Robust Output Feedback Stabilization for Two New Uncertainty Structures

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***Abstract*—In this technical note, we provide two new uncertainty structures for linear systems which admit robust output feedback stabilization. These structures are characterized by having poles or zeros at the origin. Our method is motivated by the fact that high-gain control results available to robust output feedback stabilization of an uncertain minimum phase plant $G(s,q)$ do not readily extend to plants of the form $s^m G(s,q)$. We also show that upper and lower triangular uncertainty structures in the state space, considered by many authors in the context of recursive construction of Lyapunov functions and state feedback controls, are special cases of the structures considered in this technical note.**

***Index Terms*—Backstepping, non-minimum phase systems, output feedback control, robust stabilization, uncertain systems.**

I. Introduction

The main results of this technical note bear on the large body of literature involving construction of robustly stabilizing controllers for systems which include uncertain parameters or nonlinear elements with known bounds. A principal motivation for this technical note is the fact that results for robust stabilization via state feedback do not readily admit modifications to handle the output feedback case; e.g., for state feedback solutions, see [1]–[5] for linear systems and [6]–[10] for nonlinear systems. As far as the literature on robust output feedback stabilization is concerned, results for minimum phase plants are the benchmark against which the results this technical note can be compared; e.g., see [11]–[13] for linear systems and [14], [15] for nonlinear systems.

The main objective in this technical note is to extend robust stabilization results for the output feedback case. We are primarily concerned with uncertain transfer functions of the following form:

$$G(s,q)=\frac{s^m N(s,q)}{D(s,q)} \tag{1}$$

where $N(s,q)$ and $D(s,q)$ are uncertain polynomials depending on an uncertain parameter vector q with a known bounding set Q. Notice that if $N(s,q)$ is robustly Hurwitz and $m=0$, then the plant is minimum phase and is readily stabilizable via output feedback, which can be achieved using a high-gain controller. On the other hand, with $m>0$, it turns out that the robust stabilization problem is no longer straightforward and a naive high-gain approach will fail.

Despite the simple appearance of $G(s,q)$ above, we show in this technical note it indeed covers a large class of new state space systems. More specifically, we introduce a class of uncertain systems in the state space called *Stepwise Augmentation Structures*. For such systems, in the spirit of papers such as [2], [4], [5], [16], a recursive procedure, involving a sequence of up and down augmentations, leads to construction of a robust stabilizing controller. In contrast to this existing literature, however, the possibility of zeros at the origin rules out the naive use of a high gain controller. Finally, it is also important to note that these augmentation structures also include a number of classes of systems previously addressed in the robust stabilization literature. This includes the well-known upper and lower-triangular structures.

By way of illustration, we now describe a system which is not readily addressed by existing literature but fits into the framework of this technical note. Indeed, the state variable system

$$\begin{aligned}\dot{x_1}&=-q_1x_1+x_2+q_2x_4;\\ \dot{x_2}&=x_3;\\ \dot{x_3}&=q_3x_3+x_4;\\ \dot{x_4}&=-x_1+u;\\ y&=x_3\end{aligned} \tag{2}$$

with uncertain parameters $q\doteq(q_1,q_2,q_3)$, has neither lower triangular nor upper triangular structure. In addition, although we do not show it here, it is actually not possible to transform this system to either a lower-triangular form or a upper-triangular form via a parameter-independent state transformation. For this system, the transfer function is readily verified to be of the form

$$G(s,q)=\frac{s(s+q_1)}{s^4+(q_3-q_1)s^3+(q_2-q_1q_3)s^2-q_2q_3s+1}. \tag{3}$$

Now, with arbitrarily large uncertainty bounds $q_i^-\le q_i\le q_i^+$ with $q_1^->0$, the non-minimum phase zero at $s=0$ is problematic for high-gain robust output feedback stabilization. However, the results given in this technical note will show that this system is robustly stabilizable.

Analogous to the case above, we also consider the case when the transfer function has a sign-invariant low-frequency gain (i.e., the value of $G(s,q)$ when $s\to 0$) and is of the form

$$G(s,q)=\frac{N(s,q)}{s^m D(s,q)} \tag{4}$$

with $D(s,q)$ robustly Hurwitz. While zero feedback suffices when $m=0$, the case with $m>0$ and non-minimum phase becomes challenging.

II. Preliminaries

Throughout this technical note, we consider uncertain polynomials $f(s,q)$ whose coefficients depend *continuously* on a vector q of uncertain parameters. We also assume that q belongs to a *compact* bounding set Q. We denote $I_f(q)$ to be the *maximum degree* of $f(s,q)$. The maximum degree $I_f(q)$ is said to be *invariant* if $I_f(q)=I_f$ for some constant I_f. A scalar function $a(q)$ is said to be *positively invariant* if $a(q)>0$ for all $q\in Q$. It is clear that if $a(q)$ depends on q continuously and Q is compact, then $a(q)$ being positively invariant implies $a(q)>\underline{a}>0$ for some constant $\underline{a}$. *Negative invariance* is defined in a similar fashion, and *sign invariance* means either positive invariance or negative invariance. An uncertain polynomial $f(s,q)$ is said to be *robustly Hurwitz* (over Q) if its zeros are all inside the open left-half plane for all $q\in Q$.

