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Brief paper

Stability conditions for multi-sensor state estimation over a lossy network☆

Tianju Sui[a], Keyou You[b], Minyue Fu[a,c]

[a] Department of Control Science and Engineering, Zhejiang University, Hangzhou, 310013, China

[b] Department of Automation, Tsinghua University, Beijing, 100084, China

[c] School of Electrical Engineering and Computer Science, The University of Newcastle, NSW 2308, Australia

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ABSTRACT

This paper studies a networked state estimation problem for a spatially large linear system with a distributed array of sensors, each of which offers partial state measurements. A lossy communication network is used to transmit the sensor measurements to a central estimator where the minimum mean square error (MMSE) state estimate is computed. Under a Markovian packet loss model, we provide necessary and sufficient conditions for the stability of the estimator for any diagonalizable system in the sense that the mean of the state estimation error covariance matrix is uniformly bounded. In particular, the stability conditions for the second-order systems with an i.i.d. packet loss model are explicitly expressed as simple inequalities in terms of the largest open-loop pole and the packet loss rate.

1. Introduction

This work is concerned with the networked state estimation problem for a spatially large discrete-time linear system with distributed sensing. Each sensor provides a partial state measurement with an additive noise, and each measured output is transmitted to a remote (central) estimator through a lossy communication network involving packet loss. The estimator computes a minimum mean square error (MMSE) estimator of the system state using the received measurements. The configuration is illustrated in Fig. 1, which is motivated by a wide range of applications including networked control systems, multi-agent systems, smart electricity networks and sensor networks. The main contribution of this paper is to derive necessary and sufficient conditions for the stability of the estimator in the sense that the mean of the state estimation error covariance matrix is uniformly bounded.

With the rapid development of the sensor network and communication technologies, the problem of networked state estimation has received significant attention in the recent years (Hespanha, Naghshtabrizi, & Xu, 2007; Schenato, Sinopoli, Franceschetti, Poolla, & Sastry, 2007; Sinopoli et al., 2004). One of the major difficulties comes from the packet loss in transmitting the sensor measurements. By treating the received measurements as intermittent measurements, the Kalman filter technique is applied to compute the networked MMSE state estimate for the single sensor case (Sinopoli et al., 2004). However, the stability of the state estimator is known to be seriously influenced by the packet loss model and the algebraic structure of the system in a coupled and complicated manner (Huang & Dey, 2007; Mo & Sinopoli, 2010; You, Fu, & Xie, 2011). Strictly speaking, it is still not well understood how they jointly affect the stability of the networked MMSE state estimator.

Two frameworks for the networked state estimation are proposed in the literature, by transmitting either the raw measurements directly, or the processed one instead. The former approach is easy to implement but the associated stability condition is difficult to establish, whereas the latter one yields simpler stability conditions (Schenato, 2008; Sui, You, Fu, & Marelli, 2015) but adds the processing burden to the transmitters. The latter one may not be possible when considering the constraints of the hardware and power in sensor networks. Especially, by transmitting the estimate of state in sensor side, it typically has a higher dimension than the raw measurement; the latter approach tends to transmit more data through the network. Another major drawback is that in our distributed sensing setting, pre-computing the state estimate in each sensor might not be sensible due to the access of only partial state

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E-mail addresses: suitj@zju.edu.cn (T. Sui), youky@tsinghua.edu.cn (K. You), minyue.fu@newcastle.edu.au (M. Fu).

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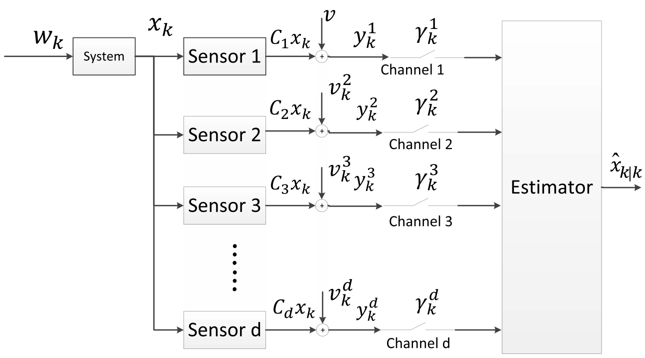

Fig. 1. Networked state estimation under distributed sensing.

measurements in each sensor. In Sun and Deng (2004), each sensor locally computes a state estimate and the central estimator aggregates these local estimators. Such an estimate is typically not optimal, and requires the stability of local estimators. This is an unnecessarily strong assumption for the distributed setting. For these reasons, it is preferable to directly transmit the raw measurements. We will adopt this framework in this paper.

To quantify the effect of packet loss, two types of channel models have been widely adopted: (1) the independent and identically distributed (i.i.d.) model where the packet loss process is modeled as an i.i.d. process (Sinopoli et al., 2004); (2) the Markovian model where the packet process is described by a binary Markov chain (Huang & Dey, 2007), which is inspired by the so-called Gilbert–Elliott (GE) channel. In the Markovian model, the packet loss rate is temporally correlated, which is more complicated.

Under the i.i.d. model, Sinopoli et al. (2004), Mo and Sinopoli (2008, 2010) and Plarre and Bullo (2009) focus on the stability of an intermittent Kalman filter with only one sensor transmitting its raw measurements, and there exists a critical packet loss rate, above which the mean of the state estimation error covariance matrix will diverge to infinity (Sinopoli et al., 2004). An upper bound and a lower bound for the critical packet loss rate are also given in Sinopoli et al. (2004). For a generic vector system, it is well known difficulty to explicitly express the critical packet loss rate. Motivated but also inspired by the limitation of Sinopoli et al. (2004), the lower bound is shown to be tight in Plarre and Bullo (2009) for the system with one-step observable, and the non-degenerate systems (Mo & Sinopoli, 2010). However, a counterexample in You et al. (2011) shows that the critical packet loss rate strictly lies between the lower and upper bounds.

For the Markovian model, the filter stability analysis was initiated in Huang and Dey (2007) where the only sensor transmits its raw measurements to the remote estimator, and a stability criterion was given in terms of an infinite sum. By exploiting the system structure, the necessary and sufficient stability condition for the second-order systems and certain classes of higher-order systems are explicitly given in You et al. (2011). In Rohr, Marelli, and Fu (2014), they derived necessary and sufficient stability conditions for a class of degenerate linear systems.

In comparison, this paper studied the networked state estimation problem with *multiple* sensors, with different sensors subject to possibly different packet losses. This is of great importance in many real-world scenarios where the system covers a large spatial domain, and is widely studied by many researchers (Deshmukh, Natarajan, & Pahwa, 2014; He, Wang, Wang, & Zhou, 2014; Hu, Wang, & Gao, 2013; Hu, Wang, Gao, & Stergioulas, 2012; Quevedo, Ahlén, & Johansson, 2013; Wei, Wang, & Shu, 2009). The stability analysis of the resulting networked MMSE estimator is challenging and Deshmukh et al. (2014), Quevedo et al. (2013) and Wei et al. (2009) worked on the sufficient condition for the estimation stability. The main difficulty caused by multiple sensors is that measurements at each sampling time may be partially lost, instead of either no loss or complete loss as in the case of single sensor.

The main contribution in our paper is to develop a new *regression matrix technique* to study the necessary and sufficient stability condition for the minimum mean square state estimator (MMSE), which is suitable for both single sensor and multi-sensor cases. Under the Markovian model, we establish a necessary and sufficient condition for the stability of the networked state estimator for diagonalizable systems with multiple sensors, which is able to characterize how the largest open-loop pole and the packet loss pattern jointly affect the stability of estimator. For a better verification, an efficient algorithm is designed to check the condition. We demonstrate, through a second-order system under the i.i.d. model, that the stability condition reduces to simple inequalities.

The rest of the paper is organized as follows. In Section 2, the problem formulation is described and the MMSE estimate for the system with multiple sensors over a lossy channel is derived. In Section 3, the stability condition for the MMSE estimator of a diagonalizable system is given. For second-order systems, stability conditions are given by simple inequalities in Section 4. Concluding remarks are drawn in Section 5. To improve the readability, some of the proofs are given in the Appendix.

