G_c(z) = K(zI - A - BK)^{-1}B$$  \hspace{1cm} (18)

Then, the largest $\delta$ for which quadratic stabilization is achievable is given by

$$\delta_{\text{sup}} = \frac{1}{\inf K \|G_c(z)\|_{\infty}}$$  \hspace{1cm} (19)

Further, the solution to (19) is given by (12).

Proof. It is well-known [12] that the quadratic stabilization for (11) is achievable if and only if

$$\delta < \frac{1}{\inf K \|G_c(z)\|_{\infty}}$$

Hence, it remains to show that the solution to (19) is given by (12). To this end, we take $(A, B)$ to be a controllable canonical form, which yields

$$G_c(z) = \frac{k(z)}{a(z) - k(z)}$$  \hspace{1cm} (20)

where $a(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + z^n = |zI - A|$ and $k(z) = k_0 + k_1z + \cdots + k_{n-1}z^{n-1}$ is the control polynomial. The optimal control gain $K$ must be such that it yields a stable $G_c(z)$ which is either all-pass or arbitrarily close to it. If $a(z)$ is strictly anti-stable, then solution to $k(z)$ is given by

$$K = \frac{a_0^2}{a_0^2 - 1} \left[ a_0 - 1, a_1 - a_0, \ldots, a_{n-1} - a_0 \right]$$  \hspace{1cm} (21)

which gives $\|G_c(z)\|_{\infty} = |a_0| = \prod |\lambda_n^e|$. Indeed, (21) comes from solving the all-pass requirement for $G_c(z)$:

$$a(z) - k(z) = \alpha z^n k(z)$$  \hspace{1cm} (22)

Combining (22)-(23) yields

$$k(z) = \frac{a(z) - \alpha z^n k(z)}{1 - \alpha^2}$$  \hspace{1cm} (24)

Setting the $n$th order coefficient of $k(z)$ to zero results in $\alpha = 1/a_0$. It is straightforward to verify that (24) is the same as (21). Further, we claim $k(z)$ is strictly anti-stable. This is because (24) can be rewritten as

$$k(z) = \frac{a(z)}{1 - \alpha^2}(1 - \frac{z^n k(z)}{a(z)})$$

Because $a(z)$ is antistable, $|a| < 1$ and $|z^n k(z)| < 1$. Hence, $k(z)$ is strictly anti-stable, which implies $G_c(z)$ is stable.

If $a(z)$ has a stable factor, we can write $a(z) = a_s(z)a_u(z)$, where $a_s(z)$ and $a_u(z)$ are the stable and unstable factors. Then, we should have $k(z) = a_s(z)k_1(z)$, which yields

$$G_c(z) = \frac{k_1(z)}{a_s(z) - k_1(z)}$$

and we can proceed as before. If $a_u(z)$ is strictly anti-stable, we still have $\|G_c(z)\|_{\infty} = \prod |\lambda_n^e|$. If $a_u(z)$ has marginally stable roots, then $k_1(z)$ can be chosen so that $\|G_c(z)\|_{\infty}$ is arbitrarily close to $\prod |\lambda_n^e|$. Hence, we have verified (12).

Proof of Theorem II.1. Define the quantization error by

$$e = u - v = f(v) - v = \Delta(v)\cdot v$$  \hspace{1cm} (25)

Then,

$$\Delta(v) \in [\delta^-, \delta^+]$$  \hspace{1cm} (26)

We can model the quantized feedback system as the following uncertain system:

$$x(k + 1) = Ax(k) + B(1 + \Delta(Kx))Kx(k)$$  \hspace{1cm} (27)

and the corresponding quadratic stabilization condition becomes

$$\nabla V(x) = V((A + B(1 + \Delta(Kx))K)x) - V(x) < 0, \forall x \neq 0$$  \hspace{1cm} (28)

Define

$$\nabla P(\Delta) = (A + B(1 + \Delta)K)^TP(A + B(1 + \Delta)K) - P$$

for $\Delta \in [\delta^-, \delta^+]$.
where $\Delta$ is independent of the state. Note that the inverse mapping of $\Delta(v)$ in (25) is a multi-branch continuous function (except at $v = 0$). Hence, for any $\Delta_0 \in [\delta^{-}, \delta^{+}]$, there exists some $v_0 \neq 0$ such that $\Delta(v_0) = \Delta_0$. By Lemma II.1, (28) is equivalent to (29). But the latter is the condition for the system (11) to be quadratically stabilizable.

For a logarithmic quantizer where $-\delta^{-} = \delta^{+} = \delta$, the above means that the problem of coarsest quantization is equivalent to finding the maximum $\delta$ for (11) to be quadratically stabilizable. By Lemma II.2, the solution to the latter is given by (12). Hence, the solution to $\rho_{out}$ is given by (9).

