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Worst-Case Properties of the Uniform Distribution and Randomized Algorithms for Robustness Analysis*

Er-Wei Bai,† Roberto Tempo,‡ and Minyue Fu§

Abstract. In this paper we study a *probabilistic approach* which is an alternative to the classical worst-case algorithms for robustness analysis and design of uncertain control systems. That is, we aim to estimate the probability that a control system with uncertain parameters q restricted to a box Q attains a given level of performance γ. Since this probability depends on the underlying distribution, we address the following question: What is a "reasonable" distribution so that the estimated probability makes sense? To answer this question, we define two worst-case criteria and prove that the uniform distribution is optimal in both cases. In the second part of the paper we turn our attention to a subsequent problem. That is, we estimate the sizes of both the so-called "good" and "bad" sets via sampling. Roughly speaking, the good set contains the parameters $q \in Q$ with a performance level better than or equal to γ while the bad set is the set of parameters $q \in Q$ with a performance level worse than γ. We give bounds on the minimum sample size to attain a good estimate of these sets in a certain probabilistic sense.

Key words. Randomized algorithms, Robustness analysis, Uncertain parameters.

1. Introduction and Preliminaries

Consider a measurable function $u(q)$: $\mathbf{R}^n \to \mathbf{R}$, where $q = [q_1, q_2, \ldots, q_n]'$ and each q_i is restricted to a bounded interval. Without loss of generality, we normalize each q_i into the interval $[-\frac{1}{2}, \frac{1}{2}]$ and define $Q = [-\frac{1}{2}, \frac{1}{2}]^n \subset \mathbf{R}^n$. The following two problems are of interest in robustness analysis:

Problem 1. To find $q_{\max} \in Q$ such that

$$u(q_{\max}) \doteq \max_{q \in Q} u(q)$$

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† Department of Electrical and Computer Engineering, University of Iowa, Iowa City, Iowa 52242, U.S.A. erwei@icaen.uiowa.edu.

‡ CENS-CNR, Politecnico di Torino, Torino, Italy. tempo@polito.it.

§ Department of Electrical and Computer Engineering, University of Newcastle, Newcastle, N.S.W. 2308, Australia. eemf@ee.newcastle.edu.au.

or, for given error bound $\varepsilon > 0$, to find $\bar{q} \in Q$ such that

$$|u(q_{\max}) - u(\bar{q})| \leq \varepsilon.$$

Problem 2. For given performance level $\gamma > 0$, to check whether

$$u(q) \leq \gamma$$

for all $q \in Q$.

Note that by proper formulation, if the vector q represents the uncertain parameters entering into a control system, many robustness analysis questions belong to either one of the above two problems. For instance, for single-input single-output systems, if $u(q)$ is equal to the maximum real part of the roots of the closed loop polynomial, then $u(q_{\max})$ determines whether the system is robustly stable. On the other hand, if $u(q)$ is equal to the H_∞ norm of the sensitivity function, then $u(q)$ smaller than γ for all $q \in Q$ implies that robust performance is attained. Several robustness problems which can be formulated in either one of the two cases described above are listed in Section 4. Additional problems of this type are formulated in [6] in the classical $M - \Delta$ setting.

With these motivations, we now introduce the *good set* and the *bad set*. For given $\gamma > 0$, define the good set $Q_g(\gamma) \subseteq Q$ and the bad set $Q_b(\gamma) \subseteq Q$ as

$$Q_b(\gamma) \doteq \{q \in Q : u(q) > \gamma\};$$
$$Q_g(\gamma) \doteq \{q \in Q : u(q) \leq \gamma\}.$$

Roughly speaking, the good set $Q_g(\gamma)$ contains the parameters $q \in Q$ with a performance level better than or equal to γ and the bad set $Q_b(\gamma)$ is the set of parameters $q \in Q$ with a performance level worse than γ. Obviously, the union of these two sets coincides with Q.

Recently, it has been shown that several key problems in robustness, including μ calculation and stability of interval matrices, are NP-hard; see, e.g., [2], [3], [8], and [10]. Therefore, several researchers took a different direction leading to a probabilistic-based approach; e.g., [1], [6], [11], [13], [14], and [16]. The key idea in this framework is to solve Problems 1 and 2 previously described in a probabilistic sense. For example, we can say that the probability that $u(q) \leq \gamma$ is at least $1 - \delta$, where $\delta \in (0, 1)$. Similarly, for $\varepsilon \in (0, 1)$, given $\bar{q} \in Q$ we estimate

$$|u(q_{\max}) - u(\bar{q})| \leq \varepsilon$$

with probability $1 - \delta$. Following the terminology in [15], we call ε the *accuracy* and $1 - \delta$ the *confidence* parameter. One interesting feature of this probabilistic setting is that, unlike its deterministic counterpart, the complexity of randomized algorithms may not increase exponentially with the number of parameters; see Lemma 3.1, the discussion in Section 5, and [6], [11], [14], and [15]. A drawback of this setting, however, is that the results obtained depend on the specific choice of the underlying probability measure. Taking $\mathscr{F}$ as a probability measure, the probability distribution function is given by

$$F(q) = \mathrm{Prob}_F\{x \le q\} = \int_{x \le q} \mathscr{F}(dx),$$

where the vector inequality applies elementwise and the subscript F indicates the underlying probability measure. The probability of the bad set $Q_{\mathrm{b}}(\gamma)$ is

$$F(Q_{\mathrm{b}}(\gamma)) \doteq \mathrm{Prob}_F\{q \in Q_{\mathrm{b}(\gamma)}\} = \int_{Q_{\mathrm{b}}(\gamma)} \mathscr{F}(dq).$$