Notation. x' is the transpose of vector x and A^* is the conjugate transpose of matrix A. $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix, and $\mathrm{col}\{C_1, \dots, C_n\} = [C_1', \ C_2', \dots, C_n']'$. The sets of real number and non-negative integer are represented by $\mathbb{R}$ and $\mathbb{N}$, respectively. For two discrete random vectors $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, let $\mathbb{P}\{X|Y\}$ denote the conditional probability mass function of X with the knowledge of Y, e.g., $\mathbb{P}\{X = x|Y = y\}$ for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

2. Problem formulation

Consider a discrete-time stochastic system

$$x_{k+1} = Ax_k + w_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state and w_k is a white Gaussian noise with covariance matrix $Q > 0$. The initial state x_0 is a Gaussian random vector with mean $\bar{x}_0$ and covariance matrix $P_0 > 0$. To remotely estimate the system state, we use a sensor network in Fig. 1 with $d \geq 2$ sensors to take noisy measurements, i.e.,

$$y_k^i = C_i x_k + v_k^i, \quad i \in \{1, 2, \dots, d\}, \tag{2}$$

where $v_k^i \in \mathbb{R}^{m_i}$ is a white Gaussian noise of sensor i with covariance matrix $R_i > 0$ and $\sum_{i=1}^d m_i = m$. In addition, x_0, w_k and v_k^i are mutually independent.

All the random variables in this paper are defined on a common probability space $(\Omega, \mathbb{P}, \mathcal{F})$, where Ω is the space of elementary events, $\mathcal{F}$ is the underlying σ-field on Ω, and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. Throughout the paper, we denote

$$y_k = \mathrm{col}\{y_k^1, y_k^2, \dots, y_k^d\}, \qquad C = \mathrm{col}\{C_1, C_2, \dots, C_d\}, \quad \text{and}$$
$$R = \mathrm{diag}\{R_1, R_2, \dots, R_d\}. \tag{3}$$

We are concerned with a networked system, where each sensor and the central estimator are linked through a communication network. Due to the channel unreliability, the transmitted packets may be randomly lost. We use a binary random process γ_k^i to describe the packet loss process. That is, $\gamma_k^i = 1$ indicates that the packet transmitted from sensor i is successfully delivered to the estimator at time k, and $\gamma_k^i = 0$ if the packet is lost. The implication of packet loss is that the estimator may fail to generate a stable state estimator.

Our objective is to study how the packet loss will affect the stability of the MMSE estimator. To this end, we denote

$$\Upsilon_k = \mathrm{diag}\{\gamma_k^1 I_1, \dots, \gamma_k^d I_d\} \in \mathbb{D}, \tag{4}$$

where $I_i \in \mathbb{R}^{m_i \times m_i}$ is an identity matrix, and $\mathbb{D}$ consists of 2^d elements.

Define the packet arrival matrix

$$S_k = \mathrm{diag}\{\Upsilon_0, \Upsilon_1, \dots, \Upsilon_{k-1}\} \in \mathbb{S}_k. \tag{5}$$

The information available to the estimator at time k is then given by

$$\mathcal{F}_k = \{(\Upsilon_0, \Upsilon_0 y_0), (\Upsilon_1, \Upsilon_1 y_1), \dots, (\Upsilon_k, \Upsilon_k y_k)\}. \tag{6}$$

Denote the MMSE (one-step-ahead) predictor and the MMSE estimator by

$$\hat{x}_{k|k-1} = \mathbb{E}[x_k|\mathcal{F}_{k-1}] \quad \text{and} \quad \hat{x}_{k|k} = \mathbb{E}[x_k|\mathcal{F}_k],$$

respectively. Their corresponding estimation error covariance matrices are defined by

$$P_{k|k-1} = \mathbb{E}[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})'|\mathcal{F}_{k-1}]$$

and

$$P_{k|k} = \mathbb{E}[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})'|\mathcal{F}_k].$$

In this paper, a Kalman like algorithm is developed to recursively compute the MMSE estimate, and we establish the packet loss condition under which the mean of the state estimation error covariance matrix is uniformly bounded, i.e.,

$$\limsup_{k\to\infty} \mathbb{E}[P_{k|k}] < \infty, \tag{7}$$

where the expectation is taken with respect to the random process $\{\Upsilon_k\}$. Here (7) is interpreted that there exists a positive-definite matrix $\bar{P} > 0$ such that $\mathbb{E}[P_{k|k}] \le \bar{P}$ for all $k \in \mathbb{N}$.

As in one sensor case (Sinopoli et al., 2004), the loss of sensor measurement is equivalent to that the measurement noise level goes to infinity. Then, a Kalman like estimator is optimal under multiple sensors as shown below.

Estimator 1. *The MMSE estimate for the networked system in (1)–(2) is recursively computed by*

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k\Upsilon_k(y_k - C\hat{x}_{k|k-1}); \tag{8}$$

$$P_{k|k} = P_{k|k-1} - K_k\Upsilon_k C P_{k|k-1}, \tag{9}$$

where

$$K_k = P_{k|k-1}C'\Upsilon_k(\Upsilon_k C P_{k|k-1}C'\Upsilon_k + R)^{-1}; \tag{10}$$

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k}; \tag{11}$$

$$P_{k+1|k} = AP_{k|k}A' + Q. \tag{12}$$

Remark 1. For the multi-sensor case, the structure of K_k depends on Υ_k explicitly, which is different with that of Sinopoli et al. (2004). This stochastic dependence makes the stability analysis more complicated.

In the sequel, we shall study the stability of the networked MMSE estimator in Theorem 1 under the following Markovian packet loss model.

Assumption 1. The packet receival process Υ_k has Markovian property of order ν in the sense that $\mathbb{P}\{\Upsilon_k|\Upsilon_{k-1},\dots,\Upsilon_{k-\nu}\} = \mathbb{P}\{\Upsilon_k|\Upsilon_{k-1},\dots,\Upsilon_0\}$, and has a homogeneously positive transition matrix, i.e., there exists a function f such that $f(\Upsilon_k,\dots,\Upsilon_{k-\nu}) = \mathbb{P}\{\Upsilon_k|\Upsilon_{k-1},\dots,\Upsilon_{k-\nu}\} > 0$.

Note that the above Markov process contains the one in Huang and Dey (2007) as a special example. For brevity, there is no loss of generality to assume that $\Upsilon_k = 0$ if $k < 0$.

3. Stability analysis of the MMSE estimator

To establish the stability condition for the MMSE estimator (8)–(9), we define an N-step *regression matrix*

$$O_N = \mathcal{L}(S_N \text{col}\{C, CA, \dots, CA^{N-1}\}), \tag{13}$$

where $\mathcal{L}(A)$ removes all possible zero row vectors of matrix A. That is, there is no zero row vector in $\mathcal{L}(A)$.

Actually, O_N determines the observability of the associated system under packet loss and is central to the stability of the networked MMSE estimator. Specifically, the larger the number of packet loss is, the higher probability O_N becomes column rank deficient, which may result in the instability of the MMSE estimator. Thus, we extensively explore the full column rankness of O_N. To this purpose, the system (1)–(2) involving packet losses is rewritten as

$$x_N = A^N x_0 + GW, \tag{14}$$

$$Y_N = O_N x_0 + FW + V, \tag{15}$$

where $G = [A^{N-1},\dots,A,I]$, $W = \text{col}\{w_0,\dots,w_{N-1}\}$, $Y_N = \mathcal{L}(S_N\text{col}\{y_0,\dots,y_{N-1}\})$, $V = \mathcal{L}(S_N\text{col}\{v_0,\dots,v_{N-1}\})$ and

$$F = \mathcal{L}\left(S_N\begin{bmatrix} 0 & 0 & \dots & 0 & 0\\ C & 0 & \dots & 0 & 0\\ CA & C & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ CA^{N-2} & CA^{N-3} & \dots & C & 0\end{bmatrix}\right).$$

Since x_0, W and V are Gaussian, the estimation error covariance matrix of the MMSE estimate of x_N using Y_N is given in a compact form as

$$P_{N|N-1} = G\Sigma_W G^* + \Sigma_{x_N} - \Sigma_{x_N Y_N'}\Sigma_{Y_N}^{-1}\Sigma_{x_N Y_N'}^*, \tag{16}$$

where $\Sigma_{x_N} = A^N P_0 (A^N)^*$, $\Sigma_{x_N Y_N'} = A^N P_0 O_N^* + G\Sigma_W F^*$, $\Sigma_{Y_N} = O_N P_0 O_N^* + F\Sigma_Q F^* + \Sigma_V$, $\Sigma_W = I_N \otimes Q$ and $\Sigma_V = \mathcal{L}(S_N(I_N\otimes R)S_N^*)$. Note that (9) and (12) can recursively compute (16).