Now consider a non-logarithmic quantizer. Define a scaled input $\hat{v} = \alpha v$, $\alpha \neq 0$, and interpret the quantizer $f(v)$ as $\hat{f}(\hat{v})$. This results in $\Delta(\hat{v}), \hat{\delta}^{-}$ and $\hat{\delta}^{+}$. If $\delta^{-} > -1$ and $\delta^{+} < \infty$, we can always find $\alpha$ such that $-\hat{\delta}^{-} = \hat{\delta}^{+}$. Such a scaling does not change the quadratic stabilizability. It is clear from the analysis above that the system (1) cannot be quadratically stabilized via quantized state feedback if $\hat{\delta}^{+} > \delta_{sup}$. If $\delta^{-} = -1$ or $\delta^{+} = \infty$, $\alpha$ can always be found to make $[-\delta_{sup}, \delta_{sup}] \subset [\hat{\delta}^{-}, \hat{\delta}^{+}]$. Again, the system (1) cannot be quadratically stabilized via quantized state feedback. Hence, we conclude that a logarithmic quantizer with $\rho_{sup}$ (or $\rho_{inf}$) is the coarsest possible.

**Remark II.2:** It is shown in [11] that the coarsest quantization density is related to the solution to the so-called “expensive” linear quadratic control problem:

$$\min_{K} \left\{ \sum_{k=0}^{\infty} |u(k)|^2 \right\}$$
subject to
$$\text{closed-loop stability with } u(k) = Kx(k)$$

More specifically, the optimal $\rho$ can be solved using the solution to the Riccati equation for the “expensive” control problem. However, the optimal control gain $K$ for the quantization problem is different from the optimal control gain for the “expensive” control problem (This is also pointed out in [11]). From the proofs above, we see that it is better to interpret the coarsest quantization problem as an $H_{\infty}$ optimization problem (19).

**Remark II.3:** We have seen that logarithmic quantizers are essential for quadratic stabilization via quantized feedback if a coarse quantization density is required. Non-logarithmic quantizers such as finite quantizers and linear quantizers are unsuitable. For this reason, we will consider logarithmic quantizers only in the rest of this paper.

### III. STABILIZATION USING QUANTIZED OUTPUT FEEDBACK

We now show how to generalize the technique for state feedback to quantized output feedback. Consider the following system:

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$

where $A$ and $B$ are the same as before and $C \in \mathbb{R}^{1\times n}$.

We consider two possible basic configurations for quantized output feedback which may lead to other more complicated settings.

**Configuration I:** The control signal is quantized but the measurement is not.

**Configuration II:** The measurement is quantized but the control signal is not.

It turns out that they result in different quantization density requirements.

**Configuration I:** This is an easy case which has an interesting result:

**Theorem III.1:** Consider the system (31) with quantized control input. Suppose $(A, C)$ is an observable pair. Then, the coarsest quantization density for quadratic stabilization by state feedback can also be achieved by output feedback. In particular, the corresponding output feedback controller can be chosen as an observer-based controller below:

$$x_c(k+1) = Ax_c(k) + Bu(k) + L(y(k) - Cx_c(k))$$
$$v(k) = Kx_c(k)$$
$$u(k) = f(v(k))$$

where $f(\cdot)$ is the quantizer as before, $K$ is the state feedback gain designed for any achievable quantization density via quantized state feedback, and $L$ is any gain which yields (32) a deadbeat observer.

**Proof.** Let $K$ be any state feedback gain that achieves any given quantization density. Choose $L$ such that the observer is deadbeat, i.e., $e(k) = x(k) - x_c(k) \neq 0$ only for a finite number of steps $N$. This can be always done because $(A, C)$ is observable. Then, after $N$ steps, the output feedback controller is the same as state feedback controller. Hence, the system is quadratically stabilized after $N$ steps. Finally, it is a simple fact (although we do not give the details) that if a (nonlinear) system is quadratically stable after $N$ steps and that the state is bounded in the first $N$ steps (which clearly holds for the system (32)), it is quadratically stable.

**Configuration II:** In this case, the controller is in the form

$$x_c(k+1) = A_cx_c(k) + B_cf(y(k))$$
$$u(k) = C_cx_c(k) + D_cf(y(k))$$

where $f(\cdot)$ is the quantizer as before.

It is straightforward to verify that the closed-loop system is given by

$$\bar{x}(k+1) = \mathcal{A}(\Delta y(k))\bar{x}(k)$$

where $\bar{x} = [x^T \bar{x}_c^T]^T$, $\Delta(\cdot)$ is the same as in (26) and

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}, \bar{C} = \begin{bmatrix} 0 & I \\ C_c & 0 \end{bmatrix}, \bar{K} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}$$

and

$$\mathcal{A}(\Delta) = \bar{A} + \bar{B}\bar{K}(\bar{C} + \bar{I}\bar{C})$$

The problem of concern is to find the coarsest quantizer for quadratic stabilization of the closed-loop system. This can be solved by generalizing the idea for the state feedback case. The result is given below.
**Theorem III.2:** Consider the system (31). For a given quantization density \( \rho > 0 \), the system is quadratically stabilizable via a quantized controller (33) if and only if the following auxiliary system:

\[
\begin{align*}
x(k+1) &= A x(k) + B u(k) \\
v(k) &= (1 + \Delta) C x(k), \quad |\Delta| \leq \delta 
\end{align*}
\]

is quadratically stabilizable via:

\[
\begin{align*}
x_c(k+1) &= A_c x_c(k) + B_c v(k) \\
u(k) &= C_c x_c(k) + D_c v(k)
\end{align*}
\]

where \( \delta \) and \( \rho \) are related by (10).

