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ROBUST H_∞ FILTERING WITH PARAMETRIC UNCERTAINTY AND DETERMINISTIC INPUT SIGNAL

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Abstract

Many dynamical signal processes involve not only process and measurement noises but also parameter uncertainty and known input signals. When an L_2 or H_∞ filter designed based on a "nominal" model of the process is applied, the presence of parameter uncertainty will not only affect the noise attenuation property of the filter but also introduce a bias proportional to the known input signal, and the later may be very appreciable. In this paper, we develop a finite horizon robust H_∞ filtering method which provides a guaranteed H_∞ bound for the estimation error in the presence of both parameter uncertainty and known input signal. This method is developed by using a game-theoretic approach, and the results generalise those for the cases either without parameter uncertainty or without a known input signal.

1 Introduction

In contrast with the conventional L_2- estimation algorithms which minimize the variance of the estimation error (see, e.g. [1]), recent advancement in signal estimation has focused on the development of H_∞ estimation methods ([2-15]) which aim at minimizing the peak of the spectral density of the estimation error. The motivations for the H_∞ approach are that the statistical assumptions and information on the noise sources are often inaccurate or unavailable and that the L_2 estimation algorithms are sensitive to parameter variations in the signal process; see [12] for a survey.

As in the L_2 case, H_∞ estimation algorithms are usually designed based on a "nominal" model of the signal process. For this reason, we will call them the "nominal" H_∞ estimation algorithms. Although a "nominal" H_∞ filter has been shown to be less sensitive to parameter variations in the signal process than an L_2 filter (see, e.g. [11] and Section 2), no guaranteed performance is provided when the true signal process deviates the assumed model.

To solve the above problem, a robust H_∞ estimation method has been developed in [4,6,13,15] to guarantee a prescribed H_∞- norm bound on the estimation error in the presence of parameter uncertainty. The key idea there is to convert the parameter uncertainty into an exogenous L_2 - noise so that an auxiliary H_∞ filtering problem is constructed which does not involve any parameter uncertainty. The solution to the auxiliary problem, if exists, guarantees the robust H_∞ performance for the perturbed process. It will be demonstrated in Section 2 that such a robust H_∞ filter outperforms a "nominal" H_∞ filter or an L_2 filter by far.

The focal point of this paper is to address the robust H_∞ filter problem for signal processes with both parameter uncertainty and a known input signal. If the signal process does not involve parameter uncertainties, the contribution of the known input signal in the estimation error can be completely cancelled for both L_2 and H_∞ filters (see, e.g. [1, 10]). This significant feature, however, is no longer valid in the presence of uncertain parameters. As a result, the estimation error will in general have components due to both the process and measurement noises and the known input signal. In Section 2, we will show via an example that the second component may be far more appreciable than the first one when the filter is designed based on the nominal values of the parameters.

In this paper, we generalize the robust H_∞ filtering approach in [4,6,13,15] to cope with the case where the signal process has a known input signal. The goal of the filter is to provide a uniformly small estimation error for any process and measurement noises and any initial state in the presence of parameter uncertainty and a known input signal. The problem will be solved in the finite horizon setting. As in [4,6], one of the key ideas is to convert the parameter uncertainty into a fictitious L_2 input noise and to formulate an auxiliary problem which does not involve any parameter uncertainty. It will be proved that the solution to the auxiliary problem, if exists, can be applied to the original problem and the prescribed performance is guaranteed. Then, a game-theoretic approach is used to solve the auxiliary problem, which gives a solution in terms of Riccati differential equations. Two types of known input signals are considered: causal and noncausal. Causal signals are those which can be measured but not predicted, while noncausal ones are those known *a priori*. Different filters are given for the two cases.

Our results will be demonstrated via an example to illustrate the significant improvement on signal estimates.

2 Motivation

To motivate the robust H_∞ filtering problem to be studied later, we show via an example that filter designs without taking into account parameter uncertainty may render a very poor signal estimate. Consider the signal generating system in Figure 1 modelled by:

$$\dot{x}(t) = \begin{bmatrix} 0 & -1+\delta \\ 1 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} -0.4545 \\ 0.909 \end{bmatrix} w(t) + \begin{bmatrix} g \\ 0 \end{bmatrix} r(t) \qquad (2.1)$$

$$y(t) = \begin{bmatrix} 0 & 100 \end{bmatrix} x(t) + v(t) \qquad (2.2)$$

$$z(t) = \begin{bmatrix} 0 & 100 \end{bmatrix} x(t) \qquad (2.3)$$

where $x(t)$ is the state, $w(t)$ is the process noise, $r(t)$ is a known deterministic input signal, $y(t)$ is the measurement, $v(t)$ is the measurement noise, $z(t)$ is the signal to be estimated, δ represents parameter uncertainty in the process which satisfies $|\delta| \le \bar{\delta} = 0.3$, and g is a known input gain to be specified later.

Both infinite horizon Kalman filter and infinite horizon "nominal" H_∞ filter are designed for the nominal plant that has been chosen to correspond to δ=0. These filters are of the following form:

$$\dot{\hat{x}}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -0.5 \end{bmatrix} \hat{x}(t) + K\left[y - \begin{bmatrix} 0 & 100 \end{bmatrix}\hat{x}(t)\right] + \begin{bmatrix} g \\ 0 \end{bmatrix} r(t) \qquad (2.4)$$

$$\hat{z}(t) = \begin{bmatrix} 0 & 100 \end{bmatrix} \hat{x}(t) \qquad (2.5)$$

where $\hat{z}(t)$ is the estimate of $z(t)$, and K is the filter gain matrix. For the Kalman filter design, the noise vector $[w(t), v(t)]^T$ was assumed to be zero mean white with identity covariance matrix, the minimization of $E\{e^T(t)e(t)\}$, where $e(t)$ denotes the estimation error $z(t) - \hat{z}(t)$, gives

$$K = K_K = [0.447 \quad 0.909]^T \qquad (2.6)$$

For the "nominal" H_∞ filter design, we take $\gamma = 1.1$=0.8db and design the filter to achieve

$$\| e(t) \|_2 < \gamma \left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_2,$$

which yields

$$K = K_\infty = [1.0350 \quad 2.1807]^T. \qquad (2.7)$$

In the above $\| \cdot \|_2$ denotes the usual norm in $L_2[0, \infty)$. We then apply the two filters to the perturbed plant (2.1) – (2.3), with $\delta = -0.3$ and $\delta = 0.3$. The magnitudes of the transfer functions from $[w(t), v(t)]^T$ and $r(t)$ to $e(t)$, denoted respectively by $[G_{ew}(s), G_v(s)]^T$ and $G_{er}(s)$, are plotted in Figures 2 and 3 for both filters. From the figures, we make the following observations:

1. The magnitude of $[G_{ew}(j\omega), G_{ev}(j\omega)]^T$ and $G_{er}(j\omega)$ are worsened for both designs when the parameter uncertainty exists (Note that $G_{er}(s)$=0 when there is no parameter uncertainty);
2. The magnitude of $G_{er}(j\omega)$, which is identically zero in the absence of parameter uncertainty may be far more significant than that of $[G_{ew}(j\omega), G_{ev}(j\omega)]^T$ for both designs even in the case of a moderate r;
3. The Kalman filter is more sensitive to parameter changes than the "nominal" H_∞ filter.