Observe now that $F(Q_{\mathrm{b}}(\gamma))$ can be interpreted as the "weighted" volume of the set $Q_{\mathrm{b}}(\gamma)$. That is,

$$\mathrm{Vol}_F(Q_{\mathrm{b}}(\gamma)) = \int_{Q_{\mathrm{b}}(\gamma)} \mathscr{F}(dq).$$

Then, for given $u(\cdot)$ and performance level γ, we may ask the following question: How do we calculate $F(Q_{\mathrm{b}}(\gamma))$? This probability can be easily estimated by using some classical results such as the Bernoulli [9] or Chernoff bounds [4]. In particular, let $q^1, q^2, \ldots, q^N$ be i.i.d. random samples in Q generated according to the given distribution F. Define an indicator function

$$z_i = \begin{cases} 1 & \text{if } \; q^i \in Q_{\mathrm{b}}(\gamma); \\ 0 & \text{otherwise.} \end{cases}$$

Then, invoking the Chernoff bound [4], for any $\varepsilon, \delta \in (0,1)$, we conclude that if

$$N \ge \frac{1}{2\varepsilon^2} \ln \frac{2}{\delta},$$

then

$$\mathrm{Prob}_F\left\{ \left| \frac{1}{N} \sum_{i=1}^{N} z_i - F(Q_{\mathrm{b}}(\gamma)) \right| \le \varepsilon \right\} \ge 1 - \delta.$$

The interpretation of this result is the following: If ε and δ are "small," the estimated probability $(1/N)\sum_{i=1}^{N} z_i$ is a very accurate estimate of the true probability $F(Q_{\mathrm{b}}(\gamma))$. We also observe that the number of samples required to compute this estimate is *linear* in both $1/\varepsilon^2$ and $\ln(1/\delta)$.

We notice that without some reasoning attached to the chosen probability measure, the quantity $F(Q_{\mathrm{b}}(\gamma))$ is meaningless. To argue this, consider two extreme cases. First, we select the probability measure $\mathscr{F}$ such that $\mathscr{F}(dq) = 0$ if $q \in Q_{\mathrm{b}}(\gamma)$. Then $F(Q_{\mathrm{b}}(\gamma)) = 0$. On the other hand, if the probability measure is chosen such that $\mathscr{F}(dq) = 0$ if $q \in Q_{\mathrm{g}}(\gamma)$, then

$$F(Q_{\mathrm{b}}(\gamma)) = 1 - \int_{Q_{\mathrm{g}}(\gamma)} \mathscr{F}(dq) = 1 - F(Q_{\mathrm{g}}(\gamma)) = 1.$$

This simple example shows that, for an arbitrarily chosen probability measure, the probability of q being in the bad set does not mean too much. This brings a key question of the randomized approach in robustness analysis: What is a "reasonable" distribution so that the results obtained make sense?

In this direction, Barmish and Lagoa [1] have shown that the uniform distribution F_{uni} is the worst-case distribution in a certain class. More precisely, let q be a vector of independent random variables and let $\mathscr{X} \subset \mathbf{R}^n$ be a closed, convex, and centrally symmetric set. Then

$$\min_F \text{Prob}_F\{q \in \mathscr{X}\} = \text{Prob}_{F_{\text{uni}}}\{q \in \mathscr{X}\},$$

where the minimization is carried out in the set of all probability distributions satisfying two conditions: (1) F is absolutely continuous so that the probability density function $f(q)$,

$$\frac{dF(q)}{dq} = f(q) = \prod_{i=1}^{n} f_i(q_i),$$

is well defined; (2) the marginal density functions $f_i(q_i)$ are nondecreasing and symmetric. In the same paper [1], this result is then applied to robustness analysis of an affine polynomial family, taking $\mathscr{X}$ as the so-called value set. However, the fact that $\mathscr{X}$ needs to be convex and centrally symmetric seems a critical requirement which is generally not satisfied for the sets $Q_{\text{g}}(\gamma)$ and $Q_{\text{b}}(\gamma)$. Given these motivations, the first objective in this paper is to show that the uniform distribution F_{uni} has several interesting properties among all distributions. In addition, in this paper, the mapping $u(\cdot)$ is nonlinear, which in turn means that the set $\mathscr{X}$ discussed above is not necessarily convex and centrally symmetric.

We now briefly summarize the main results of this paper. In Section 2 we study two worst-case optimality criteria. First, we prove a connection between the worst-case distribution, in a min-max sense, and the weighted volume of the bad set; see Theorem 2.1 and related comments. Secondly, we prove that the uniform distribution F_{uni} is "optimal" in the sense that it requires the minimum number of samples to attain a certain confidence $1-\delta$ for all functions $u(\cdot)$ in the class $\mathscr{U}_L$ of Lipschitz continuous functions. In Section 3 we compute the minimum sample size N required to estimate the probability that the volume of the bad set is smaller than a certain percentage of the volume of the set Q. As in the case of the Chernoff bound, N is independent of the number of uncertain parameters. In Section 4 we then apply these results to uncertain control systems. In particular, we show how a number of problems in robustness analysis can be reformulated in this setting and we present a numerical example showing the efficacy of this approach. In Section 5 we discuss some issues and drawbacks of the existing results and, in particular, we study cases when the bound N grows with the problem size. Finally, in Section 6 we provide conclusions. The proofs of the results of Section 2 are given in the Appendix.