The rest of this section introduces three preliminary results. The first one is quoted from [16].

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Lemma 1: Given two uncertain polynomials $f(s,q)$ and $g(s,q)$ with $q \in Q$, suppose both uncertain polynomials have invariant maximum degrees, their leading coefficients are positively invariant, $f(s,q)$ is robustly Hurwitz and $I_g \le I_f + 1$. Then, there exists $\alpha > 0$ such that $f(s,q) + \alpha g(s,q)$ is robustly Hurwitz.

The result above is based on the fact that the zeros of a polynomial depends on its coefficients continuously. As $\alpha \to 0$, the roots of $f(s,q) + \alpha g(s,q)$ approach those of $f(s,q)$ and, if $I_g = I_f + 1$, another root at $-\infty + j0$. Therefore, there must exist $\alpha > 0$ such that $f(s,q) + \alpha g(s,q)$ is robustly Hurwitz. An obvious generalization of the result above is a reversely ordered version of it [16], as stated below, obtained using the simple fact that an nth order polynomial $f(s)$ is Hurwitz implies that its reversely ordered version $s^n f(s^{-1})$ is also Hurwitz:

Lemma 2: Given two uncertain polynomials $f(s,q)$ and $g(s,q)$ with $q \in Q$, suppose both uncertain polynomials have invariant maximum degrees, their zeroth degree coefficients are both positively invariant, $f(s,q)$ is robustly Hurwitz and $I_g \le I_f + 1$. Then, there exists $\alpha > 0$ such that $sf(s,q) + \alpha g(s,q)$ is robustly Hurwitz.

In both results above, the degree of a robustly Hurwitz uncertain polynomial is extended by at most one. The following result is a case where the degree can be extended by two:

Lemma 3: Given two uncertain polynomials $f(s,q)$ and $g(s,q)$ and a compact bounding set Q, suppose both uncertain polynomials have invariant maximum degrees, $f(s,q)$ has a positively invariant leading coefficient, $g(s,q)$ has positively invariant leading coefficient and zeroth degree coefficient, $f(s,q)$ is robustly Hurwitz and $I_g = I_f + 2$. Then, there exists $\alpha > 0$ such that $sf(s,q) + \alpha g(s,q)$ is robustly Hurwitz.

Proof: It is well known [17] that a given polynomial is Hurwitz if and only if the associated Hurwitz matrix is such that all the principal minors are positive. Denoting $n = I_f$, the Hurwitz matrix for $sf(s,q) + \alpha g(s,q)$ is given by

$$H_1(\alpha,q) = \begin{bmatrix} H_{1,0}(q) + \alpha H_{1,1}(q) & 0 \\ H_{1,2}(q,\alpha) & \alpha g_0(q) \end{bmatrix}$$

where $H_{1,0}(q)$ is the Hurwitz matrix of $f(s,q)$ when viewed as an $(n+1)$-th order polynomial (i.e., the polynomial $a_{n+1}s^{n+1} + f(s,q)$ with $a_{n+1} = 0$). Also, in the expression above, $H_{1,1}(q)$ is the part of the Hurwitz matrix for $g(s,q)$ with the last row and column deleted. Further examination shows that $H_{1,0}(q)$ has the structure

$$H_{1,0}(q) = \begin{bmatrix} a_n(q) & [a_{n-2}(q)\cdots] \\ 0 & H_0(q) \end{bmatrix}$$

where $a_0(q), a_1(q), \cdots, a_n(q)$ are the coefficients of $f(s,q)$ and $H_0(q)$ is its Hurwitz matrix.

In view of the structural properties above, we claim that all the leading principal minors of $H_1(q,\alpha)$ are positively invariant for sufficiently small $\alpha > 0$. To prove the claim, we consider the highest order minor $\det H_1(q,\alpha)$, noting that a similar proof applies to the other lower order minors as well. Indeed, we write $\det H_1(q,\alpha) = \alpha g_0(q)(a_n(q)\det H_0(q) + o(q,\alpha))$, where the term $o(q,\alpha)$ vanishes uniformly in q as $\alpha \to 0$. Now using the properties of $f(s,q)$, we know that $\det H_0(q)$ and $a_n(q)$ are both positively invariant. Therefore, for suitably small $\alpha > 0$, $\det H_1(q,\alpha)$ is positively invariant. In view of this claim, we now conclude that $sf(s,q) + \alpha g(s,q)$ is robustly Hurwitz for suitably small $\alpha > 0$. □

III. Stabilizable Transfer Function Structures

In this section, we provide robust stabilization results for the two transfer function structures discussed in Section I.

A. Pseudo-Minimum Phase Uncertain Plants

Recalling the discussion in Section I, we consider a nth order proper transfer function of the form (1) with invariant relative degree r. We assume that $m \ge 0$, $N(s,q)$ is an $(n-m-r)$-th order robustly Hurwitz polynomial with a positively invariant zeroth degree coefficient and $D(s,q)$ is a non-Hurwitz uncertain polynomial with a positively invariant highest degree coefficient. When $m > 0$, it is further assumed that the zeroth order coefficient, $d_0(q)$, of $D(s,q)$, is sign-invariant so that there is no unstable zero-pole cancelation. Since the numerator of the plant has its zeros at the origin and in the open left half plane, we refer to the plant as *pseudo-minimum phase*.