To establish the stability condition, we study an upper bound of $P_{N|N-1}$ first. Denote $P_k := P_{k|k-1}$; we have the following results.

Lemma 2. *Assume that O_N has full column rank. There exists a positive definite matrix $\bar{P}_N$, independent of P_0, such that*

$$P_N \le \bar{P}_N. \tag{17}$$

Proof. Consider a direct estimate of x_N using Y_N as follows

$$\bar{x}_N = A^N O_N^\dagger Y_N, \tag{18}$$

where $O_N^\dagger$ is the Moore–Penrose inverse (Horn & Johnson, 1985) of O_N. Since O_N has full column rank, then

$$O_N^\dagger O_N = I. \tag{19}$$

By (14) and (15), it yields that

$$Y_N = O_N x_0 + FW + V. \tag{20}$$

Substituting (20) into (18), it follows from (19) that

$$\begin{aligned}\bar{x}_N &= A^N O_N^\dagger[O_N x_0 + FW + V]\\ &= A^N x_0 + A^N O_N^\dagger[FW + V]\\ &= x_N - [(G - A^N O_N^\dagger F)W - A^N O_N^\dagger V].\end{aligned} \tag{21}$$

The estimation error of the least square estimate in (18) is computed by $\bar{e}_N = x_N - \bar{x}_N = (G - A^N O_N^\dagger F)W - A^N O_N^\dagger V$. This implies that

$$\begin{aligned}\bar{P}_N &= \mathbb{E}[\bar{e}_N(\bar{e}_N)'|\mathcal{F}_N]\\ &= (G - A^N O_N^\dagger F)\Sigma_W(G - A^N O_N^\dagger F)' + (A^N O_N^\dagger)\Sigma_V(A^N O_N^\dagger)',\end{aligned} \tag{22}$$

which only associates with noises, and is independent of the initial estimation P_0. Since P_N is achieved by the MMSE estimate using Y_N, it is trivial that $\bar{P}_N \ge P_N$. ■

With the initial packet arrival process defined as $S_{-\nu,0} = 0$, we define the set of all S_N leading to column rank deficient O_N (i.e., not having full column rank) by

$$\mathcal{R}_N = \{S_N | O_N \text{ is column rank deficient}\}, \tag{23}$$

and the probability of this set is

$$\mathbb{P}(\mathcal{R}_N) \triangleq \mathbb{P}(S_N \in \mathcal{R}_N) = \sum_{S_N \in \mathcal{R}_N} \mathbb{P}(S_N). \tag{24}$$

This quantity is important to stability analysis as it characterizes the probability of the regression matrix O_N losing observability. Two cases under different system structures are discussed in the sequel.

3.1. Single eigen-block

For a system having a single eigen-block, all the open-loop poles are with the same magnitude. In particular, it has the following structure.

Assumption 2. $A = \alpha\,\mathrm{diag}(e^{\mathrm{i}\theta_1}, e^{\mathrm{i}\theta_2}, \ldots, e^{\mathrm{i}\theta_n})$ for some magnitude $\alpha > 0$, and (C, A) is observable.

Now, we are in the position to deliver our main result for the single eigen-block case.

Theorem 3. *Under Assumptions 1 and 2, the necessary and sufficient condition for* $\limsup_{N\to\infty} \mathbb{E}[P_N] < \infty$ *is that*

$$\alpha^2 \limsup_{N\to\infty} (\mathbb{P}(\mathcal{R}_N))^{1/N} < 1. \tag{25}$$

By (23), the inequality in (25) implies that the stability of the MMSE estimator is determined by the probability of packet loss patterns leading to the column rank deficiency of O_N.

Two lemmas below are needed to prove Theorem 3.

Lemma 4. *Suppose that O_N is column rank deficient. Under Assumption 2, it holds that $Tr(P_N) \geq \underline{p}\alpha^{2N}$ for any $\underline{p} > 0$ satisfying $\underline{p}I \leq P_0$ and $\underline{p}I \leq Q$.*

Proof. Consider a special case that $w_k = 0$ and $v_k^i = 0$ for all $k \in \mathbb{N}$ and $i \in \{1, 2, \ldots, d\}$. The error covariance matrix of the MMSE estimate is denoted by $\underline{P}_N$, and is computed by (16) as

$$\underline{P}_N = A^N P_0 (A^N)^* - A^N P_0 O_N^* (O_N P_0 O_N^*)^\dagger O_N P_0 (A^N)^*. \tag{26}$$

Obviously, $\underline{P}_N \leq P_N$. Since the right-hand side (RHS) of (26) is monotonically increasing in P_0 (Sinopoli et al., 2004), it follows that

$$\underline{P}_N \geq \underline{p} A^N (I - O_N^* (O_N O_N^*)^\dagger O_N)(A^N)^*. \tag{27}$$

By the singular value decomposition (Horn & Johnson, 1985), it holds that $O_N = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V$, where U and V are unitary matrices, and D is an $a \times a$ invertible matrix with $a = \mathrm{rank}(O_N) < n$. Subsequently, $O_N^*(O_N O_N^*)^\dagger O_N = V^* \begin{bmatrix} I_a & 0 \\ 0 & 0 \end{bmatrix} V$. Together with (27), it follows that

$$\underline{P}_N \geq \underline{p} A^N V^* \begin{bmatrix} 0 & 0 \\ 0 & I_{n-a} \end{bmatrix} V (A^N)^*. \tag{28}$$

By Assumption 2 and that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any compatible matrices A and B, it implies $\mathrm{Tr}(\underline{P}_N) \geq \underline{p}\alpha^{2N}\mathrm{Tr}(I_{n-a}) \geq \underline{p}\alpha^{2N}$. The proof is completed by noting that $\mathrm{Tr}(P_N) \geq \mathrm{Tr}(\underline{P}_N)$. ■

To explicitly express the dependence of P_N on S_N and P_0 (the covariance matrix of x_0), we denote it by $\phi(P_0, S_N)$. Let $P_0 = xI$ for $x > 0$, and write the associated $\mathbb{E}[P_N]$ as $\xi_N(x) := \mathbb{E}[P_N]$, it follows that

$$\xi_N(x) := \sum_{S_N \in \mathbb{S}_N} \mathbb{E}[P_N | S_N]\mathbb{P}(S_N) = \sum_{S_N \in \mathbb{S}_N} \phi(xI, S_N)\mathbb{P}(S_N). \tag{29}$$

Then, we introduce following result.

Lemma 5 (Sui, You, & Fu, 2014)**.** *For any $P_0 > 0$, if there exists $N_0 > 0$ and $\tilde{x} > 0$ such that $\tilde{x}I \geq \xi_{N_0}(\tilde{x})$ and $\tilde{x}I \geq P_0$, then* $\limsup_{N\to\infty} \mathbb{E}[P_N] < \infty$.

Proof of Theorem 3. Necessity: Denote the complement of $\mathcal{R}_N$ in (23) by $\mathcal{R}_N^c$, which contains all S_N leading to the full column rankness of O_N.

Let ℓ be the minimum integer such that $\mathbb{P}(\mathcal{R}_\ell^c) > 0$. Such a finite ℓ must exist. Indeed, it follows from Assumption 1 that the probability of receiving packets up to time n is positive. Since (C, A) is observable, it implies that $\mathbb{P}(\mathcal{R}_n^c) > 0$.

Given a sufficiently large integer N, considering a time horizon from 0 to $N\ell$, we shall use $\mathrm{Tr}(\mathbb{E}[P_{N\ell}])$ to derive the necessary condition for stability.

