Multiple Antenna Wireless Communication Systems: Limits to Capacity Growth

Leif Hanlen
Department of Electrical
and Computer Engineering
The University of Newcastle,
Callaghan, NSW 2308, Australia

Minyue Fu
Department of Electrical
and Computer Engineering
The University of Newcastle,
Callaghan NSW 2308, Australia

***Abstract*—We present a general multiple path scattering model for multiple transmit, multiple receive wireless systems. The model is generated using physical modelling of the scatterers surrounding transmit and the receive arrays. The condition of the resulting multiple-input multiple-output (MIMO) transfer matrix is then examined. A key parameter η which determines the channel condition is identified. This parameter depends on the local scatter geometries, the separation of arrays, and the wavelength of the transmitted signal. We show that there exists a critical value for this parameter at which the channel condition changes sharply. The implication is that the promised linear growth in channel capacity may not eventuate if the separation of the transmit and receive arrays is large ($\approx 10\times$ or $20\times$ at 2GHz) compared with the distance from array elements to local scatterers. Monte-Carlo simulations are used to demonstrate these claims.**

I. Introduction

The work of [1] and [2] predicted a remarkable capacity increase for multiple transmit, multiple receive wireless systems in the presence of multi-path scattering. In [1] and [2] a *linear* growth in capacity is predicted, proportional to the minimum number of transmit and receive antennas in the MIMO system. The predicted increase was shown for a practical indoor environment, where scatterers are dense in [3]. This has resulted in a large amount of literature (eg. [4], [5], [6], [7], [8] concerning MIMO wireless systems, and to the development of "Space-Time" codes, which attempt to achieve the predicted channel capacity.

Fundamental to this work is the assumption that the wireless channel may be modelled by an independent random $N \times M$ transfer matrix, where N and M are the numbers of the transmit and receive elements respectively. That is, the entries of the transfer matrix are assumed to be independent, complex random variables. The assumption of independence guarantees a (statistically) well-conditioned transfer matrix and prevents a loss of capacity due to correlations between the channels.

Independent multi-path coefficients may be ensured by assuming a random placement of many scatterers between transmitter and receiver elements. In this instance, *line-of-sight* (LOS) transmission may be present, but is not dominant. Similarly, the "rich scattering environment" in [3] corresponds to an absence of LOS transmissions. Both situations motivate the use of a *non-line-of-sight* (NLOS) model.

Besides [9] very little literature exists relating the channel model and scatterer geometries to the actual MIMO transfer matrix and few authors have addressed the physical conditions necessary for the required well-conditioned transfer matrix. In particular is local scattering, such as found in outdoor transmission, a sufficient approximation of a "rich scattering environment" in order to achieve the predicted linear increase in capacity?

The work of [9] studies a case where scatterers are placed locally in uniform arrays. From this geometry, two classes of channel model are predicted:

- an *"uncorrelated high rank model"* - the well known independent random matrix approximation; and
- an *"uncorrelated low rank"* or *"pin-hole"* model - where the channels are highly dependent and MIMO capacity is consequently low.

These models are disjoint - the transition from one model to the other is predicted (qualitatively) to occur smoothly at "large" separation of arrays.

This paper aims to provide a detailed analysis of the effects of separation of arrays for multi-antenna systems. More specifically, we study a case where scatterers are placed randomly within circular regions, surrounding the transmit and receive arrays. This corresponds to the well-known local scattering model. This model has the advantage that it provides more scope for analysis of the effects of separation, and includes the results of [9] as a subset, without a noticeable increase in model complexity. We present a channel model using simple physical principles. We then proceed to simplify this model and analyze the condition of the transfer matrix. We have identified a key parameter that determines the condition of the channel model. This parameter is dependent on the distance from array elements to local scatterers, the separation between the transmit and receive arrays, and the wavelength. We show that there exists a critical value for this parameter at which the condition of the channel model (and hence the channel capacity) alters sharply. The promised linear growth in channel capacity can not eventuate when the value of this parameter is too low, which corresponds to the case when the separation of transmit and receive arrays is large compared with the distance to local scatterers, or when the wavelength is too large.

This paper is organized as follows: The configuration of the

0-7803-7376-6/02/$17.00 (c) 2002 IEEE.

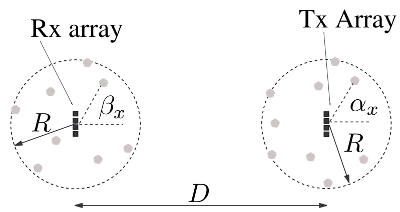

Fig. 1. Physical arrangement arrangement of scatterers and antenna elements for model

multi-antenna system is explained in Section II. Section III develops a MIMO transfer matrix for the channel. This transfer matrix is simplified and its condition is analyzed in Section IV. Some Monte-Carlo simulations are given in Section V. Section VI provides a summary of our work.

II. System Configuration

We consider a NLOS channel as shown in 1, where fading results from scatterers at both the transmit and receive ends of the link. We consider only scatterers close to either the transmitter or receiver arrays since distant scatterers tend to suffer large attenuation in signal strength, and have a correspondingly small effect on transmission. All signals are assumed to be narrow-band, and the channel is assumed to exhibit frequency-flat fading.

Consider the multi-antenna system with N transmit elements (denoted by tx) and M receive elements (denoted by rx), both placed in linear arrays.

