ROBUST FILTERING FOR UNCERTAIN LINEAR DISCRETE-TIME DESCRIPTOR SYSTEMS

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Abstract: This paper is concerned with the problem of robust filtering for uncertain linear discrete-time descriptor systems. The matrices of the system state-space model are uncertain and supposed to belong to a given polytope. A linear matrix inequality based method is proposed for designing linear stationary strictly proper filters, for the one-step prediction, that guarantee the asymptotic stability of the estimation error and an optimized upper-bound on the asymptotic error variance, irrespective of the parameter uncertainty. The proposed robust filter design is based on a parameter-dependent Lyapunov function.

Keywords: Filtering, robust LMS filtering, descriptor systems, uncertain systems, discrete-time systems, parameter-dependent Lyapunov functions.

1. INTRODUCTION

Descriptor systems (also known as singular systems or differential-/difference-algebraic systems) are an important class of dynamic system models from both a theoretical and practical point of view due to their capacity in describing algebraic constraints between physical variables (Lewis, 1986; Dai, 1989*a*). Descriptor models are encountered naturally in many areas of applications, such as economical systems, electrical networks, robotic systems, chemical processes, etc. (Dai, 1989*a*; Hill and Mareels, 1990; Lewis, 1986; Takaba and Katayama, 1997; Verghese *et al.*, 1981).

In the last decade, a great deal of interest has been devoted to signal estimation methods based on Kalman filtering for discrete-time descriptor systems; see, for instance, Bianco *et al.* (2005), Dai (1989*b*), Deng and Liu (1999), Ishihara *et al.* (2004*a*), Nikoukhah *et al.* (1992), Nikoukhah *et al.* (1999) and Zhang *et al.* (1999). These methods are based on the knowledge of a perfect model for the signal to be estimated and thus they may fail to provide a guaranteed error variance when only an approximate model is available.

On the other hand, the design of robust Kalman filters for standard linear discrete-time state space models has been attracting increasing attention and both norm-bounded and polytopic parameter uncertainties have been considered; see, for instance, Barbosa *et al.* (2002), Petersen and McFarlane (1996), Petersen and Savkin (1999)

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and the reference therein. Very recently, a robust Kalman filter for linear discrete-time descriptor systems subject to norm-bounded parameter uncertainty using a Riccati equation approach has been proposed in Ishihara *et al.* (2004*b*). To the best of the authors' knowledge, to-date the problem of robust Kalman filtering for linear discrete-time descriptor systems with convex bounded uncertainty has not yet been investigated.

This paper addresses the design of robust filters, for the one-step prediction, for linear discrete-time descriptor systems. The matrices of the system state-space model are uncertain and supposed to belong to a given polytope. A linear matrix inequality (LMI) based method is developed for designing linear stationary strictly proper filters that guarantee the asymptotic stability of the estimation error and an optimized upper-bound on the asymptotic error variance, irrespective of the parameter uncertainty. The proposed robust filter design is based on a parameter-dependent Lyapunov function.

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices, I_n is the $n \times n$ identity matrix, $\mathrm{Tr}[\cdot]$ denotes matrix trace, and $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix. For a real matrix S, S^T denotes its transpose, $\mathrm{Her}\{S\}$ stands for $S+S^T$ and $S>0$ means S is symmetric and positive definite. For a symmetric block matrix, the symbol $\star$ denotes the transpose of the symmetric blocks outside the main diagonal block. Finally, $\mathbf{E}\{\cdot\}$ stands for mathematical expectation.