For $0 \leq k_1 < k_2$, define a regression matrix O_{k_1,k_2} by

$$O_{k_1,k_2} = \mathcal{L}(S_{k_1,k_2}\mathrm{col}\{C, CA, \ldots, CA^{k_2-k_1-1}\}).$$

Accordingly, let

$$\mathcal{R}_{k_1,k_2} = \{S_{k_1,k_2} | O_{k_1,k_2} \text{ is column rank deficient}\},$$

and its complement

$$\mathcal{R}_{k_1,k_2}^c = \{S_{k_1,k_2} | O_{k_1,k_2} \text{ is full column rank}\}.$$

Define $\mathbb{P}(\mathcal{R}_{k_1,k_2}) \triangleq \sum_{S_{k_1,k_2} \in \mathcal{R}_{k_1,k_2}} \mathbb{P}(S_{k_1,k_2})$ and similarly for $\mathbb{P}(\mathcal{R}_{k_1,k_2}^c)$. Let l be any integer with $0 \leq l < \ell$. The set of all possible $S_{l,N\ell}$ is divided into the following disjoint subsets.

- Subset 1: $O_{l,N\ell}$ is column rank deficient.
- Subset 2: $O_{l,N\ell}$ has full column rank but $O_{\ell+l,N\ell}$ is column rank deficient.
- ......
- Subset N: $O_{(N-2)\ell+l,N\ell}$ has full column rank but $O_{(N-1)\ell+l,N\ell}$ is column rank deficient.

Then, the probability of events in Subset 1 is given by $\mathbb{P}(\mathcal{R}_{l,N\ell})$, and that in Subset $j+1$ is given by $\mathbb{P}(\mathcal{R}_{j\ell+l,N\ell})\mathbb{P}(\mathcal{R}_{(j-1)\ell+l,N\ell}^c | \mathcal{R}_{j\ell+l,N\ell})$, $j = 1, \ldots, N-1$.

In view of (5) and (23), it implies that $\mathcal{R}_N = \mathcal{R}_{0,N}$ and $\mathbb{P}(\mathcal{R}_t) = \mathbb{P}(\mathcal{R}_{0,t} | S_{-\nu,0} = 0)$. By the homogeneity in Assumption 1, it follows that for any $p, t \in \mathbb{N}$,

$$\mathbb{P}(\mathcal{R}_t) = \mathbb{P}(\mathcal{R}_{p,p+t} | S_{p-\nu,p} = 0). \tag{30}$$

Since $\mathbb{P}\{\Upsilon_k | \Upsilon_{k-1}, \ldots, \Upsilon_{k-\nu}\} > 0$, there exists a positive β independent with t and p such that

$$\min_{S_{p-\nu,p}} \mathbb{P}(\mathcal{R}_{p,p+t} | S_{p-\nu,p}) \geq \beta\mathbb{P}(\mathcal{R}_{p,p+t} | S_{p-\nu,p} = 0) = \beta\mathbb{P}(\mathcal{R}_t). \tag{31}$$

Then, we obtain that

$$\mathbb{P}(\mathcal{R}_{p,p+t}) = \sum_{S_{p-\nu,p} \in \mathbb{S}_\nu} \mathbb{P}(\mathcal{R}_{p,p+t} | S_{p-\nu,p})\mathbb{P}(S_{p-\nu,p}) \geq \beta\mathbb{P}(\mathcal{R}_t) \sum_{S_{p-\nu,p} \in \mathbb{S}_\nu} \mathbb{P}(S_{p-\nu,p}) = \beta\mathbb{P}(\mathcal{R}_t). \tag{32}$$

Using the above decomposition and denoting by $\{i\}$ the Subset i, we further obtain that

$$\mathrm{Tr}(\mathbb{E}[P_{N\ell}]) = \mathrm{Tr}\left(\sum_{S_{N\ell} \in \mathbb{S}_{N\ell}} \phi(P_0, S_{N\ell})\mathbb{P}(S_{N\ell})\right)$$
$$= \sum_{S_{N\ell} \in \{1\}} \mathrm{Tr}(\phi(\phi(P_0, S_{0,l}), S_{l,N\ell}))\mathbb{P}(S_{N\ell}) + \cdots$$
$$+ \sum_{S_{N\ell} \in \{N\}} \mathrm{Tr}(\phi(\phi(P_0, S_{0,l+(N-2)\ell}), S_{l+(N-2)\ell,N\ell}))\mathbb{P}(S_{N\ell})$$

$$\geq \sum_{S_{N\ell}\in\{1\}} \mathrm{Tr}(\phi(Q, S_{l,N\ell}))\mathbb{P}(S_{N\ell}) + \cdots$$
$$+ \sum_{S_{N\ell}\in\{N\}} \mathrm{Tr}(\phi(Q, S_{l+(N-2)\ell,N\ell}))\mathbb{P}(S_{N\ell}). \quad (33)$$

Based on Lemma 4, with $\underline{p}I \leq P_0$ and $\underline{p}I \leq Q$, it follows that

$$\begin{aligned}\mathrm{Tr}(\mathbb{E}[P_{N\ell}]) &\geq \underline{p}\alpha^{2(N\ell-l)}\mathbb{P}(\mathcal{R}_{l,N\ell}) + \sum_{j=1}^{N-2}\underline{p}\alpha^{2((N-j)\ell-l)} \\ &\quad \times \mathbb{P}(\mathcal{R}_{j\ell+l,N\ell})\mathbb{P}(\mathcal{R}^c_{(j-1)\ell+l,N\ell}|\mathcal{R}_{j\ell+l,N\ell}) \\ &\geq \underline{p}\beta\alpha^{2(N\ell-l)}\mathbb{P}(\mathcal{R}_{N\ell-l}) + \sum_{j=1}^{N-2}\underline{p}\beta\alpha^{2((N-j)\ell-l)} \\ &\quad \times \mathbb{P}(\mathcal{R}_{(N-j)\ell-l})\mathbb{P}(\mathcal{R}^c_{(j-1)\ell+l,N\ell}|\mathcal{R}_{j\ell+l,N\ell}). \end{aligned} \quad (34)$$

Similar to (32), there exists a positive β_1 such that

$$\mathbb{P}(\mathcal{R}^c_{(j-1)\ell+l,N\ell}|\mathcal{R}_{j\ell+l,N\ell}) \geq \beta_1\mathbb{P}(\mathcal{R}^c_\ell) > 0. \quad (35)$$

Let $\beta_2 = \underline{p}\beta\beta_1\mathbb{P}(\mathcal{R}^c_\ell)$, it follows from (34) and (35) that

$$\mathrm{Tr}(\mathbb{E}[P_{N\ell}]) \geq \beta_2 \sum_{j=2}^{N-1}\alpha^{2(j\ell-l)}\mathbb{P}(\mathcal{R}_{j\ell-l}).$$

Note that β_2 is independent of N. By

$$\limsup_{N\to\infty}\mathrm{Tr}(\mathbb{E}[P_{N\ell}]) < \infty,$$

it implies that $\alpha^2 \limsup_{j\to\infty}(\mathbb{P}(\mathcal{R}_{j\ell-l}))^{1/(j\ell-l)} < 1$. Since l is arbitrarily selected from the set $\{0, \dots, \ell-1\}$, we conclude that $\alpha^2 \limsup_{N\to\infty}(\mathbb{P}(\mathcal{R}_N))^{1/N} < 1$.