2. Worst-Case Properties of the Uniform Distribution

First, we define the set of allowable probability measures. Let $\mathscr{P}$ be the set of all probability measures that are absolutely continuous with respect to the Lebesgue measure.

As discussed in the Introduction, the probability distribution is $F(q) = \int_{x \le q} \mathscr{F}(dx)$ and the probability of the set $Q_s \subseteq Q$ is

$$\text{Prob}_F\{q \in Q_s\} = \int_{Q_s} \mathscr{F}(dq) = F(Q_s),$$

which can be interpreted as the weighted volume of the set Q_s. That is,

$$\text{Vol}_F(Q_s) = \int_{Q_s} \mathscr{F}(dq).$$

Clearly, the uniform distribution $\mathscr{F}_{\text{uni}} \in \mathscr{P}$ satisfies

$$\mathscr{F}_{\text{uni}}(dq) = \begin{cases} dq, & q \in Q, \\ 0, & q \notin Q, \end{cases}$$

and

$$\text{Vol}_{F_{\text{uni}}}(Q_s) = F_{\text{uni}}(Q_s) = \int_{Q_s} dq.$$

For given $u(\cdot)$ and performance level γ, the goal is to describe the bad set $Q_b(\gamma)$ or, at least, its size. As previously discussed, this bad set is fixed for given $u(\cdot)$ and γ and its measure is hard to determine. Thus, the idea is to use a randomized algorithm to estimate it. That is, for each distribution $\mathscr{F} \in \mathscr{P}$, we can estimate $F(Q_b(\gamma)) = \text{Prob}_F\{q \in Q_b(\gamma)\}$ and use it to evaluate the approximate size of $Q_b(\gamma)$. As discussed in the previous section, however, this probability can assume the extreme values zero or one, depending on the specific choice of the probability measure. In a more realistic setting, we have from Lemma A.1 in the Appendix that, for any $\mathscr{F} \in \mathscr{P}$, $F(Q_b(\gamma))$ always lies between

$$\inf_{Q_s \in \mathscr{Q}(\gamma)} F_1(Q_s) \le F(Q_b(\gamma)) \le \sup_{Q_s \in \mathscr{Q}(\gamma)} F_1(Q_s)$$

for any $\mathscr{F}_1 \in \mathscr{P}$ where

$$F_1(Q_s) = \int_{Q_s} \mathscr{F}_1(dq) \tag{2.1}$$

and

$$\mathscr{Q}(\gamma) = \{Q_s \subseteq Q : F(Q_s) = F(Q_b(\gamma))\}. \tag{2.2}$$

Conceptually, $\sup F_1(Q_s)$ is an overestimate of $F(Q_b(\gamma))$ and $\inf F_1(Q_s)$ is an underestimate; they both depend on the distribution F_1 but neither one is a good estimate of $F(Q_b(\gamma))$. On the other hand, these two bounds can always be achieved in a worst-case scenario. Thus, to de-emphasize the dependence on the probability measure $\mathscr{F}_1$, a better choice would be

$$\sup_{\mathscr{F}_1 \in \mathscr{P}} \inf_{Q_s \in \mathscr{Q}(\gamma)} F_1(Q_s) \quad \text{or} \quad \inf_{\mathscr{F}_1 \in \mathscr{P}} \sup_{Q_s \in \mathscr{Q}(\gamma)} F_1(Q_s).$$

The interpretation is that to approximate $F(Q_b(\gamma))$ and its size, we use either the largest underestimate, which is a lower bound, or the smallest overestimate, which is an upper bound. In fact, the next result shows that the largest underestimate coincides with the smallest overestimate and they are both equal to $F(Q_b(\gamma))$.

Theorem 2.1. *For any measurable function $u(\cdot)$, performance level γ, and probability measure $\mathscr{F} \in \mathscr{P}$, we have*

$$\sup_{\mathscr{F}_1 \in \mathscr{P}} \inf_{Q_s \in \mathscr{Q}(\gamma)} F_1(Q_s) = F(Q_b(\gamma)) = \inf_{\mathscr{F}_1 \in \mathscr{P}} \sup_{Q_s \in \mathscr{Q}(\gamma)} F_1(Q_s),$$

where the distribution F_1 and the set $\mathscr{Q}(\gamma)$ are defined in (2.1) *and* (2.2).

Proof. See the Appendix.

This result is not surprising and is intuitively clear. It basically says that the worst-case distribution, in a min-max sense, is whatever distribution is used to define "volume." It is also interesting to note that we can think of the volume $\mathrm{Vol}(Q_s)$ of the set Q_s as the integral of the uniform distribution

$$\mathrm{Vol}(Q_s) = \int_{Q_s} dq = \int_{Q_s} \mathscr{F}_{\mathrm{uni}}(dq) = \mathrm{Vol}_{F_{\mathrm{uni}}}(Q_s).$$

Then the following corollary is immediate.