2. PROBLEM FORMULATION

Consider the following linear descriptor system

$$\begin{aligned} E\zeta(k+1) &= A_a\zeta(k) + B_a w(k) \\ y(k) &= C_a\zeta(k) + D_a w(k) \\ s(k) &= L_a\zeta(k) \end{aligned} \tag{1}$$

where $\zeta \in \mathbb{R}^{n_\zeta}$ is the state, $w(k) \in \mathbb{R}^{n_w}$ is a zero-mean white noise signal (including process and measurement noises) with an identity covariance matrix and uncorrelated with $\zeta(0)$ for $k = 0, 1\ldots$, $y(k) \in \mathbb{R}^{n_y}$ is the measurement, $s(k) \in \mathbb{R}^{n_s}$ is the signal to be estimated, and E, A_a, B_a, C_a D_a and L_a are real constant matrices of appropriate dimensions with $\mathrm{rank}(E) = n < n_\zeta$.

It is assumed that system (1) satisfies the following assumptions:

A1 System (1) is *regular*, i.e. $\det(zE - A) \not\equiv 0$.

A2 $\mathrm{rank}(E) = \deg \det(zE - A)$.

A3 $\mathrm{Im}\{B_a\} \subseteq \mathrm{Im}\{E\}$.

Note that, assumption **A1** guarantees the existence and uniqueness of solution to (1) for any initial condition, **A2** ensures that the system is impulse-free and implies the causality of (1), whereas **A3** implies that the noise signal $w(k)$ does not influence the algebraic constraints in the system.

It is well known that under assumptions **A1**−**A3**, system (1) can be transformed into an equivalent difference-algebraic system of the following SVD normal form (Bender and Laub, 1987; Takaba and Katayama, 1997):

$$\begin{aligned} S\colon\ x(k+1) &= A_1x(k) + A_2\phi(k) + Bw(k) \\ 0 &= H_1x(k) + H_2\phi(k) \\ y(k) &= C_1x(k) + C_2\phi(k) + Dw(k) \\ s(k) &= L_1x(k) + L_2\phi(k) \end{aligned} \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the dynamic state, $\phi(k) \in \mathbb{R}^{n_\phi}$, $n_\phi = n_\zeta - n$, is the algebraic state, $s(k)$, $w(k)$ and $y(k)$ are as before and A_i, C_i, H_i, L_i, $i = 1, 2$, B and D are real matrices of appropriate dimensions with H_2 nonsingular. Note that $w(k)$ is uncorrelated with $x(0)$ and $\phi(0)$ for $k = 0, 1\ldots$.

Hereafter, it is assumed that system (1) is already given in the difference-algebraic form (2) and the underlying matrices are unknown but assumed to belong to the following polytope:

$$\Omega = \left\{ \Pi : \Pi = \sum_{i=1}^{\ell} \alpha_i \Pi_i,\ \alpha_i \geq 0 : \sum_{i=1}^{\ell} \alpha_i = 1 \right\} \tag{3}$$

where

$$\Pi = \begin{bmatrix} A_1 & A_2 & B \\ H_1 & H_2 & 0 \\ C_1 & C_2 & D \\ L_1 & L_2 & 0 \end{bmatrix},\quad \Pi_i = \begin{bmatrix} A_{1i} & A_{2i} & B_i \\ H_{1i} & H_{2i} & 0 \\ C_{1i} & C_{2i} & D_i \\ L_{1i} & L_{2i} & 0 \end{bmatrix} \tag{4}$$

and where A_{ki}, C_{ki}, H_{ki}, L_{ki}, $k = 1, 2$, B_i and D_i are given real constant matrices with H_{2i} nonsingular.

This paper is aimed at designing a stationary asymptotically stable linear filter $\mathcal{F}$ which provides an estimate $\hat{s}$ of the signal s with a guaranteed performance in the mean square sense, irrespective of the uncertainty. We seek a filter of order n with a state-space realization

$$\begin{aligned} \mathcal{F}\colon\ \hat{x}(k+1) &= A_f\hat{x}(k) + B_f y(k),\quad \hat{x}(0) = 0 \\ \hat{s}(k) &= C_f\hat{x}(k) \end{aligned} \tag{5}$$

where the matrices $A_f \in \mathbb{R}^{n\times n}$, $B_f \in \mathbb{R}^{n\times n_y}$ and $C_f \in \mathbb{R}^{n_s\times n}$ are to be found. The filter $\mathcal{F}$ is required to ensure an optimized upper-bound on the worst-case asymptotic variance of the estimation error over Ω, namely:

$$\max_{\Omega} \Big\{ \mathrm{var}\{e\} \Big\},$$

$$\mathrm{var}\{e\} := \lim_{k\to\infty} \mathbf{E}\Big\{ [\, s(k) - \hat{s}(k)\,]^T [\, s(k) - \hat{s}(k)] \Big\}.$$