Sufficiency: Let S_N^0 be the event that there is no packet received up to time N, i.e., $\Upsilon_k = 0$ for all $0 \leq k \leq N-1$. Take any scalar q and x satisfying $qI > Q$ and $x > 0$, it follows that

$$\begin{aligned}\phi(xI, S_N^0) &= A^N(A^N)^*x + \sum_{j=0}^{N-1}A^jQ(A^j)^* \\ &\leq \left(\alpha^{2N}x + \sum_{j=0}^{N-1}\alpha^{2j}q\right)I \\ &\leq \alpha^{2N}\left(x + \frac{q}{\alpha^2-1}\right)I. \end{aligned} \quad (36)$$

Without any measurement, the resulting estimator cannot be better that using measurements. This implies that for any $S_N \in \mathcal{R}_N$,

$$\phi(xI, S_N) \leq \phi(xI, S_N^0). \quad (37)$$

Splitting the set $\mathbb{S}_N$ into $\mathcal{R}_N$ and its complement $\mathcal{R}_N^c$, it follows from Lemma 2 that

$$\begin{aligned}\xi_N(x) &= \sum_{S_N\in\mathcal{R}_N\cup\mathcal{R}_N^c}\phi(xI, S_N)\mathbb{P}(S_N) \\ &\leq \mathbb{P}(\mathcal{R}_N^c)\bar{P}_N + \sum_{S_N\in\mathcal{R}_N}\phi(xI, S_N)\mathbb{P}(S_N) \\ &\leq \mathbb{P}(\mathcal{R}_N^c)\bar{P}_N + \mathbb{P}(\mathcal{R}_N)\phi(xI, S_N^0) \\ &\leq \mathbb{P}(\mathcal{R}_N^c)\bar{P}_N + \mathbb{P}(\mathcal{R}_N)\alpha^{2N}\left(x + \frac{q}{\alpha^2-1}\right)I. \end{aligned}$$

Since $\limsup_{N\to\infty}\alpha^{2N}\mathbb{P}(\mathcal{R}_N) < 1$, it is clear that for sufficiently large $x > 0$, we have $\xi_N(x) \leq xI$ and $xI > P_0$ for some N. By Lemma 5, it follows that $\limsup_{N\to\infty}\mathbb{E}[P_N] < \infty$ for any $P_0 > 0$. ■

3.2. Extension to multiple eigen-blocks

We now generalize the result on single eigen-block to the multiple eigen-blocks, which includes any diagonalizable system.

Assumption 3. $A = \mathrm{diag}\{A_1, A_2, \dots, A_g\} \in \mathbb{R}^{n\times n}$, where $A_i = \alpha_i \mathrm{diag}\{e^{\mathrm{i}\theta_{i1}}, e^{\mathrm{i}\theta_{i2}}, \dots, e^{\mathrm{i}\theta_{in_i}}\}$ has a single eigen-block with $\alpha_i \geq 0$, $\alpha_i \neq \alpha_j$ for any $i \neq j$, and (C, A) is observable.

In light of the structure of A with multiple eigen-blocks, we decompose O_N into $O_N = \begin{bmatrix} O_N^1 & O_N^2 & \dots & O_N^g \end{bmatrix}$, where O_N^i is a $mN \times n_i$ matrix corresponding to A_i. The main result for the multiple eigen-blocks case is given below.

Theorem 6. *Under Assumptions 1 and 3, the necessary and sufficient condition for* $\limsup_{N\to\infty}\mathbb{E}[P_N] < \infty$ *is that*

$$\alpha_i^2 \limsup_{N\to\infty}(\mathbb{P}(\mathcal{R}_N(i)))^{1/N} < 1, \quad \forall\, i \in \{1, 2, \dots, g\}, \quad (38)$$

where $\mathcal{R}_N(i) = \{S_N | O_N^i \text{ is column rank deficient}\}$.

Remark 7. It is clear that Theorem 6 covers the result in Theorem 3. However, its proof is very tedious and technical. Since the idea of proof is the same as that of Theorem 3, we only provide the sketch of proof in Appendix A.

3.3. Computation of $\mathbb{P}(\mathcal{R}_N)$

In Theorem 3, the key factor to check the stability condition for systems satisfying Assumption 2 is to compute $\mathbb{P}(\mathcal{R}_N)$. We design an algorithm to do this in this subsection. Since the column part $O_N^i, i = 1, 2, \dots, g$ is independent with structures $A_j, j \neq i$, the studied system in the computation of $\limsup_{N\to\infty}(\mathbb{P}(\mathcal{R}_N(i)))^{1/N}$ is A_i, which follows Assumption 2. By computing all $\limsup_{N\to\infty}(\mathbb{P}(\mathcal{R}_N(i)))^{1/N}$ with $i = 1, 2, \dots, N$, the stability condition under Assumption 3 can be done similarly. To this end, we only study the case under Assumption 2 as an example and the period of a matrix is introduced.

Definition 1. If there exists a finite positive integer τ such that $A^\tau = \alpha^\tau I$ and τ is the minimum one, we say that A is *periodic* with a period of τ. If τ does not exist, A is *aperiodic*, and set $\tau = \infty$.

If A has a period τ, then $CA^{k+\tau} = \alpha^\tau CA^k$ for all k. The contribution of y_k to the rank of O_N is the same as that of $y_{k+\tau}$ if $k \leq N-\tau$. This observation enables us to work on a finite-length sequence in Algorithm 1.

Algorithm 1 (*Boolean Operation*).

Step 1: For any $k \in \{1, 2, \dots, \tau\}$, define $\widetilde{\gamma}_k^i = \gamma_k^i \vee \gamma_{k+\tau}^i \vee \cdots \vee \gamma_{k+\lceil N/\tau\rceil\tau}^i$ and $\widetilde{\Upsilon}_k = \mathrm{diag}(\widetilde{\gamma}_k^1 I_1, \dots, \widetilde{\gamma}_k^d I_d)$, where $\vee$ is a Boolean OR operator, and $\lceil\cdot\rceil$ is the standard ceiling function.
Step 2: Let $\widetilde{S}_N = \mathrm{diag}(\widetilde{\Upsilon}_1, \dots, \widetilde{\Upsilon}_\tau)$.

It is clear that $\widetilde{S}_N \in \mathcal{R}_\tau$ is equivalent to that of $S_N \in \mathcal{R}_N$. Note from (23) that the cardinality of $\mathcal{R}_\tau$ is finite, and it allows us to explore its structure. In particular, $\mathbb{P}(\widetilde{S}_N \in \mathcal{R}_\tau)$ will be expressed via a specially designed matrix in this subsection and can be explicitly derived for some systems. To this purpose, we use $\zeta(\cdot)$ to count the number of measurements being received by the estimator at time τ, i.e.,

$$\zeta(S_\tau) = \sum_{k=1}^{\tau}\sum_{i=1}^{d}\gamma_k^i. \quad (39)$$

Assume that $\mathcal{R}_\tau$ has r elements, e.g. $\mathcal{R}_\tau = \{s_1, \dots, s_r\}$ where s_i is given in an ascending order in terms of $\zeta(s_i)$. Particularly, if $i < j$, then $\zeta(s_i) \leq \zeta(s_j)$. By Algorithm 1, it implies that $\mathbb{P}\{\widetilde{S}_{N+\tau} = s_i | \widetilde{S}_N = s_j\} = 0$.

Let M_{ij} be a matrix with (p, q)th element given by $\mathbb{P}(\widetilde{S}_{(N+1)\tau} = s_i, S_{(N+1)\tau-\nu,(N+1)\tau} = \mathbb{S}_\nu(p)|\widetilde{S}_{N\tau} = s_j, S_{N\tau-\nu,N\tau} = \mathbb{S}_\nu(q))$ where $\mathbb{S}_\nu(p)$ is the pth element of $\mathbb{S}_\nu$. By the homogeneity of the packet loss model in Assumption 1, M_{ij} is time-invariant, and M_{ij} is a zero matrix if $j > i$. Then, we obtain a lower triangular matrix M with $r \times r$ blocks

$$M = \begin{bmatrix} M_{11} & 0 & \dots & 0 \\ M_{21} & M_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ M_{r1} & M_{r2} & \dots & M_{rr} \end{bmatrix}. \tag{40}$$

The stability condition can be characterized by the maximum eigenvalue of M in the following result; see Appendix B for proof.

Theorem 8. *Under Assumption 1 and that A has a period τ, then*

$$\mathbb{P}(\mathcal{R}_{N\tau}) = uM^N v \tag{41}$$

for any integer $N \geq 0$, where $u = [1\ 1\ \dots\ 1]$, $v = [1\ 0\ \dots\ 0]'$. Moreover,

$$\begin{aligned} \limsup_{N\to\infty}(\mathbb{P}(\mathcal{R}_N))^{1/N} &= (\lambda_{\max}(M))^{1/\tau} \\ &= \max_{1\leq i\leq r}\{\lambda_{\max}(M_{ii})^{1/\tau}\}, \end{aligned} \tag{42}$$

where $\lambda_{\max}(M)$ and $\lambda_{\max}(M_{ii})$ are the largest eigenvalues of M and M_{ii} in magnitude, respectively.