The above observations shows that a more robust filter design is needed.

For the case where there is no deterministic input signal ($r(t) \equiv 0$), a robust H_∞ filtering theory has been developed to cope with parameter uncertainties; see [4,6,13,15]. Applying the results in [6] to the above example, a robust filter is given by the following:

$$\dot{\hat{x}}(t) = \begin{bmatrix} -0.5054 & -1.1170 \\ -0.8501 & -0.5347 \end{bmatrix}\hat{x}(t) + \begin{bmatrix} 7.938 \\ 2.3540 \end{bmatrix} \{y - [0\ \ 100]\hat{x}\} \quad (2.8)$$

$$\hat{z}(x) = [0.0770 \quad 100.0179]\ \hat{x}(t) \quad (2.9)$$

The magnitude of the corresponding transfer function $[G_{ew}(j\omega), G_{ev}(j\omega)]^T$ is plotted in Figure 4 for $\delta = -0.3$ and $\delta = 0.3$ (curves 1 and 2, respectively). It can be seen that the result is much more robust than those in Figures 2 and 3.

The above results demonstrate that it is crucial to take the parameter uncertainty into account in designing the filter. The theory in [4,6,13,15], however, are not readily applicable to processes which have a known deterministic input signal.

3 Problem Formulation and a Key Lemma

Consider uncertain linear systems, described by

$$(\Sigma): \ \dot{x}(t) = [A(t) + \Delta A(t)]x(t) + B_1(t)w(t) + B_2(t)r(t), x(0) \quad (3.1)$$

$$y(t) = [C(t) + \Delta C(t)]x(t) + v(t) \quad (3.2)$$

$$z(t) = L(t)x(t) \quad (3.3)$$

where $x(t) \in \boldsymbol{R}^n$ is the state, $w(t) \in \boldsymbol{R}^p$ is the process noise, $r(t) \in \boldsymbol{R}^r$ is a known deterministic input signal, $y(t) \in \boldsymbol{R}^m$ is the measurement, $v(t) \in \boldsymbol{R}^m$ is the measurement noise and $z(t) \in \boldsymbol{R}^q$ is the signal to be estimated. We assume that $A(t), B_1(t), B_2(t), C(t)$, and $L(t)$ and known real bounded piecewise continuous matrix functions that describe the nominal system and the matrices $\Delta A(t)$ and $\Delta C(t)$ represent parameter uncertainties in the matrices $A(t)$ and $C(t)$, respectively. These uncertainties are modelled by

$$\begin{bmatrix} \Delta A(t) \\ \Delta C(t) \end{bmatrix} = \begin{bmatrix} H_1(t) \\ H_2(t) \end{bmatrix} F(t)E(t) \quad (3.4)$$

where $H_1(t), H_2(t)$ and $E(t)$ are known real bounded piecewise continuous matrix functions of appropriate dimensions and $F(t) \in \boldsymbol{R}^{i \times j}$ is an unknown matrix function with Lebesgue measurable elements and satisfying

$$\sigma_{\max}[F(t)] \le 1, \ \forall t \in \mathbb{R}. \quad (3.5)$$

For the sake of notation simplification, in the sequel we shall omit the dependence of t in the matrices when there is no confusion.

In this paper we are concerned with obtaining an estimate $\hat{z}(t)$ of $z(t)$ over the horizon $[0,T]$ using the measurement history $\{y(\tau), 0 \le \tau \le t\}$ and the known deterministic exogenous signal, $r(\cdot)$. The filter is required to provide a uniformly small filtering error, $e(t) = z(t) - \hat{z}(t)$, for any w and v in $L_2[0,T]$ and $x_0 \in \boldsymbol{R}^n$ and for all admissible uncertainties. We shall consider the following performance index

$$J(w,v,x_0,r,\hat{z}) = \tfrac{1}{2}\Big\{\| z - \hat{z}\|_2^2 - \gamma^2\big[\| w\|_2^2 + \| v\|_2^2 + \| x_0 - \hat{x}_0\|_R^2\big]\Big\} \quad (3.6)$$

where $\gamma > 0$ is a given scalar, $\hat{x}_0$ is an *a priori* estimate of x_0 and $R = R^T > 0$ is a given weighting matrix which reflects the confidence in the estimate $\hat{x}_0$. In the above $\| x\|_A^2$ denotes x^TAx and $\| v\|_2$ means the usual norm in $L_2[0,T]$ defined as $\| v\|_2^2 = \int_0^T v^Tvdt$. Also the notation $M > N$ ($M \ge N$) means that $M-N$ is a positive definite (semi-definite) matrix.

The robust H_∞ filtering problem for system (Σ) is concerned which finding an estimate $\hat{z}(t), t \in [0,T]$ which minimises

$$\sup_{w,v,x_0,F(t)} \{J(w,v,x_0,r,\hat{z})\} \quad (3.7)$$

The key idea used here to guarantee robust stability and performance for the estimation error dynamics is to convert the uncertainties to fictitious noise sources and to solve an auxiliary H_∞ filtering problem which does not involve parameter uncertainties. The performance index for this auxiliary filtering problem (if solvable) yields an upper bound for the worst case performance criterion (3.7). Justification for this technique is provided next.

Introduce the following auxiliary system

$$(\Sigma_a): \ \dot{\eta}(t) = A\eta(t) + [B_1 \ \ \tfrac{\gamma}{\varepsilon}H_1]\tilde{w}(t) + B_2r(t), \quad \eta(0) = \eta_0 \quad (3.8)$$

$$y_a(t) = C\eta(t) + [0 \ \ \tfrac{\gamma}{\varepsilon}H_2]\tilde{w}(t) + \tilde{v}(t) \quad (3.9)$$

$$z_a(t) = \begin{bmatrix} L \\ \varepsilon E \end{bmatrix}\eta(t) \quad (3.10)$$

where $\eta(t) \in \boldsymbol{R}^n$ is the state, η_0 is an unknown initial state, $\tilde{w}(t) \in \boldsymbol{R}^{p+i}$ and $\tilde{v}(t) \in \boldsymbol{R}^m$ are noise signals, $y_a(t) \in \boldsymbol{R}^m$ is the measurement, $z_a(t) \in \boldsymbol{R}^{q+j}$ is the signal to be estimated, $r, A, B_1, B_2, C, E, H_1, H_2$ and L are as in (3.1) – (3.4) and $\varepsilon > 0$ is a scaling parameter to be chosen. Associated with system (Σ_a) we introduce an estimate for $z_a(t), t \in [0,T]$, of the form

$$\hat{z}_a(t) = \begin{bmatrix} \hat{z}(t) \\ 0 \end{bmatrix} \quad (3.11)$$

where $\hat{z}(t)$ is an estimate of $L\eta(t)$ using the measurement history $\{y_a(t), 0 \le \tau \le t\}$ and the known input signal r. Next, we define the following performance index for the estimate $\hat{z}_a(t)$:

$$J_a(\tilde{w},\tilde{v},\eta_0,r,\hat{z},\varepsilon) = \tfrac{1}{2}\{\| z_a - \hat{z}_a\|_2^2 - \gamma^2[\| \tilde{w}\|_2^2 + \| \tilde{v}\|_2^2 + \| \eta_0 - \hat{x}_0\|_R^2]\} \quad (3.12)$$

where γ, $\hat{x}_0$ and R are as in (3.6). We have the following result:

Lemma 1 Consider the systems (Σ) and (Σ_a) together with the performance indices (3.6) and (3.12). Then, we have, for any $\varepsilon > 0$,

$$\sup_{w,v,x_0,F(t)} [J(w,v,x_0,r,\hat{z})] \le \sup_{\tilde{w},\tilde{v},\eta_0} [J_a(\tilde{w},\tilde{v},\eta_0,r,\hat{z},\varepsilon)]$$

for any $\varepsilon > 0$.

Proof: For any given x_0, F, w, v, r, and $\hat{z}$ for the system (Σ) and any $\varepsilon > 0$, take

$$\eta_0 = x_0, \quad \tilde{w}(t) = \begin{bmatrix} w(t) \\ \varepsilon\gamma^{-1}FEx(t) \end{bmatrix}, \quad \tilde{v}(t) = v(t). \quad (3.13)$$

Then, for all $t \in [0,T]$,

$$\eta(t) = x(t), \quad y_a(t) = y(t), \quad z_a(t) = \begin{bmatrix} z(t) \\ \varepsilon Ex(t) \end{bmatrix},$$

which implies

$$J_a(\tilde{w},\tilde{v},\eta_0,r,\hat{z},\varepsilon) = \tfrac{1}{2}\Big\{\| z - \hat{z}\|_2^2 - \gamma^2\big[\| w\|_2^2 + \| v\|_2^2 + \| x_0 - \hat{x}_0\|_R^2\big] + \varepsilon^2\big[\| Ex\|_2^2 - \| FEx\|_2^2\big]\Big\}$$

Considering (3.5) with $\tilde{w}$ and $\tilde{v}$ as in (3.13), we obtain $J_a(\tilde{w},\tilde{v},\eta_0,r,\hat{z},\varepsilon) \ge J(w,v,x_0,r,\hat{z})$ and the result follows immediately. ∇∇∇

In view of Lemma 1, our approach for solving the robust H_∞ filtering problem involves consideration of the worst-case of the performance bound (3.12) in *lieu* of the worst-case performance (3.7). This leads to the following problem: *Find an estimate $\hat{z}_a(t), t \in [0,T]$ of the form* (3.11) *using the measurements $\{y_a(\tau), 0 \le \tau \le t\}$ and the known input signal, r, that solves the auxiliary problem*

$$\underset{\hat{z}}{\text{minimise}}\left\{\sup_{\tilde{w},\tilde{v},\eta_0}\big[J_a(\tilde{w},\tilde{v},\eta_0,r,\hat{z},\varepsilon)\big]\right\} \quad (3.14)$$

subject to (3.8) - (3.11). Note that the system (Σ_a) is parameterized by ε, which is a scaling parameter to be searched in order that an estimate, $\hat{z}$, solving (3.14) be found.

Remark 3.1 The above estimate $\hat{z}$ with y_a replaced by y will provide an estimate of z for the robust H_∞ filtering problem. Note that the case when $r(t) \equiv 0$, has been analysed in [4,6,13,15] in both the continuous and discrete-time contexts and it has been shown there that the estimate $\hat{z}$ as above guarantees the following H_∞ performance

$$\| z - \hat{z}\|_2^2 < \gamma^2\big[\| w\|_2^2 + \| v\|_2^2 + \| x_0 - \hat{x}_0\|_R^2\big] \quad (3.15)$$

for any w and v in $L_2[0,T]$ and $x_0 \in \boldsymbol{R}^n$ and for all admissible uncertainties whenever $\| w\|_2^2 + \| v\|_2^2 + \| x_0 - \hat{x}_0\|_R^2 \ne 0$.

Remark 3.2 The auxiliary problem (3.14), although it does not have any parameter uncertainty, cannot be treated via the standard H_∞ estimation techniques . This is because the estimate $\hat{z}_a(t)$ is restricted by (3.11). In other words, it is not possible in general to completely cancel the effect of the input signal r in the estimation error. Therefore, alternative solution is required.

4 Main Results

In this section, a solution to the auxiliary filtering problem introduced in Section 3 will be developed by using a game theoretic approach where the estimator plays against adversaries composed of the noise sources and the initial state.

4.1 Game Theoretic Solution to the Auxiliary Problem

The deterministic linear-quadratic game problem is to find worst case noises and initial state, $\bar{w}(\bullet),\bar{v}(\bullet)$ and η_0, respectively, and an estimate $\hat{z}(t)$ using the measurements $\{y_a(\tau), 0 \le \tau \le t\}$ and the known input signal r such that

$$\min_{\hat{z}} \max_{\bar{w},\bar{v},\eta_0} J_a(\bar{w},\bar{v},\eta_0,r,\hat{z},\varepsilon) \tag{4.1}$$

subject to (3.8) – (3.10). In view of (3.9) – (3.12), the optimization problem (4.1) can be recast into the form

$$\min_{\hat{z}} \max_{y_a,\bar{w},\eta_0} J_a = \frac{1}{2}\Big\{ \| L\eta - \hat{z}\|_2^2 + \|\varepsilon E\eta\|_2^2 - \gamma^2[\|\bar{w}\|_2^2 + \|y_a - C\eta - D_w\bar{w}\|_2^2 + \|\eta_0 - \hat{x}_0\|_R^2] \Big\} \tag{4.2}$$

where $D_w = [0 \quad \frac{\gamma}{\varepsilon}H_2]$.

Inspired by [2], the above game will be solved in two stages. We consider first the maximization of J_a with respect to $\bar{w}$ and η_0 for given $\hat{z}$ and y_a and then a min-max optimization of the resulting cost function will be performed with respect to $\hat{z}$ and y_a, respectively.