The largest sector bound \( \delta_{\text{sup}} \) (which gives \( \rho_{\text{inf}} \)) is given by

\[
\delta_{\text{sup}} = \frac{1}{\inf_{\bar{K}} ||G_c(z)||_\infty}
\]

where \( \bar{K} \) is defined in (35) and

\[
\bar{G}_c(z) = (1 - H(z)G(z))^{-1}H(z)G(z)
\]

where \( G(z) = C(zI - A)^{-1}B \) and \( H(z) = D_c + C_c(zI - A_c)^{-1}B_c \).

Further, if \( G(z) \) has relative degree equal to 1 and no unstable zeros, then the coarsest quantization density for quantized state feedback can be reached via quantized output feedback.

**Proof.** The proof is similar to the proof of Theorem II.1. The sector bound for the quantization error is done as in (25)-(26). For the given \( \rho \), the quadratic stability of the closed-loop system (31)-(33) requires the existence of some \( \bar{P} = P^T > 0 \) such that

\[
\bar{x}^T A(\Delta y)^T \bar{P} A(\Delta y) - \bar{P} \bar{x} < 0
\]

for all \( \bar{x} \neq 0 \) and \( y = Cx = \bar{C} \bar{x} \). Using Lemma II.1, the above is equivalent to

\[
A(\Delta)^T \bar{P} A(\Delta) - \bar{P} < 0, \quad \forall |\Delta| \leq \delta
\]

The latter is the same as requiring the system (37)-(38) to be quadratically stable. The solution to \( \delta_{\text{sup}} \) comes from the equivalence between quadratic stability and \( H_\infty \) optimization [12].

Suppose \( G(z) \) has relative degree 1 and no unstable zeros. Write \( G(z) = b(z)/a(z) \). From the proof of Theorem II.1, we know that the state feedback case corresponds to \( H_\infty \) optimization of \( G_c(z) \) in (20). If we choose \( H(z) = k(z)/b(z) \). Then, \( G_c(z) \) in (40) becomes \( G_c(z) \). Hence, the quantization density for the quantized state feedback can be achieved by quantized output feedback. \( \square \)

Now we give an example to show that using quantized output requires a higher quantization density than using quantized state feedback.

**Example III.1:** The system is given by (31) with \( G(z) = C(zI - A)^{-1}B = (z - 3)/z(z - 2) \). Using quantized state feedback, \( \delta = 2 \) and \( \rho = (2 - 1)/(2 + 1) = 0.3333 \). For quantized output feedback, computing (39) yields \( \delta = 10 \) and \( \rho = (10 - 1)/(10 + 1) = 0.8182 \).

**Remark III.1:** In [11], output feedback control design is done in two steps. In Step 1, coarsest quantization is solved for state estimation, which is a dual problem to the state feedback stabilization problem. In Step 2, the separation principle is applied, i.e., optimal state feedback is combined with optimal state estimation. The main result is that logarithmic quantization is sufficient for output feedback stabilization.

The drawback of the approach in [11] is that the physical meaning of the state estimation quantizer is not clear. Indeed, the problem of quantized state estimation is formulated to be:

\[
e(k + 1) = A e(k) + f_e(C e(k))
\]

where \( e(k) = x(k) - x_c(k) \) is the state estimation error and \( f_e(\cdot) \) is the state estimation quantizer. What is unsatisfactory in this formulation is that the quantizer needs to know both \( y(k) \) and its estimate \( C x_c(k) \). If the control signal is generated elsewhere using a quantized \( y(k) \), it is difficult to imagine why its estimate needs to be sent back to the measurement end to form \( C x_c(k) \) for quantization. Hence, the validity of this formulation seems to be questionable.

**IV. STABILIZATION OF MIMO SYSTEMS USING QUANTIZED FEEDBACK**

In this section, we generalize the quantization results in Section 2 to MIMO systems. We also consider Configurations I and II.

**Configuration I.** The system is still as in (31) (or (1) for state feedback) except that we now allow \( u \in \mathbb{R}^m, v \in \mathbb{R}^l \). Suppose quantized state feedback (2)-(3) is used, where \( K \in \mathbb{R}^{m \times n} \) and

\[
f(v) = \text{diag}\{f_1(v_1), f_2(v_2), \ldots, f_m(v_m)\}
\]

where \( v_j \) is the \( j \)th component of \( v \) and \( f_j(\cdot) \) is a quantizer of the form (6) but with quantization level \( 0 < \rho_j < 1 \).