Remark 9. Since M is a lower triangular matrix, its eigenvalues are the diagonal elements and much easier to get.

4. Second-order systems with multiple sensors

In this section, the necessary and sufficient stability condition for the MMSE estimator of a second-order system, i.e. $A \in \mathbb{R}^{2\times 2}$, with multiple sensors is explicitly given by simple inequalities under i.i.d. packet loss model. For brevity, we make the following assumption.

Assumption 4. (1) $\{\gamma_k^i\}$ is an i.i.d. process with its packet arrival rate $p_i = \mathbb{E}\{\gamma_k^i = 1\}$, and $\{\gamma_k^1\}, \dots, \{\gamma_k^d\}$ are spatially independent; (2) $\text{rank}(\text{col}\{C_i, C_j\}) = 2$ for any $1 \leq i \neq j \leq d$.

Remark 10. If $\text{rank}(\text{col}\{C_i, C_j\}) = 1$ and $\text{rank}(C_i) \neq 0$ for some $i \neq j$, then C_i and C_j are dependent. This means that receiving the packet from sensor i is equivalent to that of sensor j when concerning the rank of regression matrix in Eq. (13). We can combine sensors i and j, and endow a smaller packet loss probability $(1 - p_i)(1 - p_j)$.

Obviously, there is no loss of generality to renumber the sensors in three groups.

Group 1: For $1 \leq i \leq d_1$, (C_i, A) is observable and $\text{rank}(C_i) = 1$.
Group 2: For $d_1 < i \leq d_2$, (C_i, A) is unobservable and $\text{rank}(C_i) = 1$.
Group 3: For $d_2 < i \leq d$, $\text{rank}(C_i) = 2$.

Remark 11. The sensors in Group 1 are observable; it is proved in Lemma 15 that $\text{col}\{C_i, C_iA^j\}$ for any $1 \leq i \leq d_1$ and $1 \leq j < \tau$ is of full rank. This implies that any two packets received from the same sensor within a period result in a full column rank O_N. In contrast, any number of packets from the same sensor in Group 2 will lead to a column rank deficient O_N. The sensors in Group 3 are one-step observable, implying that single received packet is sufficient to get a full column rank O_N. As demonstrated before, the full rankness of O_N is essential to the stability of the networked MMSE estimator. This motivates us to divide the sensors in the above three groups.

For each sensor in Group 1, a *Congruent Set* is introduced, which is key to the rank analysis of O_N.

Definition 2. For any sensor from Group 1, e.g. sensor $i \in \{1, \dots, d_1\}$, a congruent set $\mathcal{J}_i$ is defined as $\mathcal{J}_i = \{j|\exists k_{ij} \in \mathbb{N},\ \text{s.t.}\ \text{span}\{C_i\} = \text{span}\{C_jA^{k_{ij}}\}\}$.

Since $i \in \mathcal{J}_i$, the congruent set is not empty. The probability that all sensors in $\mathcal{J}_i$ lose their packets at the same time is given by

$$p_i^* = \prod_{j\in\mathcal{J}_i}(1 - p_j). \tag{43}$$

Note that if the magnitudes of two eigenvalues of A are different, O_N^1 and O_N^2 defined in Section 3.2 are all column vector, leading to one-step observability, an easier case to handle. In view of this, we study the case having a single eigen-block.

By Theorems 3 and 8, the stability condition for the case of second-order systems can be explicitly expressed as simple inequalities as below; see Appendix C for proof.

Theorem 12. *Consider the second-order ($A \in \mathbb{R}^{2\times 2}$) system (1)–(2) under Assumptions 2 and 4.*

(a) *If $\min_{i=1}^{d_1} p_i^* \leq \min_{j=d_1+1}^{d_2}(1-p_j)^\tau$, the MMSE estimator is stable if and only if*

$$\frac{\prod_{i=1}^{d}(1 - p_i)^\tau}{\min_{j=1}^{d_1} p_j^*}\alpha^{2\tau} < 1. \tag{44}$$

(b) *If $\min_{i=1}^{d_1} p_i^* > \min_{j=d_1+1}^{d_2}(1-p_j)^\tau$, which obviously holds for an aperiodic A (i.e., $\tau = \infty$), the MMSE estimator is stable if and only if*

$$\frac{\prod_{i=1}^{d}(1 - p_i)}{\min_{j=d_1+1}^{d_2}(1 - p_j)}\alpha^2 < 1. \tag{45}$$

Remark 13. Substituting $d = 1$ into Theorem 12, our result is the same as Theorem 7 in You et al. (2011) for i.i.d. packet loss model. Thus, it generalizes a result in You et al. (2011) to the multiple sensors case.

5. Conclusion

In this paper, we have studied the networked estimation problem of a stochastic discrete-time system with multiple sensors. The networked MMSE estimator is recursively computed using a Kalman like algorithm. Then, we studied the stability of the MMSE estimator under a Markovian packet loss process, and derived the necessary and sufficient condition for the stability of MMSE estimator for diagonalizable systems. Moreover, for second-order systems under the i.i.d. packet loss model, the stability condition can be given by simple inequalities.

Appendix A. Proof of Theorem 6

We first introduce the following preliminary result, where O_N is defined in (13).

Lemma 14. *Let $\xi_i = \{e_{i,1}, e_{i,2}, \dots, e_{i,n_i}\}$ with $e_{i,j}$ being a column vector with its $(\sum_{q=1}^{i-1} n_q + j)$th element equal to 1 and all the other elements equal to zero. Suppose Assumption 3 holds and there exists $1 \leq i \leq g$ such that $\text{Ker}\{O_N\} \cap \text{span}\{\xi_i\} \neq \{0\}$. For any scalar $\underline{p} > 0$ satisfying $\underline{p}I \leq P_0$ and $\underline{p}I \leq Q$, it follows that*

$$\|P_N\| \geq \alpha_i^{2N}\underline{p}, \tag{A.1}$$

where $\|\cdot\|$ is the induced Euclidean norm.

Proof. By (27), it is straightforward to show that

$$P_N \geq \underline{p}A^N(I - O_N^\dagger O_N)(A^N)^*.$$

By the property of Moore–Penrose inverse, it holds that $(I - O_N^{\dagger} O_N)^2 = I - O_N^{\dagger} O_N$. Thus,

$$\|P_N\| \geq \underline{p}\|A^N (I - O_N^{\dagger} O_N)\|^2. \quad \text{(A.2)}$$

Take a nonzero vector $x \in \mathrm{Ker}\{O_N\} \cap \mathrm{span}\{\xi_i\}$; it follows that $O_N x = 0$, $(I - O_N^{\dagger} O_N)x = x$ and $Ax = \mathrm{diag}\{0_1, \ldots, 0_{i-1}, A_i, 0_{i+1}, \ldots, 0_g\}x$. This implies that

$$\|A^N (I - O_N^{\dagger} O_N)x\| = \|A^N x\| = \alpha_i^N \|x\|. \quad \text{(A.3)}$$

Together with $\|A^N (I - O_N^{\dagger} O_N)x\| \leq \|A^N (I - O_N^{\dagger} O_N)\|\|x\|$, we obtain that $\|A^N (I - O_N^{\dagger} O_N)\| \geq \alpha_i^N$. Substituting the above into (A.2) leads to (A.1).

Now, we are in the position to prove Theorem 6.