We apply a proper compensator $C(s) = N_c(s)/D_c(s)$ and the objective is to select the coefficients of $N_c(s)$ and $D_c(s)$ to assure that the resulting closed-loop polynomial

$$f(s,q) = s^m N(s,q)N_c(s) + D(s,q)D_c(s) \tag{5}$$

is robustly Hurwitz. When such a compensator exists, the system is said to be *robustly stabilizable via output feedback*.

Theorem 1: The pseudo-minimum phase uncertain plant $G(s,q)$ is robustly stabilizable via output feedback. Furthermore, a robustly stabilizing proper controller $C(s) = N_c(s)/D_c(s)$ can be chosen to be minimum phase and satisfying the following conditions:

(i) When $m = 0$

$$\deg N_c(s) = \deg D_c(s) = r - 1.$$

(ii) When $m > 0$ and $d_0(q) > 0$

$$\deg N_c(s) = m + r - 2;$$
$$\deg D_c(s) = \max\{m-1, m+r-2\}.$$

(iii) When $m > 0$ and $d_0(q) < 0$

$$\deg N_c(s) = \deg D_c(s) = m + r - 1.$$

In addition, the controller $C(s)$ can be designed using the following procedure:

Step 1) Choose $N_c(s)$ to be any Hurwitz polynomial with degree as given above and take

$$f_0(s,q) = N(s,q)N_c(s).$$

Step 2) If $m = 0$, for $k = 1, 2, \ldots, r$, choose $\alpha_k > 0$ such that

$$f_k(s,q) = f_{k-1}(s,q) + \alpha_k s^{k-1} D(s,q)$$

is robustly Hurwitz (by applying Lemma 1). If $m > 0$, take

$$\bar{D}(s,q) = \begin{cases} D(s,q) & \text{if } d_0(q) > 0; \\ D(s,q)(s-1) & \text{otherwise} \end{cases}$$

and, for $k = 1, 2, \ldots, m$, choose $\alpha_k > 0$ such that

$$f_k(s,q) = sf_{k-1}(s,q) + \alpha_k \bar{D}(s,q)$$

is robustly Hurwitz (by applying Lemma 3 for $k = 1$ and Lemma 2 for $k > 1$). If $r > 1$, continue with, for $k = m+1, m+2, \ldots, m+r-1$, choosing $\alpha_k > 0$ such that

$$f_k(s,q) = f_{k-1}(s,q) + \alpha_k s^{k-1} \bar{D}(s,q)$$

is robustly Hurwitz (by applying Lemma 1).

Step 3) If $r \leq 1$, take

$$\bar{D}_c(s) = \alpha_1 s^{m-1} + \alpha_2 s^{m-2} + \cdots + \alpha_m.$$

If $r > 1$, take

$$\bar{D}_c(s) = \alpha_1 s^{m-1} + \alpha_2 s^{m-2} + \cdots + \alpha_m + \alpha_{m+1} s^m + \cdots + \alpha_{m+r-1} s^{m+r-2}.$$

Then, $D_c(s)$ is given by

$$D_c(s) = \begin{cases} \bar{D}_c(s)(s-1) & \text{if } m > 0 \text{ and } d_0(q) > 0; \\ \bar{D}_c(s) & \text{otherwise.} \end{cases}$$

Proof: It is easy to verify that the specified dimensions guarantee that the controller $C(s)$ is proper. Hence, it suffices to show that the $f_k(s,q)$, constructed via the procedure above, is robustly Hurwitz for all k. We first consider the case of $m = 0$. Here, $f_0(s,q)$ has degree $n-1$ because $N_c(s)$ has degree $r-1$. Therefore, applying Lemma 1 yields $f_1(s,q) = f_0(s,q) + \alpha_1 D(s,q)$ robustly Hurwitz for some $\alpha_1 > 0$, and $f_1(s,q)$ has degree n. A similar argument applies to $k > 1$. Next, we consider the case of $m > 0$ and $d_0(q) > 0$. Here, $f_0(s,q)$ has degree $n-2$ because $N_c(s)$ has degree $m+r-2$. Also note that $\bar{D}(s,q) = D(s,q)$ has degree n with positively invariant leading coefficient and zeroth degree coefficient. Hence, applying Lemma 3 yields $f_1(s,q) = sf_0(s,q) + \alpha_1 \bar{D}(s,q)$ robustly Hurwitz for some $\alpha_1 > 0$, and its order is still n. For $k = 2$, $f_2(s,q) = sf_1(s,q) + \alpha_2 \bar{D}(s,q)$. Because $f_1(s,q)$ has degree n with positively invariant leading coefficient and $\bar{D}(s,q)$ has degree n with positively zeroth degree coefficient, Lemma 2 ensures that $f_2(s,q)$ is robustly Hurwitz for some $\alpha_2 > 0$. A similar argument applies to $2 < k \leq m$. It is easy to see that $f_m(s,q)$ has degree $n+m-1$. If $r > 1$, further construction of $f_k(s,q)$ is needed to ensure that the resulting controller is proper. Since $f_{m+1}(s,q) = f_m(s,q) + \alpha_{m+1} s^m \bar{D}(s,q)$, it is robustly Hurwitz for some $\alpha_{m+1} > 0$ by Lemma 1, and this argument continues for $k > m+1$. The case of $m > 0$ and $d_0(q) < 0$ is converted back to the previous case by noting that the zeroth degree coefficient of $\bar{D}(s,q)$ is positively invariant. The only difference here is that the degree of $N_c(s)$ must be increased by one to $m+r-1$ because of the term $(s-1)$. This implies that the degree of $D_c(s,q)$ must be increased by one as well because $C(s)$ must be proper. Hence, $\deg D_c(s) = \max\{m-1, m+r-1\}$. Since $r \geq 0$, the above simplifies down to $\deg D_c(s) = m+r-1$ as specified in the theorem.□