Because we have more than one quantizer, the notion of coarsest quantization is not well-defined. Instead, we ask the following question: Given a vector of quantization levels \( \rho = [\rho_1 \rho_2 \cdots \rho_m] \), does there exist an quantized feedback controller that quadratically stabilizes the system (31)? The main result is given below:

**Theorem IV.1:** Given the system in (31) and a quantization level vector \( \rho \), consider the following auxiliary system:

\[
x(k + 1) = A x(k) + B(1 + \Delta(k)) v(k)
\]

where \( |\Delta_j(k)| \leq \delta_j \) for all \( j = 1, 2, \ldots, m \) and \( k \), and \( \delta_j \) are converted from \( \rho_j \) using (10), and \( v(k) \) is a control input. Suppose the auxiliary system is quadratically stabilizable via state feedback (3), then (31) is quadratically stabilizable via quantized state feedback. Conversely, suppose the system (31)
is quadratically stabilizable via quantized state feedback and, in addition, suppose $\ln \rho_i / \ln \rho_j$ are irrational numbers for all $i \neq j$ when $m > 1$. Then, for any (arbitrarily small) $\epsilon > 0$, the auxiliary system (44) with $|\Delta_j(k)| \leq \delta_j - \epsilon$ is quadratically stabilizable via state feedback (3).

Further, the auxiliary system is quadratically stabilizable via state feedback (3) if the following state feedback $H_\infty$ control has a solution $K$ for some diagonal scaling matrix $\Gamma > 0$:

$$\|\Lambda \Gamma K(zI - A - BK)^{-1}B\Gamma^{-1}\|_\infty < 1 \quad (45)$$

where

$$\Lambda = \text{diag}\{\delta_1, \cdots, \delta_m\} \quad (46)$$

In particular, any $K$ that renders (45) a solution to either quadratic stabilizability problem.

Finally, if $(A, C)$ is an observable pair and (31) is quadratically stabilizable via quantized state feedback for the given $\rho$, then it is also quadratically stabilizable via observer-based quantized state feedback (32) for the same $\rho$.

**Remark IV.1:** It is easy to see that if a given $\rho$ does not satisfy the condition that $\ln \rho_i / \ln \rho_j$ are irrational for $i \neq j$, we can make it so by perturbing the $\rho_j$ slightly. That is, the condition on $\rho$ holds generically.

Three technical lemmas are required for the proof of the result above.

**Lemma IV.1:** For the quantizer (6) and any $|\Delta| \leq \delta$, the inverse function $\Delta^{-1}(v)$ is not unique, and is given by

$$\ln v/\bar{u}(0) = i \ln \rho - \ln(\Delta + 1), \quad i = 0, \pm 1, \pm 2, \cdots \quad (47)$$

**Proof.** The results follow directly from the definition of $\Delta(v)$ in (26). \hfill \Box

**Lemma IV.2:** Let $f_j(\cdot), j = 1, 2, \cdots, m$ be a set of quantizers as in (6) but with (possibly different) values $\bar{u}(0)$ and $0 < \rho_j < 1$. Suppose the ratios $\ln \rho_i / \ln \rho_j$ are irrational numbers for all $1 \leq i, j \leq m$, $i \neq j$ (This condition is void if $m = 1$). Then, given any pairs of vectors $v_j, \Delta_j^0$ with $v_j \neq 0$ and $|\Delta_j^0| \leq \delta_j, j = 1, 2, \cdots, m$, and any scalar $\epsilon > 0$ (arbitrarily small), there exists a scalar $\alpha > 0$ such that

$$|\Delta_j(\alpha v_j) - \Delta_j^0| < \epsilon, \quad j = 1, 2, \cdots, m \quad (48)$$

where $\Delta_j(\cdot)$ is as defined in (25)-(26). That is, as $\alpha$ varies from 0 to $\infty$, the vector $[\Delta_1(\alpha v_1) \cdots \Delta_m(\alpha v_m)]^T$ covers the hyperrectangle $[-\delta_1, \delta_1] \oplus \cdots \oplus [-\delta_m, \delta_m]$ densely.

**Proof.** Note that each $\Delta_j(\cdot)$ is periodic in $\ln(v/\bar{u}(0))$ with the period $\ln \rho_j$ and that within each period the mapping between $\ln(v/\bar{u}(0))$ and $\Delta_j(\cdot)$ is one-to-one. Therefore, it suffices to show that as $\alpha$ varies, $\ln(\alpha v_j / \bar{u}(0)) / \ln \rho_j \mod [\ln \alpha / \ln \rho_1, \cdots, \ln \alpha / \ln \rho_m]^T$ covers $B = [0, \ln \rho_1] \oplus \cdots \oplus [0, \ln \rho_m]$ densely. This is equivalent to that $\gamma = [\mod(\ln \alpha, \ln \rho_1) \cdots \mod(\ln \alpha, \ln \rho_m)]^T$ covers $B$ densely.