Necessity: It is clear that

$$\begin{aligned}\|\mathbb{E}[P_N]\| &= \|\sum_{S_N \in \mathbb{S}_N} \phi(P_0, S_N)\mathbb{P}(S_N)\| \\ &\geq \frac{1}{n}\mathrm{Tr}\left(\sum_{S_N \in \mathbb{S}_N} \phi(P_0, S_N)\mathbb{P}(S_N)\right) \\ &= \frac{1}{n}\sum_{S_N \in \mathbb{S}_N} \mathrm{Tr}(\phi(P_0, S_N))\mathbb{P}(S_N) \\ &\geq \frac{1}{n}\sum_{S_N \in \mathbb{S}_N} \|\phi(P_0, S_N)\|\mathbb{P}(S_N). \quad \text{(A.4)}\end{aligned}$$

Take any $i \in \{1, 2, \ldots, g\}$, and denote

$$\mathcal{R}_N^c(i) := \{S_N | O_N(i) \text{ is of full column rank}\}.$$

As in the proof of Theorem 6, there must exist a minimum and finite $\ell \in \mathbb{N}$ such that $\mathbb{P}(\mathcal{R}_\ell^c(i)) > 0$ for all $i = 1, 2, \ldots, g$. If $O_N(i)$ is column rank deficient, $\mathrm{Ker}\{O_N\} \cap \mathrm{span}\{\xi_i\} \neq \{0\}$. By Lemma 14 and following similar arguments in the necessity proof of Theorem 3, we obtain that

$$\|\mathbb{E}[P_{N\ell}]\| \geq \frac{\beta_2}{n}\sum_{j=2}^{N-1} \alpha_i^{2(j\ell - l)}\mathbb{P}(\mathcal{R}_{j\ell - l}(i)) \quad \text{(A.5)}$$

for any $l \in \{0, 1, \ldots, \ell - 1\}$. Again, β_2 is independent of N. This implies that (38) holds if $\limsup_{N\to\infty} \mathbb{E}[P_N] < \infty$.

Sufficiency: The main idea is the same as that of Theorem 3. However, it is more involved and tedious, we do not repeat here for saving space. ■

Appendix B. Proof of Theorem 8

Let

$$\begin{aligned}W_N = \mathrm{col}\{&\mathbb{P}(\widetilde{S}_{N\tau} = s_1, S_{N\tau-v,N\tau} = \mathbb{S}_v(1)), \\ &\mathbb{P}(\widetilde{S}_{N\tau} = s_1, S_{N\tau-v,N\tau} = \mathbb{S}_v(2)), \\ &\ldots, \mathbb{P}(\widetilde{S}_{N\tau} = s_1, S_{N\tau-v,N\tau} = \mathbb{S}_v(2^{d_v})), \\ &\ldots, \mathbb{P}(\widetilde{S}_{N\tau} = s_r, S_{N\tau-v,N\tau} = \mathbb{S}_v(2^{d_v}))\}. \quad \text{(B.1)}\end{aligned}$$

By the definition of M and let $v = [1\ 0\ \ldots\ 0]'$, we have that $W_{N+1} = MW_N$, and $W_N = (M)^N v$. Since the sum of all elements of W_N is $\mathbb{P}(\widetilde{S}_{N\tau} \in \mathcal{R}_\tau)$, it follows that

$$\mathbb{P}(S_{N\tau} \in \mathcal{R}_{N\tau}) = \mathbb{P}(\widetilde{S}_{N\tau} \in \mathcal{R}_\tau) = uW_N = uM^N v.$$

And the proof of (42) is divided into two parts:

(1) Find an upper bound of $\limsup_{N\to\infty} \sqrt[N\tau]{uM^N v}$:

By a similarity transformation (Horn & Johnson, 1985), we have that

$$M = V\mathrm{diag}\{B_1, B_2, \ldots, B_f\}V^{-1}, \quad \text{(B.2)}$$

where V is a non-singular matrix, B_j is a $b_j \times b_j$ Jordan block with eigenvalue λ_j and $\sum_{j=1}^f b_j = \sum_{i=1}^r t_i$.

In light of the structure of A, we decompose vectors uV and $V^{-1}v$ as $uV = \begin{bmatrix} U_1 & U_2 & \ldots & U_f \end{bmatrix}$ and $V^{-1}v = \mathrm{col}\{Z_1, Z_2, \ldots, Z_f\}$. This implies that

$$uM^N v = \sum_{j=1}^f U_j B_j^N Z_j. \quad \text{(B.3)}$$

Since B_j is in a Jordan form, the (p, q)th element of $B_j^N(p, q)$ is given by

$$B_j^N(p, q) = \begin{cases} \begin{pmatrix} N \\ q-p \end{pmatrix} \lambda_j^{N+p-q} & \text{if } p \leq q \\ 0 & \text{if } p > q, \end{cases} \quad \text{(B.4)}$$

where $\begin{pmatrix} N \\ q-p \end{pmatrix}$ is the combination number that selects $q-p$ elements from N elements. Rewrite $U_j B_j^N Z_j$ as $U_j B_j^N Z_j = \lambda_j^N \Psi_j(N)$ with

$$\Psi_j(N) = \sum_{p=1}^{b_j}\sum_{q=1}^{b_j} U_j(p)Z_j(q)B_j^N(p, q)\lambda_j^{-N},$$

where $U_j(p)$ is the pth element of U_j and $Z_j(q)$ is the qth element of Z_j.

Note that the magnitude of $B_j^N(p, q)\lambda^{-N}$ is $\begin{pmatrix} N \\ q-p \end{pmatrix}\lambda_j^{p-q}$. If $\Psi_j(N) \neq 0$, it follows that

$$\limsup_{N\to\infty} \sqrt[N\tau]{\Psi_j(N)} = 1.$$

Subsequently, $\limsup_{N\to\infty} \sqrt[N\tau]{uM^N v} \leq \sqrt[\tau]{\lambda_{\max}(M)}$.

(2) Find a lower bound of $\limsup_{N\to\infty} \sqrt[N\tau]{uM^N v}$:

We first note that

$$\mathbb{P}(\mathcal{R}_{N\tau}) \geq \max_{1\leq i\leq r}\{\mathbb{P}(\widetilde{S}_{N\tau} = s_i, \widetilde{S}_{(N-1)\tau} = s_i, \ldots, \widetilde{S}_\tau = s_i)\}. \quad \text{(B.5)}$$

Deleting the zero rows and columns in M_{ij}, it becomes $\widetilde{M}_{ij}$. Based on the properties of M_{ij} in Section 3.3, it implies that

$$\begin{aligned}&\mathbb{P}(\widetilde{S}_{N\tau} = s_i, \widetilde{S}_{(N-1)\tau} = s_i, \ldots, \widetilde{S}_\tau = s_i) \\ &= u_i M_{ii}^{N-1} M_{i1} v_1 = \tilde{u}_i \widetilde{M}_{ii}^{N-1} \widetilde{M}_{i1} \tilde{v}_1, \quad \text{(B.6)}\end{aligned}$$

where $u_i = [1\ 1\ \ldots\ 1] \in \mathbb{R}^{1\times 2^{d_v}}$, $v_1 = [1\ 0\ \ldots\ 0]' \in \mathbb{R}^{2^{d_v}\times 1}$, and $\tilde{u}_i$, $\tilde{v}_1$ are with the similar structure as u_i, v_1 with appropriate dimensions.

Using Assumption 1 and noting the definition of M_{i1} and $\underline{M}_{ii}$, we obtain that the first column of $\widetilde{M}_{i1}$ is a positive vector and $\widetilde{M}_{ii}$ is a positive matrix. By the Perron–Frobenius Theorem (Gantmacher, 1959), the eigenvector of the maximum eigenvalue of $\widetilde{M}_{ii}$ is positive. Then, replace v and u in (B.3) by $\widetilde{M}_{i1}\tilde{v}_1$ and $\tilde{u}_i$, both of which are positive vectors. Then, it follows that $\Psi_j(N)$ associated with the maximum eigenvalue of M_{ii} is positive. Thus

$$\begin{aligned}&\limsup_{N\to\infty} \sqrt[N\tau]{\mathbb{P}(\widetilde{S}_{N\tau} = s_i, \widetilde{S}_{(N-1)\tau} = s_i, \ldots, \widetilde{S}_\tau = s_i)} \\ &= \sqrt[\tau]{\lambda_{\max}(\widetilde{M}_{ii})} = \sqrt[\tau]{\lambda_{\max}(M_{ii})}. \quad \text{(B.7)}\end{aligned}$$

Substituting (B.7) into (B.5) leads to that

$$\limsup_{N\to\infty} \sqrt[N\tau]{\mathbb{P}(\mathcal{R}_{N\tau})} \geq \max_{1\leq i\leq r} \sqrt[\tau]{\lambda_{\max}(M_{ii})}.$$

Since M is a lower triangular matrix, it is obvious that $\max_{1\leq i\leq r} \sqrt[\tau]{\lambda_{\max}(M_{ii})} = \sqrt[\tau]{\lambda_{\max}(M)}$. In addition, $\mathbb{P}(\mathcal{R}_N)$ is a decreasing function of N. This implies that

$$\limsup_{N\to\infty} \sqrt[N]{\mathbb{P}(\mathcal{R}_N)} = \limsup_{N\to\infty} \sqrt[N\tau]{\mathbb{P}(\mathcal{R}_{N\tau})}.$$

Combining (1) and (2), (42) is proved. ■

Appendix C. Proof of Theorem 12

We introduce three technical Lemmas for the second-order system and their proofs are trivial and omitted.