Corollary 2.1. *For any measurable function $u(\cdot)$ and performance level γ, we have*

$$\sup_{\mathscr{F} \in \mathscr{P}} \inf_{Q_s \in \mathscr{Q}(\gamma)} F(Q_s) = F_{\mathrm{uni}}(Q_b(\gamma)) = \inf_{\mathscr{F} \in \mathscr{P}} \sup_{Q_s \in \mathscr{Q}(\gamma)} F(Q_s),$$

where

$$\mathscr{Q}(\gamma) = \{Q_s \subseteq Q : F_{\mathrm{uni}}(Q_s) = F_{\mathrm{uni}}(Q_b(\gamma))\}.$$

Next, we turn our attention to the following question: Given δ and ε, what is an "optimal distribution" in terms of requiring the minimum number of samples to meet a prescribed probability for all $u(\cdot)$ in the class of Lipschitz continuous functions? Interestingly, this optimal distribution turns out to be the uniform distribution. To state this result precisely, we need to define two sets.

Consider the set of all absolutely continuous probability distributions so that the density functions

$$f(q) = \frac{dF(q)}{dq}$$

is well defined. Now, let $\mathscr{P}_L$ be the set of all Lipschitz continuous density functions with Lipschitz constant L such that $f(\bar{q}) \leq 1 - \xi$ for some $\bar{q} \in Q$ and $\xi > 0$. Let $\mathscr{U}_L$ be the set of all Lipschitz continuous functions $u(\cdot)$ with Lipschitz constant L.

We remark that, for any nonuniform distribution, there always exists $\bar{q} \in Q$ such that $f(\bar{q}) \leq 1 - \xi$ for some $\xi > 0$. Thus, in practice, $\mathscr{P}_L$ is the set of all Lipschitz continuous density functions besides the uniform density function. We take $q^1, q^2, \ldots, q^N$ i.i.d. random samples in Q according to $f \in \mathscr{P}_L$ and denote the largest $u(q^i)$ as

$$u(q^N_{\max}) = \max_{i=1,2,\ldots,N} u(q^i).$$

Finally, for the given ε and δ, we denote by $k(f)$ the minimum number of samples required to satisfy

$$k(f) = \arg\min \left\{ \text{Prob}_F \left\{ \sup_{u \in \mathcal{U}_L} |u(q_{\max}) - u(q^N_{\max})| \leq \varepsilon \right\} \geq 1 - \delta \right\}, \tag{2.3}$$

where Prob_F emphasizes again that the probability is with respect to the underlying probability measure $\mathcal{F}$ or, equivalently, with respect to the density function $f(q)$.

Contrary to the criterion used to state the Chernoff bound, we observe that here there is only one level of probability. The result below shows that the uniform distribution gives the minimum sample size. However, this sample size is an exponential function of the number of parameters; see the comments in Section 5.

We are now ready to state the second result of this section.

Theorem 2.2. *Consider the sets $\mathcal{U}_L$ and $\mathcal{P}_L$ previously defined. For any $\delta \in (0, 1)$ and $\varepsilon \in (0, \xi]$, we have*

$$k(f_{\text{uni}}) \leq \min_{f \in \mathcal{P}_L} k(f).$$

Proof. See the Appendix.

3. The Minimum Sample Size

Motivated by the latter result of the previous section, we now elaborate on the issue of the minimum sample size. Define the sample complexity as

$$N_0 \doteq \frac{\ln(1/\delta)}{\ln(1/(1-\varepsilon))}$$

and, for completeness, recall that the minimum sample size for the problem of estimating $u_{\max}$ with sampling is given by the result below.

Lemma 3.1. *Consider a measurable function $u(\cdot)$ and let $\mathcal{F} \in \mathcal{P}$. If*

$$N \geq N_0,$$

then

$$\text{Prob}_F\{\text{Prob}_F\{u(q) > u(q^N_{\max})\} \leq \varepsilon\} \geq 1 - \delta$$

for any ε and $\delta \in (0, 1)$.

This result was independently derived in [6] and [14]. We notice that the bound given in Lemma 3.1 improves upon the Chernoff bound for the special case when estimating the maximum of a function via random search is of interest. We also observe that this result is independent of the underlying probability measure $\mathcal{F}$. We now use Lemma 3.1 to establish a connection with the volume of the bad set $Q_b(\gamma)$. A similar result has been established in [6].

Corollary 3.1. *Let $\mathcal{F} \in \mathcal{P}$ and let $q^1, q^2, \ldots, q^N$ be i.i.d. random samples according to the probability distribution F generated by $\mathcal{F}$. For any ε and $\delta \in (0,1)$, if*

$$N \geq N_0,$$

then

$$\mathrm{Prob}_F\left\{\frac{\mathrm{Vol}_F(Q_b(u(q_{\max}^N)))}{\mathrm{Vol}_F(Q)} \leq \varepsilon\right\} \geq 1-\delta.$$

This result follows immediately from Lemma 3.1 and can be interpreted in terms of the "amount of badness" of the set $Q_b(\gamma)$. If

$$\mathrm{Vol}_F(Q_b(\gamma)) \leq \varepsilon \mathrm{Vol}_F(Q)$$

we can say that $Q_b(\gamma)$ is ε-bad for $\gamma = u(q_{\max}^N)$. Then, from Corollary 3.1, we conclude that if

$$N \geq N_0 = \frac{\ln(1/\delta)}{\ln(1/(1-\varepsilon))},$$

then, with probability at least $1-\delta$, $Q_b(u(q_{\max}^N))$ is at most ε-bad.

4. Applications to Probabilistic Robustness Analysis of Control Systems

The results derived in the previous sections can be immediately applied to several problems in robustness analysis. We now list a number of them; see [6] for a similar discussion. The distribution chosen in this section is uniform and the "volume" is defined accordingly.