We first find the necessary conditions for optimality of η_0 and $\bar{w}$ for given $\hat{z}$ and y_a. To begin, we write the performance index J_a with the additional Lagrange multiplier $\gamma^2\lambda$, i.e. let the Hamiltonian

$$J_1 = \frac{1}{2}\Big\{ \|L\eta - \hat{z}\|_2^2 + \|\varepsilon^2 E\eta\|_2^2 - \gamma^2[\|\bar{w}\|_2^2 + \|y_a - C\eta - D_w\bar{w}\|_2^2 + \|\eta_0 - \hat{x}_0\|_R^2] + \gamma^2\lambda^T\int_0^T(-\dot{\eta} + A\eta + B_w\bar{w} + B_2 r)dt \Big\}$$

where $B_w = [B_1 \quad \frac{\gamma}{\varepsilon}H_1]$. By using standard optimization results, the maximizer strategies must satisfy

$$\eta_0 = \hat{x}_0 + R^{-1}\lambda(0) \tag{4.3}$$

$$\bar{w} = \bar{D}[B_w^T\lambda + D_w^T(y_a - C\eta)] \tag{4.4}$$

$$\dot{\lambda} = -A^T\lambda - \gamma^{-2}C_1^TC_1\eta - C^T(y_a - C\eta) + C^TD_w\bar{w} + \gamma^{-2}L^T\hat{z}, \quad \lambda(T) = 0 \tag{4.5}$$

where

$$C_1^T = [L^T \quad \varepsilon E^T], \quad \bar{D} = (I + D_w^TD_w)^{-1}.$$

Note that (3.8), (4.3) – (4.5) give rise to a linear two point boundary value problem whose solution (η^*, λ^*) is in the form

$$\eta^*(t) = \hat{\eta}(t) + Q(t)\lambda^*(t) \tag{4.6}$$

where $\hat{\eta}$ and Q are to be determined. In the above, η^* and λ^* represent the optimal values of respectively η and λ for any fixed admissible $\hat{z}$, y_a and r.

Differentiating (4.6) and considering (3.8), (4.4)-(4.5) we obtain

$$\dot{\hat{\eta}} - (A + \gamma^{-2}QC_1^TC_1)\hat{\eta} - (QC^T\tilde{D} + B_wD_w^T\tilde{D})(y_a - C\hat{\eta}) + \gamma^{-2}QL^T\hat{z} - B_2r = [-\dot{Q} + Q\tilde{A}^T + \tilde{A}Q + Q(\gamma^{-2}C_1^TC_1 - C^T\tilde{D}C)Q + B_w\bar{D}B_w^T]\lambda^* \tag{4.7}$$

where

$$\tilde{D} = I - D_w\bar{D}D_w^T, \quad \tilde{A} = A - B_wD_w^T\tilde{D}C.$$

We also note that $\tilde{D} = (I + D_wD_w^T)^{-1}$ and $\bar{D}D_w^T = D_w^T\tilde{D}$.

Since (4.7) should hold for arbitrary λ^*, we obtain

$$\dot{\hat{\eta}} = (A + \gamma^{-2}QC_1^TC_1)\hat{\eta} + \bar{B}_1(y_a - C\hat{\eta}) - \bar{B}_2\hat{z} + B_2r \tag{4.8}$$

$$\dot{Q} = \tilde{A}Q + Q\tilde{A}^T + Q(\gamma^{-2}C_1^TC_1 - C^T\tilde{D}C)Q + B_w\bar{D}B_w^T \tag{4.9}$$

with $\hat{\eta}(0) = \hat{x}_0$ and $Q(0) = R^{-1}$, where

$$\bar{B}_1 = QC^T\tilde{D} + B_wD_w^T\tilde{D}, \quad \bar{B}_2 = \gamma^{-2}QL^T$$

Hence, the optimal strategies of $\bar{w}$ and η_0 are

$$\bar{w}^* = \bar{D}[B_w^T\lambda^* + D_w^T(y_a - C\eta^*)] \tag{4.10}$$

$$\eta_0^* = \hat{x}_0 + R^{-1}\lambda^*(0) \tag{4.11}$$

Now substituting (4.10) and (4.11) in J_a we can get

$$J_a = \frac{1}{2}\Big[\|L(\hat{\eta} + Q\lambda^*) - \hat{z}\|_2^2 + \|\varepsilon E(\hat{\eta} + Q\lambda^*)\|_2^2 - \gamma^2[\|\bar{D}^{1/2}B_w^T\lambda^*\|_2^2 + \|\tilde{D}^{1/2}[y_a - C(\hat{\eta} + Q\lambda^*)]\|_2^2 + \|\lambda^*(0)\|_{R^{-1}}^2]\Big] \tag{4.12}$$

In the sequel we will perform the min-max optimization of J_2 with respect to $\hat{z}$ and y_a, respectively.

Adding to (4.12) the identically zero term

$$0 = \frac{\gamma^2}{2}\int_0^T \frac{d}{dt}[(\lambda^*)^TQ\lambda^*]dt + \frac{\gamma^2}{2}\|\lambda^*(0)\|_{R^{-1}}^2$$

$$= \frac{\gamma^2}{2}\Big[-\gamma^{-2}\|C_1Q\lambda^*\|_2^2 + \|\tilde{D}^{1/2}CQ\lambda^*\|_2^2 + \|\bar{D}^{1/2}B_w^T\lambda^*\|_2^2 + \int_0^T 2(\lambda^*)^TQ\big[\gamma^{-2}(L^T\hat{z} - C_1^TC_1\hat{\eta}) - C^T\tilde{D}(y_a - C\hat{\eta})\big]dt + \|\lambda^*(0)\|_{R^{-1}}^2\Big],$$

it can be easily derived that

$$J_a = \frac{1}{2}\Big[\|L\hat{\eta} - \hat{z}\|_2^2 + \|\varepsilon E\hat{\eta}\|_2^2 - \gamma^2\|\tilde{D}^{1/2}(y_a - C\hat{\eta})\|_2^2\Big].$$

Next, introducing the change of variables

$$\bar{z} = L\hat{\eta} - \hat{z} \tag{4.13}$$

$$\bar{v} = (y_a - C\hat{\eta}) \tag{4.14}$$

the min-max optimization of J_a with respect to $\hat{z}$ and y_a, respectively, results in the following min-max problem

$$\min_{\bar{z}} \max_{\bar{v}} J_a = \frac{1}{2}\Big[\|\varepsilon E\hat{\eta}\|_2^2 + \|\bar{z}\|_2^2 - \gamma^2\|\tilde{D}^{1/2}\bar{v}\|_2^2\Big] \tag{4.15}$$

subject to (4.9) and

$$\dot{\hat{\eta}} = \bar{A}\hat{\eta} + \bar{B}_1\bar{v} + \bar{B}_2\bar{z} + B_2r, \quad \hat{\eta}(0) = \hat{x}_0 \tag{4.16}$$