Observe that considering (2) and (5), the dynamics of the estimation error $e := s - \hat{s}$ can be described by the following state-space model:

$$S_e:\ \begin{aligned} \xi(k+1) &= \tilde{A}_1\xi(k) + \tilde{A}_2\phi(k) + \tilde{B}w(k) \\ 0 &= \tilde{H}_1\xi(k) + \tilde{H}_2\phi(k) \\ e(k) &= \tilde{L}_1\xi(k) + \tilde{L}_2\phi(k) \end{aligned} \tag{6}$$

where

$$\xi = \begin{bmatrix} x \\ \hat{x} \end{bmatrix},\ \ \tilde{A}_1 = \begin{bmatrix} A_1 & 0 \\ B_f C_1 & A_f \end{bmatrix},\ \ \tilde{A}_2 = \begin{bmatrix} A_2 \\ B_f C_2 \end{bmatrix} \tag{7}$$

$$\tilde{B} = \begin{bmatrix} B \\ B_f D \end{bmatrix},\ \ \tilde{H}_1 = \begin{bmatrix} H_1 & 0 \end{bmatrix},\ \ \tilde{H}_2 = H_2 \tag{8}$$

$$\tilde{L}_1 = \begin{bmatrix} L_1 & -C_f \end{bmatrix},\ \ \tilde{L}_2 = L_2. \tag{9}$$

We conclude this section by recalling a version of Finsler's lemma that will be used in the derivation of a result in the next section

Lemma 1. (Boyd *et al.*, 1994) Given matrices $\Psi_i = \Psi_i^T \in \mathbb{R}^{n\times n}$ and $\mathcal{N}_i \in \mathbb{R}^{m\times n}$, $i = 1,\dots,\nu$, then

$$x_i^T\Psi_i x_i < 0,\ \forall x_i \in \mathbb{R}^n : \mathcal{N}_i x_i = 0,\ x \neq 0;\quad i = 1,\dots,\nu \tag{10}$$

if and only if there exist matrices $L_i \in \mathbb{R}^{n\times m}$, $i = 1,\dots,\nu$, such that

$$\Psi_i + L_i\mathcal{N}_i + \mathcal{N}_i^T L_i^T < 0, \quad i = 1,\dots,\nu. \tag{11}$$

Note that the conditions of (11) remain sufficient for (10) to hold even when arbitrary constraints are imposed to the scaling matrices L_i, including setting $L_i = L$, $i = 1, 2, \dots, \nu$.

3. ROBUST FILTER

In this section we shall deal with the filter design. In order to pave the way for deriving the robust filter, initially we consider the case where the system matrix Π is perfectly known, i.e. $\ell = 1$.

First, we derive a necessary and sufficient LMI condition which ensures a prescribed upper-bound on the asymptotic variance of the estimation error for a given filter (5).