Application 1. Let $u(q)$ be the maximum real part of the eigenvalues, where $q \in Q$ denotes the uncertain parameters. Let q^i, $i = 1, 2, \ldots, N$, be i.i.d. random samples in Q generated according to a uniform distribution. If $u(q^i) < 0$ for all $i = 1, \ldots, N$ and $N \geq N_0$, then, with probability at least $1-\delta$, the volume of the unstable set $\{q \in Q : u(q) \geq 0\}$ is smaller than the volume of $Q_b(u(q_{\max}^N))$ which is no greater than $\varepsilon \mathrm{Vol}_{F_{\mathrm{uni}}}(Q)$. Thus, we conclude that with probability at least $1-\delta$, the volume of the unstable set it at most ε-bad. The same argument clearly holds for discrete time systems. In this case, it suffices to take $u(q)$ as the maximum magnitude of the eigenvalues and $|u(q^i)| < 1$ for all $i = 1, \ldots, N$.

Application 2. Let $u(q) = \|S(s,q)\|_\infty \doteq \sup_\omega |S(j\omega, q)|$ be the H_∞ norm of the sensitivity function $S(s,q)$ of a SISO system. As in the first example of this section, let q^i, $i = 1, 2, \ldots, N$, be i.i.d random samples in Q generated according to a uniform distribution in Q. If $u(q^i) < \gamma$ for all $i = 1, \ldots, N$ and $N \geq N_0$, then, with probability at least $1-\delta$, the volume of the set of "bad" plants with a performance level greater than γ is smaller than the volume of $Q_b(u(q_{\max}^N))$ which is no greater than $\varepsilon \mathrm{Vol}_{F_{\mathrm{uni}}}(Q)$. We conclude that, with probability at least $1-\delta$, the volume of the set of bad plants is at most ε-bad. For discrete time systems, the same argument holds taking $u(q) = \|S(z,q)\|_\infty \doteq \sup_{\theta \in [0,2\pi]} |S(e^{j\theta}, q)|$.

Application 3. Let $u(q)$ be equal to the inverse of the structured singular value μ; see, e.g., [5] and [12]. If the samples q^i, $i = 1, 2, \ldots, N$, are randomly generated in Q according to a uniform distribution and if $u(q^i) < 1/\mu$ for all $i = 1, 2, \ldots, N$ and $N \geq N_0$, then, with a probability at least $1 - \delta$, the volume of the set of plants with the robustness margin no greater than $1/\mu$ is at most ε-bad.

Application 4. We now further elaborate on Application 3. In particular, we discuss how to calculate the maximum allowable perturbation for a SISO control system with a stable nominal plant. Let $u(q)$ be the maximum real part of the roots of the closed loop polynomial of a control system with a family of plants $G_P(s, q)$ in the forward loop with uncertain parameters $q \in Q$. In addition, suppose that $G_P(s, 0)$ is the nominal plant that is assumed to be stable, i.e., $u(q) < 0$ for $q = 0$. In order to obtain an interval polynomial, we perform parameter overbounding. Subsequently, we apply the Theorem of Kharitonov [7] obtaining a box $Q_r \doteq \{q \in Q: -r \leq q_i \leq r, i = 1, 2, \ldots, n\}$ of radius $r > 0$ so that $u(q) < 0$ for all $q \in Q_r$. In this case, $G_P(s, q)$ is stable for all $q \in Q_r$; usually, this set is much smaller than the set of all stable plants. In other words, application of the Theorem of Kharitonov may lead to very conservative results. This is especially true if the uncertain parameters enter into the plant coefficients in a nonlinear fashion. We then ask the following question: Can we estimate a box bigger than Q_r so that only a small number of plants in this box are unstable? The rationale behind this question is that if a "large" increase in the size of the box can be established, a small risk may be justified, at least in some applications. The problem of determining such a box can be immediately solved by applying the results of Corollary 3.1.

We now give a numerical example to illustrate these applications.

Example. Consider the polynomial

$$g(s, q) = s^4 + a_3(q)s^3 + a_2(q)s^2 + a_1(q)s + a_0(q),$$

where

$$a_0(q) = (100q_1 - 1)^2(50q_2 + 0.5)^2(q_3 + 1)(q_4 + 2);$$

$$a_1(q) = [(100q_1 - 1)^2 + (50q_2 + 0.5)^2](q_3 + 1)(q_4 + 2)$$
$$+ (100q_1 - 1)^2(50q_2 - 0.5)^2(q_3 + q_4 + 3);$$

$$a_2(q) = (q_3 + 1)(q_4 + 2) + (100q_1 - 1)^2(50q_2 + 0.5)^2$$
$$+ [(100q_1 - 1)^2 + (50q_2 - 0.5)^2](q_3 + q_4 + 3);$$

$$a_1(q) = (100q_1 - 1)^2 + (50q_2 + 0.5)^2 + q_3 + q_4 + 3$$

with $q = [q_1, q_2, q_3, q_4]' \in Q = [-\frac{1}{2}, \frac{1}{2}]^4$. First, we notice that the box containing only stable plants is smaller than $Q_r = [-0.01, 0.01]^4$; this follows immediately by observing that $a_0(q) = 0$ for $q_1 = 0.01$. Clearly, $\mathrm{Vol}_{F_{uni}}(Q_r) = 0.02^4$. Next, taking