Let $\beta = [\beta_1, \cdots, \beta_m]^T \in B$ be any given vector. We need to find $\alpha$ such that $\gamma$ is arbitrarily close to $\beta$. The assumption that $\ln \rho_i / \ln \rho_j$ are irrational implies that quantizers $f_i(\cdot)$ and $f_j(\cdot), i \neq j$, do not share a common period (in the logarithmic scale), which is the key to the analysis below. If $m = 1$, we can simply take

$$\ln \alpha = \beta_1 + \ln \rho_1 \quad (49)$$
as a solution with any integer $i_1$. If $m = 2$, we keep $\ln \alpha$ as in (49) but let $i_1$ vary. Because $f_1(\cdot)$ and $f_2(\cdot)$ do not share a common period, as the integer $i_1$ varies from $-\infty$ to $\infty$, $\mod(\ln \alpha, \ln \rho_2)$ will cover the set $[0, \ln \rho_2]$ densely. Let $I_1$ and $I_2$ be the infinite sequences of $i_1$ and the corresponding $i_2$, respectively, which make the corresponding set of $\mod(\ln \alpha, \ln \rho_2)$ sufficiently close to $\beta_2$. For $m = 3$, because $f_1(\cdot), f_2(\cdot)$ and $f_3(\cdot)$ do not share a common period pair-wise, there is an infinite sequence $I_1$ for $i_1$ (a subsequence of $I_1$) which generates the corresponding infinite sequence $I_2$ for $i_2$ (a subsequence of $I_2$) and infinite sequence $I_3$ for $i_3$ such that $\mod(\ln \alpha, \ln \rho_3)$ is sufficiently close to $\beta_3$. This process can continue for $m > 3$. Hence, we have proved the needed result. \hfill \Box

**Lemma IV.3:** Let $f_j(\cdot), j = 1, \cdots, m$, be a set of quantizers satisfying the conditions in Lemma IV.2. Given constant matrices $K \in \mathbb{R}^{m \times n}$ and $\Omega = \Omega^T \in \mathbb{R}^{n \times n}$, and a matrix function $\Omega(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times m}$, define

$$\Omega(\cdot) = \Omega_0 + \Omega_1(\cdot)K + K^T \Omega^T_1(\cdot) \quad (50)$$

Suppose $\Omega(\cdot)$ is strictly convex. Then,

$$x^T \Omega(\Delta(Kx))x < 0, \forall x \neq 0, x \in \mathbb{R}^n \quad (51)$$

if

$$\Omega(\Delta) < 0, \forall |\Delta_j| \leq \delta_j, j = 1, \cdots, m \quad (52)$$

Conversely, (51) implies

$$\Omega(\Delta) < 0, \forall |\Delta_j| \leq \delta_j - \epsilon, j = 1, \cdots, m \quad (53)$$

for any $\epsilon > 0$.

**Proof.** It is obvious that (52) implies (51). To see the converse, we assume (51) holds but (52) fails. Then, there exists some $x_0 \neq 0$ and $\Delta^0 = \text{diag}\{\Delta^0_1, \cdots, \Delta^0_m\}$ with $|\Delta^0_j| \leq \delta_j, j = 1, \cdots, m$, such that

$$x_0^T \Omega(\Delta^0)x_0 \geq 0 \quad (54)$$

If such $\Delta^0$ is only a boundary point, i.e., $|\Delta^0_i| = \delta_i$ for some $i$, then, (53) holds for any $\epsilon > 0$. In the sequel, we assume that $\Delta^0$ is an interior point.

We claim that $Kx_0 \neq 0$. Indeed, if $Kx_0 = 0$, then

$$x_0^T \Omega(\Delta(Kx_0))x_0 = x_0^T \Omega_0 x_0 = x_0^T \Omega(\Delta^0)x_0 \geq 0 \quad (55)$$

by (50) and (54), which contradicts (51). So, $Kx_0 \neq 0$.

Because of the strict convexity of $\Omega(\cdot)$, there exists $\Delta^1$ with $|\Delta^1_j| \leq \delta_j - \epsilon_1, j = 1, \cdots, m$, for some small $\epsilon_1 > 0$ such that

$$x_0^T \Omega(\Delta^1)x_0 > 0 \quad (56)$$
Because the above is continuous in $x_0$, we may perturb $x_0$ slightly such that (56) still holds and every element of $Kx_0$ is nonzero.

Now using Lemma IV.2, we know that $\Delta(\alpha Kx_0)$ covers $[-\delta_1, \delta_1] \oplus \cdots \oplus [-\delta_m, \delta_m]$ densely as $\alpha$ varies from $-\infty$ to $\infty$. Hence, there exists $\alpha \neq 0$ such that

$$x_0^T \Omega(\Delta(\alpha Kx_0))x_0 > 0$$

Define $x_1 = \alpha x_0$, we get

$$x_1^T \Omega(\Delta(Kx_1))x_1 > 0$$

which contradicts (51). That is, $\Delta^0$ cannot be an interior point. Hence, (51) implies (53).