Example 1: To illustrate how Theorem 1 works, we revisit the motivating example given in Section I. Recall the uncertain transfer function $G(s,q)$ in (3). We assume $q_1^- = 1, q_1^+ = 2, q_2^- = q_3^- = -1$ and $q_2^+ = q_3^+ = 1$, noting that the analysis to follow could equally well be carried out with arbitrarily large uncertainty bounds with the proviso that $q_1^- > 0$. Since $m = 1, r = 2$ and $d_0(q) = 1 > 0$, using Theorem 1, the controller is chosen to be of the form

$$C(s) = \frac{N_c(s)}{D_c(s)} = \frac{n_{c1}s + n_{c0}}{d_{c1}s + d_{c0}}.$$

For illustrative purposes, we take $N_c(s) = s+1$. Accordingly

$$\begin{aligned} f_0(s,q) &= (s+1)(s+q_1); \\ f_1(s,q) &= sf_0(s,q) + \alpha_1 D(s,q) \\ &= s(s+1)(s+q_1) \\ &\quad + \alpha_1 \left(s^4 + (q_3 - q_1)s^3 \right. \\ &\qquad \left. + (q_2 - q_1 q_3)s^2 - q_2 q_3 s + 1\right). \end{aligned}$$

It is straightforward to verify, using Routh-Hurwitz criterion, that $\alpha_1 = 0.25$ will make $f_1(s,q)$ robustly Hurwitz. Now, since $r = 2 > 1$, we continue with

$$f_2(s,q) = f_1(s,q) + \alpha_2 s D(s,q).$$

Similarly, it is found using the Routh-Hurwitz criterion that $\alpha_2 = 0.01$ will make $f_2(s,q)$ robustly Hurwitz. It follows that $D_c(s) = \alpha_2 s + \alpha_1 = 0.01s + 0.25$ and the resulting robustly stabilizer is

$$C(s) = \frac{s+1}{0.05s + 0.25}.$$

Remark 1: As seen in the example above, our robust stabilizer design involves verification of robust Hurwitz property for a given uncertain polynomial. Although we used the standard Routh-Hurwitz test in the example, such a test is inefficient for high order polynomials. For better robust stability tests, the reader is referred to [21] and [22] where various techniques are available for polynomials with parametric uncertainties.

B. Pseudo-Stable Uncertain Plants

Recalling the discussion in Section I, we consider a proper transfer function of the form

$$G(s,q) = \frac{N(s,q)}{s^m D(s,q)}$$

where $m > 0$, $D(s,q)$ is an n-th order robustly Hurwitz polynomial. Without loss of generality, we assume that $D(s,q)$ has positively invariant coefficients. Finally, the uncertain polynomial $N(s,q)$ is assumed to have a sign-invariant zeroth degree coefficient. The degree, $v(q)$, of $N(s,q)$ is allowed to vary with q, provided that $G(s,q)$ remains proper. Since the denominator has all its roots at the origin and in the open left half plane, we refer to the plant as being *pseudo-stable*.

Theorem 2: The pseudo-stable uncertain plant $G(s,q)$ above is robustly stabilizable via output feedback. Furthermore, a robustly stabilizing proper controller $C(s) = N_c(s)/D_c(s)$ can be chosen to be stable satisfying

$$\deg N_c(s) = \deg D_c(s) = m - 1$$

and designed using the following procedure:

Step 1) Choose $D_c(s)$ to be any $(m-1)$-th order Hurwitz polynomial and take $f_0(s,q) = D(s,q)D_c(s,q)$.

Step 2) For $k = 1, 2, \ldots, m$, choose $\alpha_k > 0$ such that

$$f_k(s,q) = sf_{k-1}(s,q) + \alpha_k N(s,q) S_N$$

is robustly Hurwitz (by applying Lemma 2), where S_N is the sign of the zeroth degree coefficient of $N(s,q)$.