$\varepsilon = 0.001$ and $\delta = 0.01$, we compute

$$N = 4603 > \frac{\ln(1/\delta)}{\ln(1/(1-\varepsilon))}$$

and we generate $q^1, q^2, \ldots, q^{4603} \in Q$, randomly using the uniform density function f_{uni}. Subsequently, we calculate the maximum real part $u(q)$ of the roots of $g(s,q)$ for each q^i, $i = 1, 2, \ldots, 4603$, obtaining

$$\max_{i=1,2,\ldots,4603} u(q^i) = -0.0000042395 < 0.$$

Hence, from Corollary 3.1, it follows that, with probability at least $1 - \delta = 0.99$, the volume of the bad set $Q_b(\max_{i=1,2,\ldots,4603} u(q^i))$ is no greater than $\varepsilon \operatorname{Vol}_{F_{\text{uni}}}(Q) = \varepsilon = 0.001$. We conclude that with a small risk of being unstable, we obtain an increase in size by at least

$$\frac{\operatorname{Vol}_{F_{\text{uni}}}(Q)}{\operatorname{Vol}_{F_{\text{uni}}}(Q_r)} = \frac{1}{0.02^4} = 6{,}250{,}000.$$

5. Discussions and Remarks

The results given in this paper may have applications broader than robustness analysis. The fact that the results are independent of the problem dimension seems powerful even though it is well known in the Monte Carlo literature. This is a consequence of the fact that the minimum sample size in Corollary 3.1 is stated in terms of the ratio

$$\frac{\operatorname{Vol}_F(Q_b(u(q_{\max}^N)))}{\operatorname{Vol}_F(Q)}.$$

If the size and/or the dimension of Q increases, the size of $\operatorname{Vol}_{F_{\text{uni}}}(Q_b(u(q_{\max}^N)))$ increases as well. On a negative side, we remark that the fact that $Q_b(u(q_{\max}^N))$ is ε-bad does not necessarily imply that $u(q_{\max}^N)$ is "close" to $u(q_{\max})$. In other words, except for some simple cases, it is not possible to estimate accurately the difference between $u(q_{\max}^N)$ and $u(q_{\max})$ or the difference between q^i and $q_{\max}$ taking only N_0 samples in Q. To elaborate, we study the two cases $|u(q_{\max}) - u(q_{\max}^N)| \le \varepsilon$ and $||q^i - q_{\max}|| \le \varepsilon$ separately.

For the case $||q^i - q_{\max}|| \le \varepsilon$, let $u(\cdot)$ achieve the maximum $q_{\max} \in Q$, consider a uniform distribution, and take the norm $||\cdot||$ as ℓ_∞; the same conclusion holds if a different norm is used. Then $||q^i - q_{\max}|| \le \varepsilon$ if q^i is in the box of center $q_{\max}$ and radius ε

$$B(q_{\max}, \varepsilon) = \{q : ||q - q_{\max}|| \le \varepsilon\}.$$

The volume of this box is

$$\operatorname{Vol}_{F_{\text{uni}}} B(q_{\max}, \varepsilon) = (2\varepsilon)^n.$$

For small ε, $\operatorname{Vol}_{F_{\text{uni}}}(B(q_{\max}, \varepsilon) \cap Q)$ converges to zero exponentially as the dimension of q increases. In order to have at least one q^i in the box $B(q_{\max}, \varepsilon) \cap Q$, the number of samples required *has* to increase exponentially, except for some pathological cases. One exception to this exponential growth is when the measure of the set of maximizers increases at a rate faster than the decreasing rate of $\mathrm{Vol}_{F_{uni}}(B(q_{\max}, \varepsilon) \cap Q)$. Only in such pathological cases, does the number of samples required to meet $\|q^i - q_{\max}\| \leq \varepsilon$ with probability $1 - \delta$ not depend on the dimension of q.

For the second case, when $|u(q_{\max}) - u(q_{\max}^N)| \leq \varepsilon$, let $u(\cdot)$ be a Lipschitz continuous function with Lipschitz constant L. Note that $|u(q_{\max}) - u(q_{\max}^N)| \leq \varepsilon$ if some samples q^i are in the box $B(q_{\max}, \varepsilon/L) \cap Q$. As previously discussed, for ε/L small, the volume of $B(q_{\max}, \varepsilon/L) \cap Q$ converges to zero exponentially. In turn, this implies that the number of samples needed to satisfy $|u(q_{\max}) - u(q_{\max}^N)| \leq \varepsilon$ grows exponentially as the dimension of q increases and is no longer given by N_0.

6. Conclusions

In this paper we have shown some new results for probabilistic robustness analysis of uncertain systems. A subsequent and promising line of research is focused on adaptive instead of passive randomized algorithms [17] with the specific goal to quantify the size of the "bad" set. A different research area which is worthy of investigation is related to the so-called *learning theory* and provides a framework for performing probabilistic robust design [16].

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Appendix. Proofs

To prove Theorem 2.1, we first need to state a supporting lemma.