Lemma 15. *Under Assumption 2 and (A, C_i) is observable, then for any $A \in \mathbb{R}^{2\times 2}$ with period τ and $k \neq h\tau$, $h \in \mathbb{N}$, it holds that*

$$rank \begin{bmatrix} C_i \\ C_iA^k \end{bmatrix} = 2. \tag{C.1}$$

Lemma 16. *Suppose Assumptions 2 and 4 hold and that (A, C_j) is not observable. For any $A \in \mathbb{R}^{2\times 2}$, $1 \le i \neq j \le d$ and $k \in \mathbb{N}$, it holds that*

$$rank \begin{bmatrix} C_i \\ C_jA^k \end{bmatrix} = 2. \tag{C.2}$$

Lemma 17. *Suppose that Assumption 2 holds, both (A, C_i) and (A, C_j) are observable and $A \in \mathbb{R}^{2\times 2}$. There exists at most one positive integer $1 \le k_{ij} < \tau$ such that*

$$rank \begin{bmatrix} C_i \\ C_jA^{k_{ij}} \end{bmatrix} = 1. \tag{C.3}$$

Proof of Theorem 12. Since the i.i.d. packet loss process satisfies Assumption 1, we use Theorems 3 and 8 here.

Recall that $\mathcal{R}_\tau = \{s_1, s_2, \ldots, s_r\}$, each of its element makes the regression matrix column rank deficient. In particular, the order of the elements are detailed below:

Let s_1 denote that no packet has been received, and by Lemma 16, $\{s_2, \ldots, s_r\}$ is divided into two sets: $\{s_2, s_3, \ldots, s_{\bar{d}+1}\}$ denote the events that the received measurements are from sensor 1 to sensor d_1. Similarly, $\{s_{\bar{d}+2}, s_{\bar{d}+3}, \ldots, s_r\}$ denote the events that the received measurements are from one of sensor $d_1 + 1$ to sensor d_2. Note the received measurement cannot be from any sensor d_2 to sensor d.

By i.i.d. packet loss process, $\mathbb{P}(\widetilde{S}_{N\tau} = s_i|\widetilde{S}_{(N-1)\tau} = s_j)$ is independent of $S_{(N-1)\tau-\nu,(N-1)\tau}$, where $\widetilde{S}_{N\tau}$ is given by Algorithm 1. In light of Lemmas 15–17, the diagonal elements of M in Theorem 8 can be given below:

(a) $\lambda_{\max}(M_{11}) = \mathbb{P}(s_1|s_1) = \prod_{q=1}^{d}(1 - p_q)^\tau$.

(b) $\max_{i=2}^{\bar{d}+1} \lambda_{\max}(M_{ii}) = \max_{i=2}^{\bar{d}+1} \mathbb{P}(s_i|s_i) = \max_{i=1}^{d_1}\{\prod_{q=1}^{d}(1 - p_q)^\tau / p_i^*\}$.

(c) $\max_{j=\bar{d}+2}^{r} \lambda_{\max}(M_{jj}) = \max_{j=\bar{d}+2}^{r} \mathbb{P}(s_j|s_j) = \max_{j=d_1+1}^{d_2}\{\prod_{q\neq j}(1 - p_q)^\tau\}$.

Using Theorem 8, we obtain that

$$\limsup_{N\to\infty} \sqrt[N]{\mathbb{P}(\mathcal{R}_N)} = \sqrt[\tau]{\lambda_{\max}(M)}. \tag{C.4}$$

Combining the above, we obtain that $\lambda_{\max}(M) = \max_{i=1}^{r} \lambda_{\max}(M_{ii})$. This implies that

(a) If $\min_{i=1}^{d_1} p_i^* \le \min_{j=d_1+1}^{d_2}(1 - p_j)^\tau$, then

$$\lambda_{\max}(M) = \frac{\prod_{i=1}^{d}(1 - p_i)^\tau}{\min_{i=1}^{d_1} p_i^*}.$$

(b) If $\min_{i=1}^{d_1} p_i^* > \min_{j=d_1+1}^{d_2}(1 - p_j)^\tau$, then

$$\lambda_{\max}(M) = \frac{\prod_{i=1}^{d}(1 - p_i)^\tau}{\min_{j=d_1+1}^{d_2}(1 - p_j)^\tau}.$$

Following Theorem 3, it completes the proof. ■

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Tianju Sui received B.S. degree from Zhejiang University, Hangzhou, China, in 2012. He is currently a doctoral candidate at Zhejiang University. He was the winner of the National Energy Saving Equipment Design competition and Outstanding Youth of Zhejiang Province, China, in 2012. He won the Chunhui Scholarship and Outstanding Graduate Award from Zhejiang University in 2012. His main research area includes networked estimation and distributed estimation.

Keyou You was born in Jiangxi Province, China, in 1985. He received the B.S. degree in statistical science from Sun Yat-sen University, Guangzhou, China, in 2007, and the Ph.D. degree in electrical and electronic engineering from Nanyang Technological University, Singapore, in 2012. He was with the ARC Center for Complex Dynamic Systems and Control, the University of Newcastle, Australia, as a visiting scholar from May 2010 to July 2010, and with the Sensor Network Laboratory at Nanyang Technological University as a Research Fellow from June 2011 to June 2012. Since July 2012, he has been with the Department of Automation, Tsinghua University, China, as a Lecturer. From May 2013 to July 2013, he held a visiting position in The Hong Kong University of Science and Technology, Hong Kong. His current research interests include control and estimation of networked systems, distributed control and estimation, and sensor network. Dr. You won the Guan Zhaozhi best paper award at the 29th Chinese Control Conference, Beijing, China, in 2010, and was selected to the "National 1000 Young Talents Plan" in 2013.

Minyue Fu received his Bachelor's Degree in Electrical Engineering from the University of Science and Technology of China, Hefei, China, in 1982, and M.S. and Ph.D. degrees in Electrical Engineering from the University of Wisconsin–Madison in 1983 and 1987, respectively. From 1987 to 1989, he served as an Assistant Professor in the Department of Electrical and Computer Engineering, Wayne State University, Detroit, Michigan. For the summer of 1989, he was employed by the Universite Catholoque de Louvain, Belgium, as a Maitre de Conferences Invited. He joined the Department of Electrical and Computer Engineering, the University of Newcastle, Australia, in 1989 and was promoted to a Chair Professor in Electrical Engineering in 2002. He has served as the Head of Department for Electrical and Computer Engineering and Head of School of Electrical Engineering and Computer Science over a period of 7 years. In addition, he was a Visiting Associate Professor at University of Iowa in 1995–1996, a Visiting Professor at Nanyang Technological University, Singapore, 2002, and Visiting Professor at Tokyo University in 2003. He has held a ChangJiang Visiting Professorship at Shandong University, a visiting Professorship at South China University of Technology, and a Qian-ren Professorship at Zhejiang University in China. He was elected to Fellow of IEEE in late 2003. His main research interests include control systems, signal processing and communications. His current research projects include networked control systems, multi-agent systems, smart electricity networks and super-precision positioning control systems. He has been an Associate Editor for the IEEE Transactions on Automatic Control, IEEE Transactions on Signal Processing, Automatica, and Journal of Optimization and Engineering.