Step 3) Form

$$N_c(s) = S_N(\alpha_1 s^{m-1} + \alpha_2 s^{m-2} + \cdots + \alpha_m).$$

Proof: It is easy to verify that $C(s)$ is proper. Hence, it suffices to show that $f_k(s,q)$ constructed in the procedure above is robustly Hurwitz for all k. Note that $f_0(s,q)$ has degree $n+m-1$, thus $\deg sf_0(s,q) \geq \deg N(s,q)$. Applying Lemma 2 ensures that $f_1(s,q)$ is robustly Hurwitz for some $\alpha_1 > 0$. The argument above can continue for $k > 1$ since the degree of $f_k(s,q)$ increases as k increases, implying that $\deg sf_k(s,q) \geq \deg N(s,q)$ continues to hold for $k \geq 1$. □

Example 2: We consider the following uncertain system: tac-fu-2036296 55tac01-hbai-2036301 55tac01-jlim-2036304 55tac01-abartoszewicz-2036305 55tac01-aamthor-2036307 tac-conte-2037249

$$\begin{aligned}\dot{x}_1 &= x_2 + q_1 x_3;\\ \dot{x}_2 &= x_3;\\ \dot{x}_3 &= q_3 x_3 + x_4;\\ \dot{x}_4 &= -q_2 x_3 + u;\\ y &= x_3\end{aligned}$$

with uncertainty bounds $q_i^- \leq q_i \leq q_i^+$ for $i = 1, 2$.

Computing the transfer function

$$G(s,q) = \frac{q_1 s + 1}{s^2(s+q_2)}$$

for this system, it is immediate that with $q_2^- > 0$ and arbitrarily large bounds q_1^-, q_1^+ and q_2^+, this pseudo-stable system is in the correct form for application of Theorem 2. Therefore, this system is robustly stabilizable with a compensator of order $m - 1 = 1$. Accordingly, we take

$$C(s) = \frac{N_c(s)}{D_c(s)} = \frac{n_{c1}s + n_{c0}}{d_{c1}s + d_{c0}}$$

To illustrate the construction of a robustly stabilizing controller, we assume uncertainty bounds $q_1^- = -1$, $q_1^+ = 1$, $q_2^- = 1$ and $q_2^+ = 2$, and follow the procedure specified in Theorem 2. We select $D_c(s) = s + 1$ begin the recursive design with

$$f_0(s,q) = (s+q_2)(s+1).$$

Now making the identification $\alpha_1 = n_{c1}$ and forming

$$f_1(s,q) = sf_0(s,q) + n_{c1}(q_1 s + 1)$$

It is straightforward to determine that $n_{c1} = 0.5$ will suffice by inspection using Routh-Hurwitz criterion. With this choice of n_{c1}, we obtain

$$f_1(s,q) = s^4 + (1+q_2)s^3 + (0.5q_1 + q_2)s^2 + 0.5s.$$

Now making the identification $\alpha_2 = n_{c0}$, we have

$$f_2(s,q) = sf_1(s,q) + n_0(q_1 s + 1)$$

It is straightforward to determine that $n_{c0} = .05$ will suffice by inspection. The corresponding robustly stabilizing controller is given by

$$C(s) = \frac{0.5s + 0.05}{s+1}.$$

Remark 2: In Theorems 1–2, we specified the order of the stabilizing controller. Here, two examples are given to show that stabilizing controllers may not exist in general if the order is lower than those given in the theorems. For the pseudo-minimum phase case, we consider the uncertain plant

$$G(s,q) = \frac{s^2}{s^3 + q}$$

with uncertainty bound $1 \leq q \leq 2$. Since $m = 2$ and $r = 1$, Theorem 1 guarantees that we can robustly stabilize the plant using a first order controller. However, with a zeroth order controller $C(s) = c_0$, since the closed loop polynomial $f(s,q) = s^3 + c_0 s^2 + q$ is missing a first order term, robust stabilization is precluded. Similarly, for the pseudo-stable case, we consider the uncertain plant

$$G(s,q) = \frac{1+qs}{s^2}$$

with uncertainty bound $-1 \leq q \leq 1$. Noting that $m = 2$, $r(q) = 1$ when $q \neq 0$ and $r(q) = 2$ when $q = 0$, Theorem 2 guarantees that we can robustly stabilize $G(s,q)$ using a first order controller. Again, using a zeroth order controller, it is a straightforward to see by inspection that robust stabilization is ruled out because the closed loop polynomial has a sign-indefinite first order term.

Remark 3: Readers familiar with μ synthesis [18] may wonder how our results compare with this method. We note that the μ synthesis method is conservative in general [19] and it does not offer *a priori* decision whether a given uncertainty structure admits robust stabilization or not; it often has trouble to find a robust stabilizer when the parameter range is large.

IV. Stabilizable State-Space Structures

In this section, we show that the pseudo-minimum phase uncertainty structure given in the previous section covers a large class of uncertain systems in the state-space framework. These systems admit a so-called *Stepwise Augmentation Structure* which can be generated recursively using the so-called *down augmentations* and *up augmentations*. Such structures, first introduced in [2], were called the *admissible shuffles*. Later in [16], the term *anti-symmetric stepwise configuration* was used to describe a similar class of systems. A particular important feature of the Stepwise Augmentation Structure is that it includes two well-studied uncertainty structures, the so-called lower-triangular structure and upper triangular structure, as special cases. These special structures are also known as back-stepping structure and forwarding structure in the nonlinear control literature when the uncertainties are replaced by nonlinearities.