Lemma A.1. *Let* $\mathscr{F} \in \mathscr{P}$. *For any* $0 \leq \sigma \leq 1$, *define*

$$\mathscr{Q} = \left\{ Q_s \subseteq Q \colon \int_{Q_s} \mathscr{F}(dq) = \sigma \right\}.$$

Then, for any $\mathscr{F}_1 \in \mathscr{P}$,

$$\sup_{Q_s \in \mathscr{Q}} \int_{Q_s} \mathscr{F}_1(dq) \geq \sigma \geq \inf_{Q_s \in \mathscr{Q}} \int_{Q_s} \mathscr{F}_1(dq).$$

Proof. We first show the left-hand side of the inequality. For any $\mathscr{F}_1 \in \mathscr{P}$, define the set $Q_0 = \{q \in Q : \mathscr{F}_1(dq) \geq \mathscr{F}(dq)\}$. For any $0 \leq \sigma \leq 1$, either $\int_{Q_0} \mathscr{F}(dq) \geq \sigma$ or $\int_{Q_0} \mathscr{F}(dq) < \sigma$. For the first case, there exists a subset $Q_1 \subseteq Q_0$ so that $\int_{Q_1} \mathscr{F}(dq) = \sigma$. This implies $Q_1 \subseteq \mathscr{Q}$ and

$$\sup_{Q_s \in \mathscr{Q}} \int_{Q_s} \mathscr{F}_1(dq) \geq \int_{Q_1} \mathscr{F}_1(dq).$$

Now, observe that $\mathscr{F}_1(dq) \geq \mathscr{F}(dq)$ whenever $q \in Q_1$. It follows that

$$\sup_{Q_s \in \mathscr{Q}} \left[\int_{Q_s} \mathscr{F}_1(dq) - \int_{Q_s} \mathscr{F}(dq) \right] \geq \int_{Q_1} \mathscr{F}_1(dq) - \int_{Q_1} \mathscr{F}(dq)$$

$$= \int_{Q_1} (\mathscr{F}_1(dq) - \mathscr{F}(dq)) \geq 0,$$

i.e.,

$$\sup_{Q_s \in \mathscr{Q}} \int_{Q_s} \mathscr{F}_1(dq) \geq \int_{Q_s} \mathscr{F}(dq) = \sigma.$$

For the second case $\int_{Q_0} \mathscr{F}(dq) < \sigma$, there exists a subset Q_1 such that $Q_0 \subseteq Q_1 \subseteq Q$, $\int_{Q_1} \mathscr{F}(dq) = \sigma$, and $\mathscr{F}_1(dq) < \mathscr{F}(dq)$ whenever $q \in Q/Q_1$. Since

$$0 = \int_{Q} (\mathscr{F}_1(dq) - \mathscr{F}(dq)) = \int_{Q_1} (\mathscr{F}_1(dq) - \mathscr{F}(dq)) + \int_{Q/Q_1} (\mathscr{F}_1(dq) - \mathscr{F}(dq)),$$

we have

$$\int_{Q_1} \mathscr{F}_1(dq) = \int_{Q_1} \mathscr{F}(dq) - \int_{Q/Q_1} (\mathscr{F}_1(dq) - \mathscr{F}(dq)) \geq \int_{Q_1} \mathscr{F}(dq) = \sigma.$$

Therefore,

$$\sup_{Q_s \in \mathscr{Q}} \int_{Q_s} \mathscr{F}_1(dq) \geq \int_{Q_1} \mathscr{F}(dq) = \sigma.$$

This proves the left-hand side of the inequality. The proof for the right-hand side is similar and is therefore omitted. ■

Proof of Theorem 2.1. From the above lemma, we have

$$\sup_{Q_s \in \mathscr{Q}(\gamma)} F_1(Q_b(\gamma)) \geq \sigma = \mathrm{Vol}_F(Q_b(\gamma)) = \int_{Q_b(\gamma)} \mathscr{F}(dq); \tag{8.1}$$

$$\inf_{Q_s \in \mathscr{Q}(\gamma)} F_1(Q_b(\gamma)) \leq \sigma = \mathrm{Vol}_F(Q_b(\gamma)) = \int_{Q_b(\gamma)} \mathscr{F}(dq) \tag{8.2}$$

for any $\mathscr{F}_1 \in \mathscr{P}$. Moreover, the equalities are achieved when $F_1 = F$ and therefore the conclusion follows. ■

Proof of Theorem 2.2. For any $\varepsilon \in (0, \xi]$, $\delta \in (0, 1)$, and $f \in \mathscr{P}_L$, there exists a $\bar{q} \in Q$ such that $f(\bar{q}) \leq 1 - \xi$ and a ball $B(\bar{q}, \varepsilon/L) = \{q : \|q - \bar{q}\| \leq \varepsilon/L\}$ such that

$$\int_{B(\bar{q},\varepsilon/L)\cap Q} f(q)\, dq \leq \mathrm{Vol}_{F_{\mathrm{uni}}}(B(\bar{q}, \varepsilon/L) \cap Q) = \mathrm{Vol}_{F_{\mathrm{uni}}}(D),$$

where $D = B(\bar{q}, \varepsilon/L) \cap Q$. Let $u(\cdot)$ be a monotone function that achieves the unique maximum at $\bar{q}$ and $u(q) = u(\bar{q}) - \varepsilon$ for all q at the boundary of the ball

$B(\bar{q}, \varepsilon/L)$. Then

$$|u(q^i) - u(\bar{q})| \le \varepsilon$$

if and only if

$$q^i \in B(\bar{q}, \varepsilon/L) \cap Q = D.$$

Notice that

$$\mathrm{Prob}_F\{q^i \in D\} = \int_{B(\bar{q},\varepsilon/L)\cap Q} f(q)\,dq \le \mathrm{Vol}_{F_{\mathrm{uni}}}(D)$$

or

$$\mathrm{Prob}_F\{q^i \notin D\} \ge 1 - \mathrm{Vol}_{F_{\mathrm{uni}}}(D).$$

Now,

$$\begin{aligned} 1 - \delta &\le \mathrm{Prob}_F\{|u(q_{\max}) - u(q_{\max}^N)| \le \varepsilon\} = \mathrm{Prob}_F\{\text{at least one of } q^i \in D\} \\ &= 1 - \mathrm{Prob}_F\{q^i \notin D\}^N \le 1 - (1 - \mathrm{Vol}_{F_{\mathrm{uni}}}(D))^N. \end{aligned}$$