where

$$\bar{A} = A + \varepsilon^2\gamma^{-2}QE^TE.$$

We now decompose $\hat{\eta}$ as

$$\hat{\eta} = \eta_1 + \eta_r \tag{4.17}$$

$$\dot{\eta}_1 = \bar{A}\eta_1 + \bar{B}_1\bar{v} + \bar{B}_2\bar{z}, \quad \eta_1(0) = \hat{x}_0 \tag{4.18}$$

$$\dot{\eta}_r = \bar{A}\eta_r + B_2r, \quad \eta_r(0) = 0 \tag{4.19}$$

and introduce the following Riccati equation

$$-\dot{X} = \bar{A}^TX + X\bar{A} + X(\gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T - \bar{B}_2\bar{B}_2^T)X + \varepsilon^2E^TE \tag{4.20}$$

with $X(T) = 0$. Hence, it can be easily established that

$$0 = \int_0^T \frac{d}{dt}(\eta_1^TX\eta_1)dt + \hat{x}_0^TX(0)\hat{x}_0$$

$$= -\|\varepsilon E\eta_1\|_2^2 + \|\bar{z} + \bar{B}_2^TX\eta_1\|_2^2 - \gamma^2\|\tilde{D}^{1/2}(\bar{v} - \gamma^{-2}\tilde{D}^{-1}\bar{B}_1^TX\eta_1)\|_2^2 - \|\bar{z}\|_2^2 + \gamma^2\|\tilde{D}^{1/2}\bar{v}\|_2^2 + \hat{x}_0^TX(0)\hat{x}_0.$$

Adding the above zero quantity to J_a and considering (4.17) gives

$$J_a = \frac{1}{2}\Big\{\int_0^T 2\varepsilon^2\eta_r^TE^TE\eta_1dt + \|\varepsilon E\eta_r\|_2^2 + \|\zeta_z\|_2^2 - \gamma^2\|\tilde{D}^{1/2}\zeta_v\|_2^2 + \hat{x}_0^TX(0)\hat{x}_0\Big\}$$

where

$$\zeta_z = \bar{z} + \bar{B}_2^TX\eta_1 \tag{4.21}$$

$$\zeta_v = \bar{v} - \gamma^{-2}\tilde{D}^{-1}\bar{B}_1^TX\eta_1. \tag{4.22}$$

Noting that η_r is independent of both ζ_z and ζ_v, we introduce

$$\hat{J}_a = J_a - \frac{1}{2}\Big[\|\varepsilon E\eta_r\|_2^2 + \hat{x}_0^TX(0)\hat{x}_0\Big].$$

Hence, the game problem (4.15) can be converted to

$$\min_{\zeta_z} \max_{\zeta_v} \hat{J}_a = \frac{1}{2}\Big\{\int_0^T[2\varepsilon^2\eta_r^TE^TE\eta_1 + \zeta_z^T\zeta_z - \gamma^2\zeta_v^T\tilde{D}\zeta_v]dt\Big\} \tag{4.23}$$

subject to (4.9), (4.19) and

$$\dot{\eta}_1 = \hat{A}\eta_1 + \bar{B}_1\zeta_v + \bar{B}_2\zeta_z, \quad \eta_1(0) = \hat{x}_0 \tag{4.24}$$

where

$$\hat{A} = \bar{A} + (\gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T - \bar{B}_2\bar{B}_2^T)X.$$

For further solution of the min-max problem (4.23) we shall consider two cases. The first one is when $r(t)$ is known *a priori* for the whole time horizon $[0,T]$ and the optimal solution is not required to be causal with respect to r. The other case corresponds to the situation when $r(t)$ is given on line and a causal filter with respect to both y_a and r is required.

Case 1: Non-causal signal $r(\cdot)$

If $r(t)$ is known a priori for all $t \in [0,T]$, the optimal ζ_z^* and ζ_v^* can be found using standard variational techniques. We obtain

$$\zeta_z^* = -\bar{B}_2^T\theta$$

$$\zeta_v^* = \gamma^{-2}\tilde{D}^{-1}\bar{B}_1^T\theta$$

where

$$\dot{\theta} = -\hat{A}^T\theta - \varepsilon^2E^TE\eta_r, \quad \theta(T) = 0. \qquad (4.25)$$

Substituting the above result in (4.13), (4.21), (4.22) and (4.24), the optimal solution for $\hat{z}$ and $\bar{v}$ is then

$$\hat{z}^* = L\hat{\eta}^* + \bar{B}_2^T(X\eta_1^* + \theta) \qquad (4.26)$$

$$\bar{v}^* = \gamma^{-2}\tilde{D}^{-1}\bar{B}_1^T(X\eta_1^* + \theta) \qquad (4.27)$$

where

$$\hat{\eta}^* = \eta_1^* + \eta_r$$

$$\dot{\eta}_1 = \hat{A}\eta_1 + (\gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T - \bar{B}_2\bar{B}_2^T)\theta, \quad \eta_1^*(0) = \hat{x}_0$$

and η_r is as in (4.19). Moreover, the optimal value of J_a is given by

$$J_a(\bar{w}^*, \bar{v}^*, \eta_0^*, r, \hat{z}^*, \varepsilon) = \frac{1}{2}\int_0^T[\varepsilon^2\eta_r^TE^TE\eta_r$$

$$+ \theta^T(\gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T - \bar{B}_2\bar{B}_2^T)\theta]dt + \frac{1}{2}\hat{x}_0^TX(0)\hat{x}_0 + \theta^T(0)\hat{x}_0$$

Note that by defining $\alpha = \theta - X\eta_r$, it is easy to see that

$$\dot{\alpha} = -\hat{A}^T\alpha - XB_2r, \quad \alpha(T) = 0$$

and the optimal strategy $\hat{z}^*$ and $\bar{v}^*$ simplifies to

$$\hat{z}^* = (L + \bar{B}_2^TX)\hat{\eta}^* + \bar{B}_2^T\alpha \qquad (4.28)$$

$$\bar{v}^* = \gamma^{-2}\tilde{D}^{-1}\bar{B}_1^T(X\hat{\eta}^* + \alpha) \qquad (4.29)$$

where

$$\dot{\hat{\eta}}^* = \hat{A}\hat{\eta}^* + (\gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T - \bar{B}_2\bar{B}_2^T)\alpha + B_2r, \quad \hat{\eta}^*(0) = \hat{x}_0. \qquad (4.30)$$