For systems admitting the stepwise augmentation structure, it is shown in [2], [16] that a robust linear, time-invariant state feedback stabilizer can be constructed. Such structures were also studied recently in [20] in the context of output regulation control via state feedback.

We prove this section that a large class of such structures is robustly stabilizable via output feedback if a suitably chosen output is available. This is done by showing that stepwise augmentation structure corresponds precisely to the pseudo-minimum phase structure of Theorem 1.

In the construction to follow, we begin with an uncertain system:

$$\begin{aligned}\dot{x} &= A(q)x + b(q)u\\ y &= c^T(q)x\end{aligned}$$

where $q \in Q$ represents uncertain parameters as before, $A(q)$ is an $n \times n$ continuous matrix function, $b(q)$ and $c(q)$ are $n \times 1$ continuous vector functions, and u, x and y are the input, state and output of the system, respectively. We call $\Sigma = (A(q), b(q), c(q))$ a *generating system.*

Definition 1: Given a generating system $\Sigma = (A(q), b(q), c(q))$, the system

$$\begin{aligned}\dot{x} &= A(q)x + b(q)x_{n+1};\\ \dot{x}_{n+1} &= \beta^T(q)x + \alpha(q)x_{n+1} + \theta(q)u;\\ y &= c^T(q)x\end{aligned}$$

with $n+1$ state variables is said to be a *down augmentation* of Σ if the added vectors and scalars $\alpha(q)$, $\beta(q)$ and $\theta(q)$ depend continuously on q and $\theta(q)$ is sign-invariant. We call x_{n+1} the *augmenting state variable*. Similarly, the system

$$\begin{aligned}\dot{x}_0 &= \beta^T(q)x;\\ \dot{x} &= A(q)x + b(q)\left(\alpha(q)x_0 + u\right);\\ y &= c^T(q)x\end{aligned}$$

with $n+1$ state variables is said to be an *up augmentation* of Σ if the added vector and scalar $\alpha(q)$ and $\beta(q)$ depend continuously on q and the first entry of $\beta(q)$ is sign-invariant. In this case, x_0 is called the *augmenting state variable*.

Definition 2: Let $\Sigma = (A(q), b(q), c(q))$ be a generating system with a robustly minimum phase transfer function. Then, a system is said to be a *stepwise augmentation structure* if it is obtained from Σ via a sequence of up/down augmentations, and in addition, if up augmentations are involved, the $A(q)$-matrix of the augmented system is nonsingular for all $q \in Q$.

Example 3: To illustrate the stepwise augmentation structure, we list some of the uncertain systems which fit into this framework. Using the notation

$$M(q) \doteq [A(q)|b(q)]$$

we consider the four possible structures for $M(q)$ associated with 4th order systems

$$\left[\begin{array}{cccc|c} * & \theta & 0 & 0 & 0\\ \hline * & * & \theta & 0 & 0\\ * & * & * & \theta & 0\\ * & * & * & * & \theta \end{array}\right]; \quad \left[\begin{array}{cccc|c} 0 & \theta & * & * & 0\\ 0 & 0 & \theta & * & 0\\ 0 & 0 & 0 & \theta & 0\\ \hline * & * & * & * & \theta \end{array}\right];$$

$$\left[\begin{array}{cccc|c} 0 & \theta & * & 0 & 0\\ 0 & 0 & \theta & 0 & 0\\ \hline * & * & * & \theta & 0\\ * & * & * & * & \theta \end{array}\right]; \quad \left[\begin{array}{cccc|c} 0 & \theta & * & * & 0\\ 0 & 0 & \theta & 0 & 0\\ \hline 0 & 0 & * & \theta & 0\\ * & * & * & * & \theta \end{array}\right]$$

where $*$ denotes entries that are arbitrary functions of q and θ denotes the entries which are sign-invariant. For each matrix, the underlined state variable corresponds to the generating system. For example, for the third matrix $M(q)$ above, the generating system is described by $\dot{x} = \theta(q)u$. The sequences of augmentations for the structures above are respectively down-down-down, down-up-up, down-up-down and down-down-up. In all of the examples above, the generating system is a scalar system of the form

$$\begin{aligned}\dot{x}_k &= a(q)x_k + \theta(q)u;\\ y &= x_k\end{aligned}$$

which is clearly robustly minimum-phase.

Theorem 3: Let $\Sigma = (A(q), b(q), c(q))$ be a generating system. Then, a down augmentation does not introduce any new zeros and each up augmentation introduces at most one zero at the origin. Furthermore, if m up augmentations are involved and the final A-matrix for the augmented system is nonsingular for all $q \in Q$, then the augmented system has exactly m new zeros at the origin.