That is, $(1 - \mathrm{Vol}_{F_{\mathrm{uni}}}(D))^N \le \delta$. Let N_1 be the minimum integer satisfying the above inequality. Note that N_1 is derived for a particular $u(\cdot)$; in turn, this implies that the minimum number $k(f)$ to satisfy (2.3) has to be

$$k(f) \ge N_1.$$

Since f is arbitrary in $\mathscr{P}_L$, we have $k(f) \ge N_1$ for any $f \in \mathscr{P}_L$. On the other hand, $f_{\mathrm{uni}} = 1$. Let $u(\cdot)$ be any function in $\mathscr{U}_L$ that achieves the maximum for some $\bar{q} \in Q$. For any $\varepsilon \in (0, \xi]$, if $q^i \in D$, then

$$|u(q^i) - u(\bar{q})| \le L\|q^i - \bar{q}\| \le \frac{L\varepsilon}{L} = \varepsilon.$$

Thus,

$$\mathrm{Prob}_{F_{\mathrm{uni}}}\{|u(q_{\max}) - u(q_{\max}^N)| \le \varepsilon\} \ge 1 - \delta$$

if

$$\mathrm{Prob}_{F_{\mathrm{uni}}}\{\text{some } q^i \in D\} = 1 - (1 - \mathrm{Vol}_{F_{\mathrm{uni}}}(D))^N \ge 1 - \delta$$

or

$$\delta \ge (1 - \mathrm{Vol}_{F_{\mathrm{uni}}}(D))^N.$$

The same N_1 derived for any $f \in \mathscr{P}_L$ is sufficient for this inequality to hold. Notice further that the derivation of N_1 for the uniform distribution $f_{\mathrm{uni}} = 1$ is in fact independent of $u(\cdot) \in \mathscr{U}_L$. Thus,

$$k(f_{\mathrm{uni}}) \le \inf_{f \in \mathscr{P}_L} k(f).$$

This completes the proof. ■

References

[1] B. R. Barmish and C. M. Lagoa, The Uniform Distribution: A Rigorous Justification for Its Use in Robustness Analysis, *Mathematics of Control, Signals, and Systems*, vol. 10, pp. 203–222, 1997.

[2] V. Blondel and J. N. Tsitsiklis, NP-Hardness of Some Linear Control Design Problems, *SIAM Journal of Control and Optimization*, vol. 35, pp. 2118–2127, 1997.

[3] R. P. Braatz, P. M. Young, J. C. Doyle, and M. Morari, Computational Complexity of μ Calculation, *IEEE Transactions on Automatic Control*, vol. AC-39, pp. 1000–1002, 1994.

[4] H. Chernoff, A Measure of Asymptotic Efficiency for Test of Hypothesis Based on the Sum of Observations, *Annals of Mathematical Statistics*, vol. 23, pp. 493–507, 1952.

[5] J. Doyle, Analysis of Feedback Systems with Structured Uncertainty, *IEE Proceedings*, part D, vol. 129, pp. 242–250, 1982.

[6] P. P. Khargonekar and A. Tikku, Randomized Algorithms for Robust Control Analysis Have Polynomial Time Complexity, *Proceedings of the Conference on Decision and Control*, Kobe, Japan, 1996.

[7] V. L. Kharitonov, Asymptotic Stability of an Equilibrium Position of a Family of Systems of Linear Differential Equations, *Differencial'nye Uravnenija*, vol. 14, pp. 2086–2088, 1978.

[8] A. Nemirovskii, Several NP-Hard Problems Arising in Robust Stability Analysis, *Mathematics of Control, Signals, and Systems*, vol. 6, pp. 99–105, 1993.

[9] E. Parzen, *Modern Probability Theory and Its Applications*, Wiley, New York, 1992.

[10] S. Poljak and J. Rohn, Checking Robust Nonsingularity is NP-Hard, *Mathematics of Control, Signals, and Systems*, vol. 6, pp. 1–9, 1993.

[11] L. R. Ray and R. F. Stengel, A Monte Carlo Approach to the Analysis of Control System Robustness, *Automatica*, vol. 29, pp. 229–236, 1993.

[12] M. G. Safonov, Stability Margins of Diagonally Perturbed Multivariable Systems, *IEE Proceedings*, part D, vol. 129, pp. 251–256, 1982.

[13] R. F. Stengel and L. R. Ray, Stochastic Robustness of Linear Time-Invariant Control Systems, *IEEE Transactions on Automatic Control*, vol. 36, pp. 82–87, 1991.

[14] R. Tempo, E.-W. Bai, and F. Dabbene, Probabilistic Robustness Analysis: Explicit Bounds for the Minimum Number of Samples, *Systems and Control Letters*, vol. 30, pp. 237–242, 1997.

[15] M. Vidyasagar, *A Theory of Learning and Generalization with Applications to Neural Networks and Control Systems*, Springer-Verlag, Berlin, 1996.

[16] M. Vidyasagar, Statistical Learning Theory: An Introduction and Applications to Randomized Algorithms, *Proceedings of the European Control Conference*, Brussels, 1997.

[17] A. A. Zhigljavsky, Theory of Global Random Search, Kluwer Academic, Boston, 1991.