**Proof of Theorem IV.1.** The “equivalence” between the quantized feedback problem and the quadratic stabilization problem for the auxiliary system (44) follows from Lemma IV.3. The $H_\infty$ condition for the latter comes from [13]. The result on observer-based feedback is identical to Theorem III.1.

**Configuration II.** When quantized measurements are available, we have the following result:

**Theorem IV.2:** Given the system in (31) and a quantization level vector $\rho$, consider the following auxiliary system:

$$
\begin{align*}
  x(k+1) &= Ax(k) + Bu(k) \\
  y(k) &= Cx(k) \\
  v(k) &= (I + \Delta(k))y(k)
\end{align*}
$$

(57)

where $|\Delta_j(k)| \leq \delta_j$ for all $j = 1, 2, \ldots, m$ and $k$, and $\delta_j$ are converted from $\rho_j$ using (10), and $v(k)$ is the output available for feedback. Suppose the auxiliary system is quadratically stabilizable, then (31) is quadratically stabilizable via (33). Conversely, suppose the system (31) is quadratically stabilizable via (33) and, in addition, suppose $\ln \rho_i / \ln \rho_j$ are irrational numbers for all $i \neq j$ when $m > 1$. Then, for any (arbitrarily small) $\epsilon > 0$, the auxiliary system (57) with $|\Delta_j(k)| \leq \delta_j - \epsilon$ is quadratically stabilizable.

Further, the auxiliary system is quadratically stabilizable if the following state feedback $H_\infty$ control has a solution $H(z)$ for some diagonal scaling matrix $\Gamma > 0$:

$$
\|\Gamma(\bar{I} - G(z)H(z))^{-1}G(z)H(z)\Gamma^{-1}\| < 1
$$

(58)

where $\Lambda$ is given in (46). In particular, any $H(z)$ that renders (45) is a solution to either quadratic stabilization problem.

**Proof.** The “equivalence” between the quantized feedback problem and the quadratic stabilization problem for the auxiliary system (57) follows from Lemma IV.3. The proof for the relation to $H_\infty$ optimization is similar to the proof of Theorem III.2.

**V. QUANTIZED QUADRATIC PERFORMANCE CONTROL**

The purpose of this section is to extend the results in the previous sections to include a quadratic performance objective. Consider the system in (31). Suppose the output $y(k)$ needs to be quantized. We now want to design a controller in (33) such that the following performance cost function

$$
J(x(0)) = \sum_{k=0}^\infty x^T(k)Q x(k) + u^T(k)Ru(k), \\
Q = Q^T \geq 0, R = R^T > 0
$$

(59)

is minimized in the sense below:

$$
\min EJ(x(0))
$$

(60)

In the above, $x(0)$ is assumed to be a white noise with covariance $Ex(0)x^T(0) = \sigma^2 I$ for some $\sigma > 0$.

Because the state of the closed-loop system is $\dot{x}(k)$, we may rewrite the performance cost as

$$
J(\dot{x}(0)) = \sum_{k=0}^\infty \dot{x}^T(k)Q \dot{x}(k) + u^T(k)Ru(k)
$$

(61)

where

$$
\dot{x}(0) = \begin{bmatrix} x(0) \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}
$$

(62)

Suppose we want the closed-loop system to be quadratically stable. Let $V(\dot{x}) = \dot{x}^T(P\dot{x})$, $P = \dot{P}^T > 0$, be the associated Lyapunov function. Define

$$
\nabla V(\dot{x}(k)) = V(\dot{x}(k+1)) - V(\dot{x}(k))
$$

(63)

Then, using (34), the performance cost is given by

$$
J(\dot{x}(0)) = \dot{x}^T(0)\dot{P}\dot{x}(0)
$$

$$
+ \sum_{k=0}^\infty \nabla V(\dot{x}(k)) + \dot{x}^T(k)Q \dot{x}(k) + u^T(k)Ru(k)
$$

$$
= \dot{x}^T(0)\dot{P}\dot{x}(0) + \sum_{k=0}^\infty \dot{x}^T(k)\dot{\Omega}(\Delta(y(k)))\dot{x}(k)
$$

(64)

where

$$
\dot{\Omega}(\Delta) = A(\Delta)^T P A(\Delta) - \bar{P} + \bar{Q} + (C + \bar{\Delta})\bar{C} + \bar{Q} + \bar{C}\bar{C}^T \bar{K}^T \bar{I} \bar{R}
$$

$$
\times \bar{P} \bar{K}(C + \bar{\Delta})\bar{C}
$$

(65)