Proof: For notational simplicity, we suppress the dependence of the system on q and denote the transfer function of the generating uncertain system by $G(s) = N(s)/D(s)$. Taking Laplace transforms, the transfer function of the down-augmented system is computed

$$Y(s) = \frac{c^T(sI-A)^{-1}b}{s - \alpha - \beta^T(sI-A)^{-1}b}U(s).$$

Noting that $c^T(sI-A)^{-1}b = N(s)/D(s)$ and expressing the transfer function $\beta^T(sI-A)^{-1}b$ as $N_\beta(s)/D(s)$, we obtain

$$Y(s) = \frac{N(s)}{D(s)(s-\alpha) - N_\beta(s)}U(s).$$

Hence, the down augmentation does not introduce new zeros.

The transfer function of the up-augmented system is similarly computed

$$Y(s) = c^T(sI-A)^{-1}b\frac{s}{s - \alpha\beta^T(sI-A)^{-1}b}U(s).$$

Again, denoting $\beta^T(sI-A)^{-1}b$ as a ratio of two polynomials $N_\beta(s)/D(s)$, we obtain

$$Y(s) = \frac{sN(s)}{sD(s) - \alpha N_\beta(s)}U(s).$$

Hence, at most one new zero at $s = 0$ can be introduced by each up augmentation. Finally, if m up augmentations are involved (regardless of the number of down augmentations), the numerator of the augmented transfer function will be $s^m N(s,q)$. The new factor s^m can not be canceled if the denominator of the augmented transfer function has a sign-invariant zeroth degree coefficient. This is guaranteed if the A-matrix of the augmented system is nonsingular for all $q \in Q$. □

Combining Theorems 1 and 3, we have the following result:

Corollary 1: A stepwise augmentation structure is robustly pseudo-minimum phase, and thus robustly stabilizable via output feedback.

V. Conclusion

We have introduced two new classes of uncertain linear systems, namely, pseudo-minimum phase and pseudo-stable structures, which admit robust output feedback stabilization. We have also established the connections of these structures to an uncertainty structure in the state space called stepwise augmentation structure. We complete this technical note by noting that, although specific recursive design procedures for robust stabilizers are given, our intention was only to demonstrate and verify *robust stabilizatbility*, rather than suggesting that these procedures are optimal in any sense or attempting to characterizing all robust stabilizers. Further research is needed for designing robust stabilizing controllers which can also deliver certain guaranteed performances.

References

[1] G. Leitmann, "Guaranteed asymptotic stability for some linear systems with bounded uncertainties," *J. Dyn. Syst. Meas. Control*, vol. 101, pp. 212–216, 1979.

[2] B. R. Barmish, "On enlarging the class of stabilizable linear systems with uncertainty," in *Proc. IEEE Asilomar Conf. Comput., Circuits Syst.*, Monterey, CA, 1982, [CD ROM].

[3] J. S. Thorp and B. R. Barmish, "On guaranteed stability of uncertain linear systems via linear control," *J. Optim. Theory Appl.*, vol. 35, no. 4, pp. 559–579, 1981.

[4] K. Wei, "Stabilization of linear time-invariant interval systems via constant state feedback control," *IEEE Trans. Autom. Control*, vol. AC-39, no. 1, pp. 22–32, Jan. 1994.

[5] K. Wei and B. R. Barmish, "Making a polynomial Hurwitz-invariant by choice of feedback gains," *Int. J. Control*, vol. 50, no. 4, pp. 1025–1038, 1989.

[6] C. Byrnes and A. Isidori, "New results and examples in nonlinear feedback stabilization," *Syst. Control Lett.*, vol. 12, pp. 437–442, 1989.

[7] A. Teel, "Feedback Stabilization: Nonlinear Solutions to Inherently Nonlinear Problems," Ph.D. dissertation, College Eng., Univ. California, Berkeley, 1992.

[8] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos, *Nonlinear and Adaptive Control Design*. New York: Wiley, 1995.

[9] R. Sepulchre, M. Jankovic, and P. V. Kokotovic, "Integrator forwarding: A new recursive nonlinear robust design," in *Proc. IFAC World Congress*, San Fransisco, CA, 1996, [CD ROM].

[10] W. Su and M. Fu, "Robust nonlinear control: Beyond backstepping and nonlinear forwarding," in *Proc. IEEE Conf. Decision Control*, Phoenix, AC, 1999, pp. 831–837.

[11] B. R. Barmish and K. H. Wei, "Simultaneous stabilizability of single input-single output systems," in *Proc. Int. Symp. Math. Theory Network Syst.*, Stockholm, Sweden, 1985, [CD ROM].