Lemma 2. Consider system (2) with known matrices. Given a filter (5) and a scalar $\gamma > 0$, the estimation error system (6) is asymptotically stable and $\mathrm{var}\{e\} < \gamma$ if and only if any of the following equivalent conditions hold:

(a) There exist matrices $P > 0$, Ξ, N_1 and N_2 satisfying the following LMIs:

$$\begin{bmatrix} -P + \mathrm{Her}\{N_1\tilde{H}_1\} & \star & \star & \star \\ N_2\tilde{H}_1 + \tilde{H}_2^T N_1^T & \mathrm{Her}\{N_2\tilde{H}_2\} & \star & \star \\ P\tilde{A}_1 & P\tilde{A}_2 & -P & \star \\ \tilde{L}_1 & \tilde{L}_2 & 0 & -I \end{bmatrix} < 0 \tag{12}$$

$$\begin{bmatrix} \Xi & \tilde{B}^T P \\ P\tilde{B} & P \end{bmatrix} > 0 \tag{13}$$

$$\gamma - \mathrm{Tr}[\,\Xi\,] > 0. \tag{14}$$

(b) There exist matrices $P > 0$, G, Ξ, N_1 and N_2 satisfying the following LMIs:

$$\begin{bmatrix} -P + \mathrm{Her}\{N_1\tilde{H}_1\} & \star & \star & \star \\ N_2\tilde{H}_1 + \tilde{H}_2^T N_1^T & \mathrm{Her}\{N_2\tilde{H}_2\} & \star & \star \\ G\tilde{A}_1 & G\tilde{A}_2 & P - G - G^T & \star \\ \tilde{L}_1 & \tilde{L}_2 & 0 & -I \end{bmatrix} < 0 \tag{15}$$

$$\begin{bmatrix} \Xi & \tilde{B}^T G^T \\ G\tilde{B} & G + G^T - P \end{bmatrix} > 0 \tag{16}$$

$$\gamma - \mathrm{Tr}[\,\Xi\,] > 0. \tag{17}$$

Proof. (a) First, since the matrix $\tilde{H}_2$ is nonsingular, the estimation error system (6) is equivalent to the following system without algebraic constraints:

$$\begin{aligned} \xi(k+1) &= \tilde{A}\xi(k) + \tilde{B}w(k) \\ e(k) &= \tilde{L}\xi(k) \end{aligned} \tag{18}$$

where

$$\tilde{A} = \tilde{A}_1 - \tilde{A}_2\tilde{H}_2^{-1}\tilde{H}_1,\ \ \tilde{L} = \tilde{L}_1 - \tilde{L}_2\tilde{H}_2^{-1}\tilde{H}_1. \tag{19}$$

It is well known that system (18), or equivalently (6), is asymptotically stable and $\mathrm{var}\{e\} < \gamma$ if and only if the there exist matrices $P > 0$ and $\Xi > 0$ satisfying the following inequalities (Boyd *et al.*, 1994):

$$\tilde{A}^T P\tilde{A} - P + \tilde{L}^T\tilde{L} < 0 \tag{20}$$

$$\Xi - \tilde{B}^T P\tilde{B} > 0 \tag{21}$$

$$\gamma - \mathrm{Tr}[\,\Xi\,] > 0. \tag{22}$$

In view of (19), it can be readily verified that (20) is equivalent to:

$$\eta^T\Lambda\eta < 0,\ \eta = \Phi x,\ \forall x \in \mathbb{R}^n,\ x \neq 0 \tag{23}$$

where

$$\Lambda = \begin{bmatrix} \tilde{A}_1^T P\tilde{A}_1 - P + \tilde{L}_1^T\tilde{L}_1 & \tilde{A}_1^T P\tilde{A}_2 + \tilde{L}_1^T\tilde{L}_2 \\ \tilde{A}_2^T P\tilde{A}_2 + \tilde{L}_2^T\tilde{L}_1 & \tilde{A}_2^T P\tilde{A}_2 + \tilde{L}_2^T\tilde{L}_2 \end{bmatrix} \tag{24}$$

$$\Phi = \begin{bmatrix} I_n \\ \tilde{H}_2^{-1}\tilde{H}_1 \end{bmatrix}. \tag{25}$$