Case 2: Causal signal $r(\cdot)$

In this case, the optimal ζ_z and ζ_v are required to be causal with respect to both y_a and r. In order for the estimator to be unbiased for all $t \in [0,T]$, we have to choose $\zeta_2(t)$ and $\zeta_v(t)$ properly. For simplicity, we assume in the sequel that the biased estimate for $r(\tau), \forall\tau \in [t,T]$ is zero. With this assumption, the unbiased estimate of $\eta_r(\tau)$ is given by

$$\hat{\eta}_r(\tau) = \Phi(\tau,t)\eta_r(t), \quad \forall\tau \geq t,$$

where $\Phi(\cdot)$ is the transition of (4.19). Then, the terminal conditions $\zeta_Z(t)$ and $\zeta_v(t)$ should be given by solving the following problem:

$$\min_{\zeta_z(\tau)}\max_{\zeta_v(\tau)}\frac{1}{2}\int_t^T[2\varepsilon^2\hat{\eta}_r^TE^TE^T\eta_1 + \zeta_z^T\zeta_z - \gamma^2\zeta_v^T\tilde{D}^{-1}\zeta_v]d\tau$$

subject to (4.9) and the following constraint:

$$\frac{d\eta_1}{d\tau} = \hat{A}\eta_1 + \bar{B}_1\zeta_v + \bar{B}_2\zeta_z, \quad t \leq \tau \leq T.$$

Note that the min max problem above is actually a non-causal estimation problem (with $\eta_r(t)$ replaced by $\hat{\eta}_r(\tau)$). Using the results in Case 1, we obtain the optimal $\zeta_z^*(t)$ and $\zeta_v^*(t)$ as follows (see (4.25)):

$$\zeta_z^*(t) = -\bar{B}_2^T\theta(t)$$

$$\zeta_v^*(t) = \gamma^{-2}\tilde{D}^{-1}B_1^T\theta(t)$$

where $\theta(t)$ is given by

$$\frac{d\theta}{d\tau} = -A^T\theta(\tau) - \varepsilon^2E^TE\Phi(\tau,t)\eta_r(t), \quad \theta(T) = 0.$$

Defining $P(t) = \int_t^T\varepsilon^2\Phi(\tau,t)E^TE\Phi(\tau,t)d\tau$, then we have

$$-\frac{dP}{d\tau} = A^TP(\tau) + P(\tau)A + \varepsilon^2E^TE, \quad P(T) = 0. \qquad (4.31)$$

Then, $\zeta_z^*(t)$ and $\zeta_v^*(t)$ reduce to

$$\zeta_z^*(t) = -\bar{B}_2^TP(t)\eta_r(t),$$

$$\zeta_v^*(t) = \gamma^{-2}\bar{D}^{-1}\bar{B}_1^TP(t)\eta_r(t).$$

Substituting these results in (4.13), (4.21), (4.22) and (4.24) gives

$$\hat{z}^* = L\hat{\eta}^* + B_2^T(X\eta_1^* + P\eta_r), \qquad (4.32)$$

$$\bar{v}^* = \gamma^{-2}\tilde{D}^{-1}\bar{B}_1^T(X\eta_1^* + P\eta_r),$$

where η_r is as in (4.19) and η_1^* is given by

$$\dot{\eta}_1^* = \hat{A}\eta_1^* + (\gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T - \bar{B}_2\bar{B}_2^T)P\eta_r, \ \eta_1^*(0) = \hat{x}_0.$$

Furthermore, the optimal value of J_a is given by

$$J_a = \frac{1}{2}\|\eta_r\|_\Sigma^2 + \frac{1}{2}\hat{x}_0^TX(0)\hat{x}_0 + \eta_r^T(0)P(0)\hat{x}(0)$$

where

$$\Sigma = \varepsilon^2E^TE - P(\gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T - \bar{B}_2\bar{B}_2^T)P. \qquad (4.33)$$

4.2 The Robust H_∞ Filter

The following robust H_∞ filters are obtained by combining Lemma 1 and the results in Section 4.1.

Theorem 1: (Noncausal signal $r(\cdot)$) Consider the system (Σ) where the exogenous signal $r(\cdot)$ is known in advance for the whole time horizon $[0,T]$ and let $\gamma > 0$ be a given scalar suppose for some $\varepsilon > 0$ there exist solutions $Q(t) = Q^T(t) > 0$ and $X(t) = X^T(t) \geq 0$ $\forall t \in [0,T]$ to (4.9) and (4.20), respectively. Then the filter

$$\hat{z} = (L + \bar{B}_2^TX)\hat{x} + \bar{B}_2^T\alpha, \qquad (4.34)$$

$$\dot{\alpha} = -\hat{A}^T\alpha - XB_2r, \quad \alpha(T) = 0 \qquad (4.35)$$

$$\dot{\hat{x}} = A_e\hat{x} + \bar{B}_1(y - C\hat{x}) - \bar{B}_2\bar{B}_2^T\alpha + B_2r, \quad \hat{x}(0) = \hat{x}_0 \qquad (4.36)$$

where

$$A_e = \bar{A} - \bar{B}_2\bar{B}_2^TX \qquad (4.37)$$

will guarantee the performance

$$\|z - \hat{z}\|_2^2 < \gamma^2[\|w\|_2^2 + \|v\|_2^2 + \|x_0 - \hat{x}_0\|_R^2] + \hat{x}_0^TX(0)\hat{x}_0$$

$$+ 2\alpha^T(0)\hat{x}_0 + \|\varepsilon E\eta_r\|_2^2 + \int_0^T\theta^T(\gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T - \bar{B}_2\bar{B}_2^T)\theta dt \qquad (4.38)$$

where $\theta(\cdot)$ is as in (4.25).

Remark 4.1 Note that when no *a priori* estimate of the initial state is assumed, i.e. $\hat{x}_0 = 0$, the filter (4.34) – (4.36) achieves the robust performance

$$\|z - \hat{z}\|_2^2 < \gamma^2[\|w\|_2^2 + \|v\|_2^2 + \|x_0\|_R^2] + \|\varepsilon E\eta_r\|_2^2$$

$$+ \int_0^T\theta^T(\gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T - \bar{B}_2\bar{B}_2^T)\theta dt$$

The above suggests that a way of reducing the effect of the input $r(\cdot)$ on the performance measure is to use a 'small' $\varepsilon > 0$. It should be emphasized that the solution of the two Riccati differential equations (4.9) and (4.20) depends on ε and thus it sometimes cannot be made arbitrarily small. Furthermore, a small ε may also induce a high bandwidth for the estimation error, making the estimate more vulnerable to high frequency noises. Also, observe that when (3.1) has no deterministic input, i.e. $r(t) \equiv 0$, $\eta_r(\cdot), \alpha(\cdot)$ and $\theta(\cdot)$ will all be identically zero over $[0,T]$. In this case it is easy to see that the filter (4.34) – (4.36) recovers the robust H_∞ filter of [7] and provides the robust performance.

$$\|z - \hat{z}\|_2^2 < \gamma^2[\|w\|_2^2 + \|v\|_2^2 + \|x_0\|_R^2] \qquad (4.39)$$

Remark 4.2 When there is no parameter uncertainties, i.e. E=0, (4.20) yields $X(t)$=0, $\forall t \in [0,T]$. This implies that both $\alpha(\cdot)$ and $\theta(\cdot)$ are identically zero over $[0,T]$ and the filter (4.34) – (4.36) recovers the standard H_∞ filter with a known deterministic input $r(\cdot)$. Moreover, (4.38) reduces to the H_∞ performance of the standard filter, namely

$\| z - \hat{z} \|_2^2 < \gamma^2[\| w \|_2^2 + \| \nu \|_2^2 + \| x_0 - \hat{x}_0 \|_R^2]$.