[12] K. Wei and B. R. Barmish, "An iterative design procedure for simultaneous stabilization of MIMO systems," *Automatica*, vol. 24, no. 5, pp. 643–652, 1988.
[13] K. Wei and R. K. Yedavalli, "Robust stabilizability for linear systems with both parameter variation and unstructured uncertainty," *IEEE Trans. Autom. Control*, vol. AC-34, no. 2, pp. 149–156, Feb. 1989.
[14] A. Isidori, "A Remark on the problem of semiglobal nonlinear output regulation," *IEEE Trans. Autom. Control*, vol. AC-42, no. 12, pp. 1734–1738, Dec. 1997.
[15] L. Praly, "Asymptotic stabilization via output feedback for lower triangular systems with output dependent incremental rate," *IEEE Trans. Autom. Control*, vol. AC-48, no. 6, pp. 1103–1108, Jun. 2003.
[16] K. Wei, "Quadratic stabilizability of linear systems with structural independent time-varying uncertainties," *IEEE Trans. Autom. Control*, vol. AC-35, no. 3, pp. 268–277, Mar. 1990.
[17] F. R. Gantmacher, *The Theory of Matrices*. New York: Chelsea, 1959, vol. I and II.
[18] G. J. Balas, J. C. Doyle, K. Glover, A. Packard, and R. S. Smith, *Analysis and Synthesis Toolbox: Users Guide*. Natick, MA: Mathworks, 1991.
[19] G. Meinsma, Y. Shrivastava, and M. Fu, "A dual formulation of mixed μ and the losslessness of (D, G)-scaling," *IEEE Trans. Autom. Control*, vol. 42, no. 7, pp. 1032–1036, Jul. 1997.
[20] M. Fu and J. Huang, "Robust regulation of linear systems with structural uncertainties," in *Proc. IFAC World Congress*, Jul. 2005, [CD ROM].
[21] B. R. Barmish, *New Tools for Robustness of Linear Systems*. New York: MacMillan, 1994.
[22] S. P. Battacharyya, H. Chappellat, and L. H. Keel, *Robust Control—The Parametric Approach*. Englewood Cliffs, NJ: Prentice Hall, 1995.

Instability Mechanisms in Cooperative Control

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***Abstract*—We consider a motion coordination problem with second order agent dynamics and examine the closed-loop robustness with respect to switching topology, variation of link gain, and unmodeled dynamics. In each case, we illustrate with examples possible instability mechanisms and discuss under what conditions stability is maintained.**

***Index Terms*—Cooperative control, instability, Mathieu equation, switched system, unmodeled dynamics..**

I. Introduction

Motion coordination problems have been intensively studied during the past years, leading to significant results in formation control, flocking, and consensus [1]–[9]. One of the challenges in the coordination problem is the design of local rules that guarantee the desired group behavior. The design and analysis of such rules make use of graph theory and potential function methods. The communication topology between agents is represented by a graph while the interactions between agents are modelled as artificial attraction/repulsion forces. The stability results follow from a Lyapunov function constructed from potential functions with the help of spectral properties of the graph Laplacian.

When the velocities of the agents are directly manipulatable, first-order kinematic models [1], [3] are appropriate. However, in many applications, only the acceleration of the agents can be controlled by input forces and torques, thereby leading to second or higher order dynamics [2], [5], [7], [8] with mass inertia incorporated.

In this paper, we consider double integrator agent dynamics with an undirected communication topology. We first analyze a cooperative system with a switching communication topology. Such switching may occur due to the vehicles joining or leaving a formation, transmitter/receiver failures, limited communication/sensor range, or physical obstacles temporarily blocking sensing between vehicles. For single integrator dynamics, switching topology has been studied in [1], [3] and stability under arbitrary switching has been ascertained for classes of coordination algorithms. In contrast, for second order dynamics, we illustrate with an example that a destabilizing switching sequence that triggers instability exists. We then show that stability is maintained when switching is sufficiently fast or slow so that it does not interfere with the natural frequencies of the group dynamics.

We next investigate stability properties when the link weights are perturbed by small sinusoidal oscillations. To illustrate instability in its most basic form, we make a simplifying assumption that the perturbation is sinusoidal and transform the group dynamics into a form that reveals a parametric resonance mechanism [10]–[12]. This transformation employs the spectral properties of the graph Laplacian and decouples the relative motion from the motion of the center of the agents. When mass inertia and damping terms are identical for all agents, we obtain decoupled *Mathieu equations* [11], which make parametric resonance explicit. For broader classes of mass and damping matrices, we obtain coupled Mathieu equations and discuss which frequencies lead to parametric resonance. Next, we show that sinusoidal perturbations do not destabilize the system if they are slow or fast enough.

We finally study the effect of input unmodeled dynamics, such as fast actuator dynamics. Following standard singular perturbation arguments [13], we prove that the stability of the nominal design that ignores the effects of unmodeled dynamics is preserved when the stable unmodeled dynamics are sufficiently fast. As we illustrate with an example, how fast the unmodeled dynamics must be is dictated by the graph structure and the mass inertia matrix.

The subsequent sections are organized as follows: Section II introduces the nominal system and discusses its stability properties. We illustrate our instability example due to switching in Section III-A, followed by a discussion on when stability is maintained in Section III-B. We present a parametric resonance example in Section IV-A, which exhibits decoupled Mathieu equations, and generalize it to coupled Mathieu equations in Section IV-B. We then investigate the effects of fast and slow sinusoidal perturbations in Sections IV-C and D. One of the contributions of Section IV is to introduce parametric resonance, which is a well-studied topic in structural mechanics, to cooperative control. Section V studies unmodeled dynamics.

II. Nominal Cooperative System and its Stability

We consider a group of agents which are represented by the vectors $x_i \in \mathbb{R}^p$, $i = 1, \ldots, n$ and their communication structure is represented with a graph. If the ith and jth agents have access to the relative information $x_i - x_j$, then the nodes i and j in the graph are connected by a link. To simplify our analysis, we assign an orientation to the graph

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