Considering that

$$\Big\{\eta : \eta = \Phi x,\ \forall x \in \mathbb{R}^n,\ x \neq 0\Big\} = \Big\{\eta : \tilde{H}\eta = 0,\ \eta \neq 0\Big\}$$

where

$$\tilde{H} = \begin{bmatrix} \tilde{H}_1 & \tilde{H}_2 \end{bmatrix}$$

it follows from (23) that (20) is also equivalent to:

$$\eta^T\Lambda\eta < 0,\ \forall \eta \in \mathbb{R}^{n_\eta} : \tilde{H}\eta = 0,\ \ \eta \neq 0. \tag{26}$$

By Lemma 1, (26) holds if and only if there exists a matrix N of appropriate dimensions such that:

$$\Lambda + N\tilde{H} + \tilde{H}^T N^T < 0\,. \tag{27}$$

Next, letting $N = [\, N_1^T \quad N_2^T \,]^T$, it can be readily verified that (27) is identical to (12).

On the other hand, by applying Schur's complement to (21), it follows that this inequality is equivalent to (13).

(b) The equivalence of conditions (a) and (b) will be established.

(a) $\Rightarrow$ (b): If there exist matrices P, Ξ, N_1 and N_2 satisfying (12)-(14), then it follows that (15)-(17) hold with $G = P$ and the same matrices Ξ, N_1 and N_2.

(b) $\Rightarrow$ (a): Pre- and post multiplying (15) by

$$\begin{bmatrix} I & 0 & \tilde{A}_1^T & \tilde{L}_1^T \\ 0 & I & \tilde{A}_2^T & \tilde{L}_2^T \end{bmatrix}$$

and its transpose, respectively, leads to

$$\begin{bmatrix} -P+\mathrm{Her}\{N_1\tilde{H}_1\}+\tilde{L}_1^T\tilde{L}_1 & \star \\ N_2\tilde{H}_1+\tilde{H}_2^T N_1^T+\tilde{L}_2^T\tilde{L}_1 & \mathrm{Her}\{N_2\tilde{H}_2\}+\tilde{L}_2^T\tilde{L}_2 \end{bmatrix} + \begin{bmatrix} \tilde{A}_1^T P\tilde{A}_1 & \star \\ \tilde{A}_2^T P\tilde{A}_1 & \tilde{A}_2^T P\tilde{A}_2 \end{bmatrix} < 0. \tag{28}$$

Next, pre- and post multiplying (16) by $[I \quad -\tilde{B}^T]$ and its transpose, respectively, implies that

$$\Xi - \tilde{B}^T P\tilde{B} > 0. \tag{29}$$

As $P > 0$, by applying Schur's complement to (28) and (29) one concludes that these inequalities are equivalent to (12) and (13), respectively. ▽▽▽

It should be remarked that $V(\xi) = \xi^T(k)P\xi(k)$, with P satisfying part (a) or (b) of Lemma 2, is a Lyapunov function for the unforced system of S_e.

Note that as the inequalities (15)-(17) are affine in the system matrices, by convexity condition (b) of Lemma 3 can be readily extended to the case where the matrices of system (2) are uncertain and belong to the polytope Ω. In such a situation, the LMIs (15)-(17) need to be satisfied at all vertices of the polytope Ω for fixed matrices G, N_1 and N_2 and with different matrices P and Ξ, namely P_i and Ξ_i, $i = 1, \ldots, \ell$ for each of the ℓ vertices of Ω. This robust variance analysis result is presented in the next lemma. Notice that since fixed matrices G, N_1 and N_2 are used for all vertices of Ω, this lemma provides only sufficient conditions.