We now present a robust H_∞ filter which is causal with respect to both the measurements, $y(\bullet)$ and the deterministic input $r(\bullet)$.

Theorem 2: (Causal signal $r(\bullet)$) Consider the system (Σ) where the exogenous signal $r(\bullet)$ is causally measured and let $\gamma > 0$ be a given scalar suppose for some $\varepsilon > 0$ there exist solutions $Q(t) = Q^T(t) > 0,\ X(t) = X^T(t) \geq 0,\ P(t) = P^T(t) > 0 \forall t \in [0,T]$ to (4.9), (4.20), (4.31), respectively. Then the filter is given by

$$\hat{z} = (L + B_2^T X)\hat{x} + (P - B_2^T X)\eta_r \tag{4.40}$$

$$\dot{\hat{x}} = A_e\hat{x} + \overline{B}_1(y - C\hat{x}) + \overline{B}_2\overline{B}_2^T(X - P)\eta_r, \tag{4.41}$$

where η_r and A_e are as in (4.19) and (4.37), respectively, and the guaranteed performance is

$$\begin{aligned} \| z - \hat{z} \|_2^2 < \gamma^2[\| w \|_2^2 + \| \nu \|_2^2 + \| x_0 - \hat{x}_0 \|_2^P] \\ + \hat{x}_0^T X(0)\hat{x}_0 + \| \eta_r \|_\Sigma^2 + z\eta_r^\tau(0)P(0)\hat{x}_0 \end{aligned} \tag{4.42}$$

where Σ is given by (4.33).

5 An Example

We consider the example in Section 2 and will show that a filter designed by using the proposed robust H_∞ filter method will yield improved robustness properties, compared to the Kalman and "nominal" H_∞ filters based on the nominal process.

For simplicity, we assume that $r(t)$ is a unit step input which is known *a priori* and we consider the infinite horizon. Take $\hat{x}(0) = 0, R = 0, \gamma = 1.1$ and $\varepsilon = 0.1$. For the infinite horizon, equations (4.9) and (4.20), become

$$\bar{A}Q + Q\bar{A}^T + Q(\gamma^{-2}C_1^T C_1 - C^T\bar{D}C)Q + B_w\overline{D}B_w^T = 0 \tag{5.1}$$

and

$$\overline{A}^T X + X\overline{A} + X(\gamma^{-2}\overline{B}_1\bar{D}^{-1}\overline{B}_1^T - \overline{B}_2\overline{B}_2^T)X + \varepsilon^2 E^T E = 0. \tag{5.2}$$

Solutions to these two equations give

$$Q = \begin{bmatrix} 3.7197 & 0.0794 \\ 0.0794 & 0.0235 \end{bmatrix} > 0 \tag{5.3}$$

$$X = \begin{bmatrix} 0.0119 & -0.0006 \\ -0.0006 & 0.0113 \end{bmatrix} > 0 \tag{5.4}$$

Since r is a constant, (4.35) simplifies to

$$a = -(\hat{A}^T)^{-1}XB_2 r \tag{5.5}$$

Therefore, the filter in (4.34) – (4.36) becomes

$$\dot{\hat{x}} = A_e\hat{x} + \overline{B}_1(y - C\hat{x}) + B_r\, r \tag{5.6}$$

$$\hat{z} = (L + \overline{B}_2^T X)\hat{x} + D_r\, r \tag{5.7}$$

where

$$B_r = [I + \overline{B}_2\overline{B}_2^T(A^T)^{-1}X]B_2 \tag{5.8}$$

and

$$D_r = -\overline{B}_2^T(\hat{A}^T)^{-1}XB_2 \tag{5.9}$$

Computation of the matrices above yields

$$\dot{\hat{x}} = \begin{bmatrix} 0.5054 & -1.1170 \\ 0.8501 & -0.5347 \end{bmatrix}\hat{x} + \begin{bmatrix} 7.9387 \\ 2.3540 \end{bmatrix}(y - [0 \quad 100]\hat{x}) + \begin{bmatrix} 64.2972 \\ -1.690 \end{bmatrix} r \tag{5.10}$$

$$\hat{z} = [0.0770 \quad 100.0179]\ \hat{x} + 0.8692r \tag{5.11}$$

or equivalently,

$$\hat{z}(s) = G_y(s)y(s) + G_r(s)r(s) \tag{5.12}$$

where

$$G_y(s) = \frac{236.055 + 794.1375}{s^2 + 236.43645 + 795.0906} \tag{5.13}$$

$$G_r(s) = \frac{0.1242s^2 + 5.9053s + 1049.3}{s^2 + 236.43645 + 795.0906} \tag{5.14}$$

Assuming that δ is constant, $y(s)$ and $z(s)$ can be written as

$$y(s) = G_{yw}(s)w(s) + \nu(s) + G_{yr}(s)r(s) \tag{5.15}$$

$$z(s) = G_{zw}(s)w(s) + G_{zr}(s)r(s) \tag{5.16}$$

where all the transfer functions are related to δ and $G_{zw}(s) = G_{yw}(s)$, $G_{zr}(s) = G_{yr}(s)$. The estimation error is given by

$$e(s) = z(s) - \hat{z}(s) = G_{ew}(s)w(s) + G_{ev}(s)\nu(s) + G_{er}(s)\ r(s) \tag{5.17}$$

where

$$G_{ew}(s) = G_{zw}(s) - G_y(s)G_{yw}(s) = (1 - G_y(s))G_{yw}(s); \tag{5.18}$$

$$G_{ev}(s) = -G_y(s); \tag{5.19}$$

$$G_{yr}(s) = G_{zr}(s) - G_y(s)G_{yr}(s) - G_r(s). \tag{5.20}$$

The plots of $10\log(|G_{ew}(j\omega)|^2 + |G_{ev}(j\omega)|^2|$ and $20\log(|G_{er}(j\omega)|)$ are shown in Figure 4 for $g = 10$ and different values of δ. Obviously this filter performs far better than the Kalman filter and the "nominal" H_∞ filter in Section 2. The improvement for $G_{er}(s)$ is more significant in low frequency range than mid frequencies. This is because our design was done for constant r and steady state.

It is worth noting that the function $G_{ew}(s)$ is actually identical to that of the robust filter (2.8) – (2.9) in Section 2 when r is not considered. This is natural because the auxiliary problem (3.14) is identical to the one in [6] when $r \equiv 0$, $\hat{x}_0(0)$, and $R \equiv 0$.