For the case without quantization, i.e. $\Delta(\cdot) = 0$, it is well-known (and easy to see from above) that the optimal solution for $K$ is such that $\dot{x}^T(k)\dot{\Omega}(0)\dot{x}(k) = 0$ for all $k$, which leads to $J(\dot{x}(0)) = \dot{x}^T(0)\dot{P}\dot{x}(0)$ and minimization of $\text{tr}{\dot{P}}$. In the presence of the quantizer, we can formulate the performance control problem as follows: Given a performance bound $\gamma > 0$ and $\rho > 0$, find $\dot{P}$, $\bar{K}$, if exist, such that

$$
\text{tr}{\dot{P}} < \gamma
$$

(66)

subject to

$$
\dot{x}^T x(0)\dot{\Omega}(\Delta(\bar{\Delta}))\dot{x} < 0, \forall \bar{x} \neq 0
$$

(67)
This problem will be called Quantized Quadratic Performance Control (QQPC) problem. The solution to this problem is related to the so-called guaranteed-cost control (GCC) problem for the auxiliary system (31) and (57), i.e., we want to find $P, K$ such that (66) holds subject to
\[ \Omega(\Delta) < 0, \ \forall |\Delta_j| \leq \delta_j \] (68)
where $\delta_j$ and $\rho_j$ are related by (10).

**Theorem VI.1.** Consider the system in (31), the performance cost in (59), the controller structure in (33), some performance bound $\gamma > 0$ and quantization level vector $0 < \rho < 1$. Suppose the GCC problem has a solution. Then, there exists a solution to the QQPC problem. Conversely, if the QQPC problem has a solution and in addition (when $m > 1$), $\ln \rho_i / \ln \rho_j$ are irrational numbers for all $i \neq j$, then, given any (arbitrarily small $\epsilon > 0$), the GCC problem for (68) has a solution for $|\Delta_j(k)| \leq \delta_j - \epsilon$.

**Proof.** The proof is similar to that of Theorem IV.1. The key is to show the relationship between (67) and (68). Obviously, (68) implies (67). The fact that (67) implies (68) but with $|\Delta_j| \leq \delta_j - \epsilon$ is proved using Lemma IV.3. The details are omitted here. \( \square \)

When quantized state feedback is used instead, we have the following result:

**Theorem VI.2.** Consider the system (1) with $B \in \mathbb{R}^{n \times m}$ and quantized state feedback as in (2)-(3), where $f(\cdot) = [f_1(\cdot), \ldots, f_m(\cdot)]^T$ with quantization levels $0 < \rho_1, \ldots, \rho_m < 1$. Given the performance cost function in (59) and a performance bound $\gamma > 0$, the QQPC problem becomes to finding $P = P^T > 0$ and $K$, if exist, such that
\[ \text{tr} P < \gamma \] (69)
subject to
\[ x^T \Omega(\Delta(v)) x < 0, \ \forall x \neq 0 \] (70)
where $v = K x$ and
\[ \Omega(\Delta) = (A + B(I + \Delta)K)^T P (A + B(I + \Delta)K) - P + Q + K^T(I + \Delta)R(I + \Delta)K \] (71)
The related GCC problem becomes to finding $P = P^T > 0$ and $K$, if exist, such that (69) holds subject to
\[ \Omega(\Delta) < 0, \ \forall |\Delta_j| \leq \delta_j \] (72)
Further, the GCC problem has a solution if the following linear matrix inequalities
\[ \text{tr} \bar{P} < \gamma, \left[ \begin{array}{cc} -\bar{P} & I \\ I & -S \end{array} \right] \leq 0 \] (73)
\[ \left[ \begin{array}{cccc} -S & * & * & * \\ AS + BW & -S + BA \bar{G} \bar{A}^T & * & * \\ W & \Lambda \Gamma \bar{A}^T & -\bar{R} & * \\ Q^{1/2} S & 0 & 0 & -I \end{array} \right] < 0 \] (74)
have a solution for some $\bar{P} = \bar{P}^T$, $S = S^T$, $W$ and a diagonal scaling matrix $\Gamma$, where $\bar{R} = R^{-1} - \Lambda \Sigma \Lambda$, $\Lambda$ is given in (46), and $*$ denotes the symmetric part in the matrix. Also, $P$ and $K$ are related to $S$ and $W$ as follows:
\[ P = S^{-1}, \quad K = WP \] (75)

**Proof.** The simplification of the QQPC and GCC problems is easy to check. We proceed to verify (74) as a sufficient condition for the GCC problem. Indeed, (72) holds if and only if
\[ \left[ \begin{array}{ccc} -P + Q & * & * \\ A + B(I + \Delta)K & -P^{-1} & * \\ (I + \Delta)K & 0 & -R^{-1} \end{array} \right] < 0 \] (76)
for all $|\Delta_j| \leq \delta_j$. Using (75), the above becomes
\[ \left[ \begin{array}{ccc} -S + QS & * & * \\ AS + BW & -S & * \\ (I + \Delta)W & 0 & -R^{-1} \end{array} \right] < 0 \] (77)
which is equivalent to
\[ \left[ \begin{array}{cc} -S + QS & * \\ AS + BW & -S \\ W & 0 \end{array} \right] + \left[ \begin{array}{c} 0 \\ B \\ I \end{array} \right] \Delta [W 0 0] \] (78)
Taking $\Gamma > 0$ to be any diagonal scaling matrix, (78) holds if
\[ \left[ \begin{array}{cc} -S + QS & * \\ AS + BW & -S \\ W & 0 \end{array} \right] + \left[ \begin{array}{c} 0 \\ B \\ I \end{array} \right] \Delta [W 0 0] \] (79)
which is equivalent to (74) using Schur complement. \( \square \)