Lemma 3. Consider system (2) with uncertain matrices and let Ω be a given polytope of admissible system matrix Π. Given a filter (5) and a scalar $\gamma > 0$, the estimation error system (6) is asymptotically stable and $\mathrm{var}\{e\} < \gamma$ over Ω if there exist matrices $P_i > 0$, $\Xi_i > 0$, $i = 1, \ldots,$ G, N_1 and N_2 satisfying the following LMIs for $i = 1, \ldots, \ell$:

$$\begin{bmatrix} -P_i+\mathrm{Her}\{N_1\tilde{H}_{1i}\} & \star & \star & \star \\ N_2\tilde{H}_{1i}+\tilde{H}_{2i}^T N_1^T & \mathrm{Her}\{N_2\tilde{H}_{2i}\} & \star & \star \\ G\tilde{A}_{1i} & G\tilde{A}_{2i} & P_i-G-G^T & \star \\ \tilde{L}_{1i} & \tilde{L}_{2i} & 0 & -I \end{bmatrix} < 0 \tag{30}$$

$$\begin{bmatrix} \Xi_i & \tilde{B}_i^T G^T \\ G\tilde{B}_i & G + G^T - P_i \end{bmatrix} > 0 \tag{31}$$

$$\gamma - \mathrm{Tr}[\,\Xi_i\,] > 0. \tag{32}$$

where $\tilde{A}_{1i}$, $\tilde{A}_{2i}$, $\tilde{B}_i$, $\tilde{H}_{1i}$, $\tilde{H}_{2i}$, $\tilde{L}_{1i}$ and $\tilde{L}_{2i}$ denote the matrices $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}$, $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{L}_1$ and $\tilde{L}_2$, respectively, at the i-th vertex of the polytope Ω. Moreover, $V(\xi) = \xi^T(k)P(\alpha)\xi(k)$, with

$$P(\alpha) = \sum_{i=1}^{\ell} \alpha_i P_i$$

and where α_i are the scalars as in (3), is an uncertainty-dependent Lyapunov function for the unforced system of S_e. □

In the sequel we shall develop LMI methods to design filters with an optimized asymptotic error variance for system (2). The next theorem deals with the case where the system matrix Π is perfectly known and is derived from conditions (15)-(17) of Lemma 2 (b) by using appropriate congruent transformations and parameterizations of the filter matrices and matrix G.

Theorem 1. Consider system (2) with known matrices. Given a scalar $\gamma > 0$, there exists a filter (5) such that the estimation error system (6) is asymptotically stable and $\mathrm{var}\{e\} < \gamma$ if and only if there exist matrices $X > 0$, $\Xi > 0$, F_1, F_2, Q, R, S, Z and W satisfying the following LMIs:

$$\begin{bmatrix} -X+\mathrm{Her}\{F_1\tilde{H}_1\} & \star & \star & \star \\ F_2\tilde{H}_1+H_2^T F_1^T & \mathrm{Her}\{F_2H_2\} & \star & \star \\ \mathcal{A}_1 & \mathcal{A}_2 & X-\Upsilon & \star \\ \mathcal{L}_1 & L_2 & 0 & -I \end{bmatrix} < 0 \tag{33}$$

$$\begin{bmatrix} \Xi & \mathcal{B}^T \\ \mathcal{B} & \Upsilon - X \end{bmatrix} > 0 \tag{34}$$

$$\gamma - \mathrm{Tr}[\,\Xi\,] > 0 \tag{35}$$

where

$$\mathcal{A}_1 = \begin{bmatrix} RA_1 & 0 \\ SA_1 + YC_1 + Q & Q \end{bmatrix} \tag{36}$$

$$\mathcal{A}_2 = \begin{bmatrix} RA_2 \\ SA_2 + YC_2 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} ZB \\ YB + YD \end{bmatrix} \tag{37}$$

$$\mathcal{L}_1 = [\, L_1 - Z \quad -Z \,], \quad \Upsilon = \begin{bmatrix} R+R^T & W^T+S^T \\ W+S & W+W^T \end{bmatrix} \tag{38}$$

Moreover, the filter with transfer function matrix

$$H_{\hat{s}y}(z) = Z(zI - W^{-1}Q)^{-1}W^{-1}Y \qquad (39)$$

solves the filtering problem.

Proof. Sufficiency: In the sequel, it will be shown that if the LMIs of (33)-(35) hold, then the filter (39) ensures that the conditions (15)-(17) of Lemma 2 are satisfied. Initially, note that since $X > 0$, (33) ensures that $\Upsilon > 0$, which implies that R and W are nonsingular matrices.