For comparison purposes, we finally consider an alternative method for designing G_{er} which we call the *cancellation method*. The idea is simply as follows: Because $G_r(s)$ only effects $G_{er}(s)$, it is therefore obvious that the role of $G_r(s)$ is to minimize $G_{er}(s)$ in certain sense. One possibility is to choose $G_r(s)$ such that $G_{er}(s)$ is completely cancelled (i.e., $G_{er}(s) \equiv 0$) for the nominal process. However, we show using the example above that such a cancellation method may not give an optimal worst-case solution. This observation is made from the dotted lines in Figure 5 which correspond to the spectrum of $G_{er}(s)$ designed by the cancellation method at different δ values.

6 Conclusion

A new robust H_∞ filtering method has been developed via a game theoretic approach for signal estimation of processes with both parametric uncertainty and a known input signal in the finite horizon setting. The solution to the robust H_∞ filtering involves two Riccati differential equations with a scaling parameter ε, which can be solved as for the standard finite horizon H_∞ filtering problems. The robust H_∞ filter contains two components, one for process and measurement noise attenuation and another for bias attenuation. The former turns out to be the same as in [6] for the case without known input signal, but the latter depends on the *a priori* information on the known input signal. It should be noted that when the parameter uncertainty vanishes, our robust H_∞ filter simplifies to the standard H_∞ filtering, and in particular, the bias in the estimation error due to the known input can be completely cancelled.

We expect that this new method can also be applied to the following problems: i) robust L_2 filtering with parametric uncertainty and known input signal; ii) robust tracking control for systems with parametric uncertainty; and iii) discrete-time problems.

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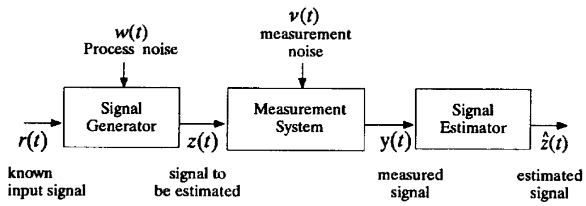

Figure 1: Signal Generating System

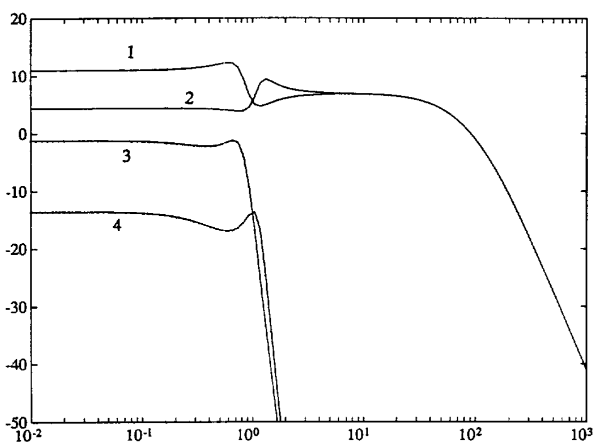

Figure 2: Kalman Filter for Process (2.1) – (2.3) with $g = 1$
curve 1: $10\log(|G_{ew}(j\omega)|^2 + |G_{ev}(j\omega)|^2)$ for $\delta = \bar{\delta}$
curve 2: $10\log(|G_{ew}(j\omega)|^2 + |G_{ev}(j\omega)|^2)$ for $\delta = -\bar{\delta}$
curve 3: $20\log(|G_{er}(j\omega)|)$ for $\delta = \bar{\delta}$
curve 4: $20\log(|G_{er}(j\omega)|)$ for $\delta = -\bar{\delta}$

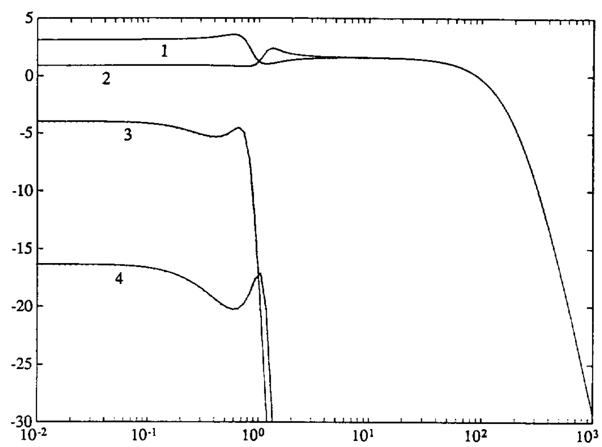

Figure 3: "Nominal" H_∞ Filter for Process (2.1) – (2.3) with $g = 2$
curve 1: $10\log(|G_{ew}(j\omega)|^2 + |G_{ev}(j\omega)|^2)$ for $\delta = \bar{\delta}$
curve 2: $10\log(|G_{ew}(j\omega)|^2 + |G_{ev}(j\omega)|^2)$ for $\delta = -\bar{\delta}$
curve 3: $20\log(|G_{er}(j\omega)|)$ for $\delta = \bar{\delta}$
curve 4: $20\log(|G_{er}(j\omega)|)$ for $\delta = -\bar{\delta}$

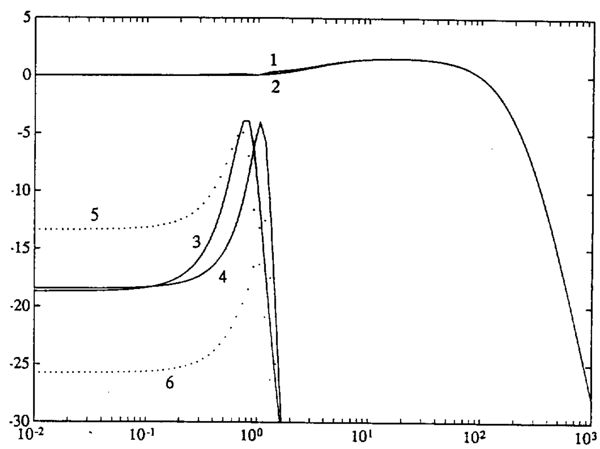

Figure 4: Robust H_∞ Filter for Process (2.1) – (2.3) with $g = 10$
curve 1: $10\log(|G_{ew}(j\omega)|^2 + |G_{ev}(j\omega)|^2)$ for $\delta = \bar{\delta}$
curve 2: $10\log(|G_{ew}(j\omega)|^2 + |G_{ev}(j\omega)|^2)$ for $\delta = -\bar{\delta}$
curve 3: $20\log(|G_{er}(j\omega)|)$ for $\delta = \bar{\delta}$
curve 4: $20\log(|G_{er}(j\omega)|)$ for $\delta = -\bar{\delta}$
curve 5: $20\log(|G_{er}(j\omega)|)$ for $\delta = \bar{\delta}$
designed with the cancellation method
curve 6: $20\log(|G_{er}(j\omega)|)$ for $\delta = -\bar{\delta}$
designed with the cancellation method