**VI. Quantized $H_\infty$ Control**

Here we extend the quantization results to $H_\infty$ control. For simplicity, only quantized state feedback is considered. The system of interest is as follows:
\[ x(k + 1) = A x(k) + B u(k) + B_1 w(k) \\ z(k) = C x(k) + D u(k) + D_1 w(k) \] (80)
where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^{m_1}$, $z \in \mathbb{R}^r$ and the control signal is in the form of (2)-(3). Given a quantization level vector $\rho$ and $H_\infty$ performance bound $\gamma > 0$, the design objective is to find $K$ such that the induced $L_2$-gain from $w$ to $z$ is less than $\gamma$.

It is easy to verify that the closed-loop system is given by
\[ x(k + 1) = [A + B(I + \Delta(u))K] x(k) + B_1 w(k) \\ z(k) = [C + D(I + \Delta(u))K] x(k) + D_1 w(k) \] (81)
As in the quadratic performance control problem, we consider the following relaxed $H_\infty$ control problem: Find $P = P^T > 0$ and $K$ such that
\[ x^T \Pi(\Delta(Kx))x < 0, \quad \forall x \neq 0 \] (82)
where
\[
\Pi(\Delta) = A^T PA^T - P + \gamma^{-2}(A^T PB_1 + C^T D_1) \\
\times [I - \gamma^{-2}(D_1^T D_1 + B_1^T PB_1)]^{-1} \\
\times (B^T PA_\Delta + D_1^T C_\Delta) + C_\Delta^T C_\Delta < 0
\]
(83)
\[
A_\Delta = A + B(I + \Delta(v))K, \quad C_\Delta = C + D(I + \Delta(v))K
\]
(84)

**Theorem VI.1:** Consider the given system (80), controller structure (2)-(3), quantization level vector $\rho$ and a $H_\infty$ performance bound $\gamma > 0$. Suppose there exist $P = P^T > 0$ and $K$ such that (82) holds, then the induced $L_2$-norm from $w$ to $z$ is less than $\gamma$.

Further, for any $P = P^T > 0$ and $K$, (82) holds if $\Pi(\Delta) < 0$ for all $|\Delta_j| \leq \delta_j$, where $\delta_j$ are related to $\rho_j$ by (6). Conversely, if (82) holds, $\Pi(\Delta) < 0$ for all $|\Delta_j| \leq \delta_j - \epsilon$, where $\epsilon > 0$ is arbitrarily small.

In addition, there exist $P = P^T > 0$ and $K$ such that $\Pi(\Delta) < 0$ for all $|\Delta_j| \leq \delta_j$ if the following linear matrix inequality
\[
\begin{bmatrix}
-S + B\Gamma A^T & * & * & * \\
(AS + BW)^T & -S & * & * \\
B_1^T & 0 & -\Gamma I & * \\
D\Gamma A^T & CS + D_1 W & D_1 - \Gamma I & * \\
0 & W & 0 & 0 & -\Gamma
\end{bmatrix} < 0
\]
(85)
has a solution for $S = S^T$, $W$ and diagonal scaling matrix $\Gamma$, where $\Lambda$ and the relationship between $(S, W)$ and $(P, K)$ are the same as in Theorem V.2.

**Proof.** The proof is similar to that of Theorem V.2. The details are omitted. \qed

**VII. CONCLUSIONS**

We have shown that the classical sector bound method can be used to study quantized feedback control problems in a non-conservative manner. Various cases have been considered: quantized state feedback control, quantized output feedback control, MIMO systems, and control with performances. In all these problems, the key result is that quantization errors can be converted into sector bound uncertainties without conservatism. By doing so, quantized feedback control problems become well-known robust control problems.

For quadratic stabilization of SISO systems (using either quantized state feedback or quantized output feedback), complete solutions are available by solving related $H_\infty$ optimization problems. For MIMO systems or SISO systems with a performance control objective, the resulting robust control problems usually do not have simple solutions, thus sufficient conditions on quantization densities are derived. These conditions are expressed either in terms of $H_\infty$ optimization or linear matrix inequalities. Note that these conditions are for a given set of quantization densities. But because these conditions are convex in the sector bounds associated with the quantization densities, optimal quantization densities can be easily computed numerically.

Finally, we note that the use of the sector bound method also explains why it is difficult to find the coarsest quantization densities in the cases of MIMO stabilization and/or performance control problems. More precisely, the difficulties are the same as finding non-conservative solutions to the related robust control problems, which are known to be very difficult.

**REFERENCES**