Let the $2n \times 2n$ matrix G in (15) and (16) be parameterized as follows:

$$G = \begin{bmatrix} R^{-1} & R^{-1}V^T \\ -U^{-1}SR^{-1} & U^{-1}V^T - V^{-1}SR^{-1}V^T \end{bmatrix}^{-1} \qquad (40)$$

where U and V are $n \times n$ nonsingular matrices such that $UV = W$. Moreover, let the matrix

$$\mathcal{T} = \begin{bmatrix} R^T & S^T \\ 0 & U^T \end{bmatrix}. \qquad (41)$$

We now show that the matrix G as above is well defined and both $\mathcal{T}$ and G are nonsingular matrices. First, note that

$$\hat{\mathcal{T}} := G^{-T}\mathcal{T} = \begin{bmatrix} I & 0 \\ V & V \end{bmatrix}. \qquad (42)$$

Since R and W are nonsingular matrices, it follows that U and V are well defined. Hence, $\mathcal{T}$ and $G^{-T}\mathcal{T}$ are nonsingular and thus G is a nonsingular matrix as well.

Consider the following state-space realization for the filter (39):

$$A_f = VW^{-1}QV^{-1}, \;\; B_f = VW^{-1}Y, \;\; C_f = ZV^{-1}.$$

By performing straightforward matrix manipulations, it can be readily verified that

$$\hat{\mathcal{T}}^T G\tilde{A}_1\hat{\mathcal{T}} = \mathcal{A}_1, \;\; \hat{\mathcal{T}}^T G\tilde{A}_2 = \mathcal{A}_2, \;\; \tilde{H}_1\hat{\mathcal{T}} = \tilde{\mathcal{H}}_1 \quad (43)$$

$$\tilde{L}_1\hat{\mathcal{T}} = \mathcal{L}_1, \;\; \hat{\mathcal{T}}^T G\tilde{B} = \mathcal{B}, \;\; \hat{\mathcal{T}}^T(G+G^T)\hat{\mathcal{T}} = \Upsilon. \quad (44)$$

Next, let the matrices

$$P = \hat{\mathcal{T}}^{-T}X\hat{\mathcal{T}}^{-1}, \;\; N_1 = \hat{\mathcal{T}}^{-T}F_1, \;\; N_2 = F_2. \quad (45)$$

Pre- and post-multiplying (33) by $\mathbb{T}_1^T$ and $\mathbb{T}_1$, respectively, where

$$\mathbb{T}_1 = \text{diag}\{\,\hat{\mathcal{T}}^{-1}, \; I_{n_\phi}, \; \hat{\mathcal{T}}^{-1}, \; I_{n_s}\}$$

and considering (43)-(45), one gets (15). Thus, (15) is satisfied with the matrices A_f, B_f, C_f, G, P, N_1 and N_2 defined as above.

In the sequel, it will be shown that (34) implies the feasibility of (16). To this end, introduce the transformation matrix

$$\mathbb{T}_2 = \text{diag}\{\, I_{n_w}, \; \hat{\mathcal{T}}^{-1}\}.$$

Pre- and post-multiplying (34) by $\mathbb{T}_2^T$ and $\mathbb{T}_2$, respectively, and considering (44) and (45) implies that (16) holds with the matrices G and P defined as above.

Necessity: Given a filter with state-space realization (A_f, B_f, C_f) such that $\text{var}\{e\} < \gamma$, by Lemma 2 (b), there exist matrices $P > 0$, G, Ξ, N_1 and N_2 such that the inequalities (15)-(17) hold. Note that (15) implies that the matrix G is nonsingular.

Let the matrix G^{-1} be partitioned as follows

$$G^{-1} := \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}$$

where all the blocks are $n \times n$ matrices and G_1 and G_4 are nonsingular matrices. In addition, since the inequality of (15) is strict, without loss of generality, G_2 and G_3 can be assumed to be nonsingular; see, e.g. de Souza and Trofino (2000). This implies that $G_4G_2^{-1}G_1 - G_3$ is a nonsingular matrix as well.

Next, define the matrices

$$R = G_1^{-1}, \quad U = (G_4G_2^{-1}G_1 - G_3)^{-1}$$

$$V = G_2G_1^{-T}, \quad S = -UG_3G_1^{-1}, \quad W = UV$$

$$Q = UA_fV, \quad Y = UB_f, \quad Z = C_fV$$

$$\mathcal{T} = \begin{bmatrix} R & 0 \\ S & U \end{bmatrix}^T, \;\; \hat{\mathcal{T}} = G^{-T}\mathcal{T}.$$

Note that in view of the latter definitions, it can be easily shown that the matrices G and $\hat{\mathcal{T}}$ are of the form as in (40) and (42), respectively. Hence, the result can be readily obtained by inverting the arguments used in the sufficiency part of the proof. $\nabla\nabla\nabla$

Remark 1. Theorem 1 provides necessary and sufficient LMI conditions for the design of a filter $\mathcal{F}$ for the difference-algebraic system (2) with known matrices that achieves a prescribed upper-bound on the asymptotic variance of the estimation error. Note that finding the filter $\mathcal{F}$ with a minimum asymptotic error variance is a convex optimization problem with LMI constraints. $\square$

The next theorem presents the robust filter design method. In the light of Lemma 3 and considering that the inequalities (33)-(35) are affine in the matrices of system (2), the following result can be readily derived from Theorem 1.

Theorem 2. Consider system (2) with uncertain matrices and let Ω be a given polytope of admissible system matrix Π. Then there exists a filter (5) that minimizes the upper-bound on the asymptotic error variance of Lemma 3 if there exist matrices $X_i > 0$, $\Xi_i > 0$, $i = 1, \ldots, \ell$, F_1, F_2, Q, R, S, Z and W solving the following convex optimization problem:

min γ subject to:

$$\begin{bmatrix} -X_i + \mathrm{Her}\{F_1\tilde{H}_{1i}\} & \star & \star & \star \\ F_2\tilde{H}_{1i} + H_{2i}^T F_1^T & \mathrm{Her}\{F_2 H_{2i}\} & \star & \star \\ \mathcal{A}_{1i} & \mathcal{A}_{2i} & X_i - \Upsilon & \star \\ \mathcal{L}_{1i} & L_{2i} & 0 & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} \Xi_i & \mathcal{B}_i^T \\ \mathcal{B}_i & \Upsilon - X_i \end{bmatrix} > 0$$

$$\gamma - \mathrm{Tr}[\,\Xi_i\,] > 0$$

for $i = i, \ldots, \ell$, where $\mathcal{A}_{1i}$, $\mathcal{A}_{2i}$, $\mathcal{B}_i$, $\tilde{H}_{1i}$ and $\mathcal{L}_{1i}$ denote the matrices $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}$, $\tilde{H}_1$ and $\mathcal{L}_1$ of Theorem 1 at the i-th vertex of the polytope Ω. Moreover, $\mathrm{var}\{e\} < \gamma$ and the filter transfer function matrix is as in (39). □

4. CONCLUSIONS

This paper has addressed the design of minimum variance filters, for the one-step prediction, for uncertain linear discrete-time descriptor systems represented by a decoupled difference-algebraic state-space model. The matrices of the system state-space model are uncertain and assumed to belong to a given polytope. First, the case of systems without uncertainty is treated and necessary and sufficient LMI conditions are derived for the design of a linear stationary strictly proper filter that minimizes the asymptotic variance of the estimation error. This result is then extended for designing linear filters that guarantee the asymptotic stability of the estimation error and an optimized upper-bound on the asymptotic error variance, in spite of significant parameter uncertainty. The proposed robust filter design method is based on a parameter-dependent Lyapunov function.

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