The arrays are separated by a distance D. Without loss of generality, we may place the arrays parallel to each other. This removes the need to introduce orientation angles of one array relative to the other. This simplification is accurate if the waves transmitted may be approximated as plane waves and the signals are narrow-band. In this case, rotation of arrays is equivalent to adding extra phase offsets at each element. The vertical position for transmit (resp. receive) element i is denoted by $\delta_{t,i}$ (resp. $\delta_{r,i}$). For simplicity it is assumed that $\delta_{t,0} = \delta_{r,0} = 0$.

Surrounding each array is a disc of local scatterers, denoted *tx scatterers* and *rx scatterers* respectively. These scatterers are placed on an annulus of maximum radius R. The placement is assumed to be random, with a uniform distribution. It is assumed that R is large compared with the array length, and that D is large compared with R.

Using spherical coordinates, each tx scatterer is placed at (Rt_s, α_s) where $t_s \in (0, 1]$ and $\alpha_s \in [0, 2\pi)$ chosen at random. Similarly, each *rx scatterer* is placed at (Rr_s, β_s) with the appropriate translation of origin.

Each scatterer is assumed to be a small, lossless and memoryless reflector. The scatterer receives all transmitted signals isotropically, and re-transmits them isotropically. We will consider two types of scatterers: circular and linear.

III. Modelling

A. Scatterer Model

We consider circular and linear scatterers, as depicted in 2 and 3, respectively. The amount of power received by a scatterer depends only on the surface area exposed toward the transmitter.

For a circular scatterer 2, let the lines from the surface of the scatterer s, drawn to the transmitter tx, subtend an angle θ_T. The radius of the scatterer is $r_{\rm scat}$. The scatterer is placed at distance R_0 from the transmitter. The power received at scatterer s, will be a fraction, f, of the power transmitted by tx. We have

$$f \stackrel{\rm def}{=} \frac{\theta_T}{2\pi} = \frac{1}{\pi} \arcsin\left(\frac{r_{\rm scat}}{R_0}\right) \qquad (1)$$

We will denote f by $f(r_{\rm scat}/R_0)$ to show its explicit dependence.

Using a similar analysis, we can consider the case where a scatterer s is a line segment 3. The scatterer is placed at coordinates (R_0, α_s) from the transmitter tx. It has a length $2r_{\rm scat}$ and an angle ϕ_s.

The proportion of the power f_s received at scatterer s will depend on the length presented to the transmitter, i.e.,

$$f = \frac{1}{\pi} \arcsin\left(\frac{r_{\rm scat}\|\cos(\phi_s - \alpha_s)\|}{R_0}\right) \qquad (2)$$

The term $r_{\rm scat}\|\cos(\phi_s - \alpha_s)\|$ can be viewed as the *effective radius* of a circular scatterer. Therefore, in our future discussions, we will consider all scatterers to be circular, for which (1) applies.

It should be noted that f is the power attenuation factor. Therefore, the signal attenuation factor is proportional to $\sqrt{f}$.

B. Channel Model

Consider the channel in 1. Define X and Y as vectors of transmitted signals and received signals, respectively:

$$X \stackrel{\rm def}{=} \begin{bmatrix} x_0 & \dots & x_{N-1} \end{bmatrix}^T; \; Y \stackrel{\rm def}{=} \begin{bmatrix} y_0 & \dots & y_{M-1} \end{bmatrix}^T$$

Then, the transfer matrix H from tx to rx consists of three factors: a transfer matrix H_t from tx to the tx scatterers, a transfer matrix H_r from the rx scatterers to rx, and a transfer

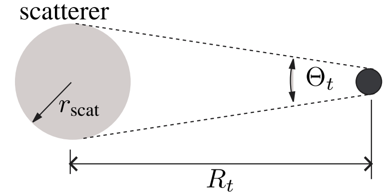

Fig. 2. Circular scatterer with effective reception/transmit area.

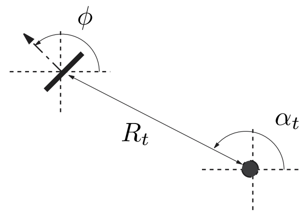

Fig. 3. Linear scatterer with effective reception/transmit area.

matrix H_s between the two sets of local scatterers. That is, we have

$$Y = HX = aH_rH_sH_tX \tag{3}$$

where a is a constant attenuation factor which is independent of N and M.

To determine H_t, we consider the x^{th} transmit element and the y^{th} local scatterer surrounding the transmit array. From 1, we see that the distance between the transmit element and the scatterer is given by

$$R_1(y,x) = \sqrt{R^2t_y^2 - 2Rt_y\delta_{t,x}\sin\alpha_y}$$

Hence, the $(y,x)^{th}$ element of H_t is given by

$$H_t(y,x) \stackrel{\text{def}}{=} \sqrt{f\left(\frac{r_{t,y}}{R_1(x,y)}\right)}\exp\left(i\frac{2\pi}{\lambda}R_1(x,y)\right) \tag{4}$$

where $r_{t,y}$ is the effective radius of the *tx scatterer*.

In a similar fashion, we may compute H_r by considering the x^{th} local scatterer surrounding the receive array and the y^{th} receive element. The $(y,x)^{th}$ element of H_r is given by

$$H_r(y,x) \stackrel{\text{def}}{=} \sqrt{f\left(\frac{r_y}{R_3(y,x)}\right)}\exp\left(i\frac{2\pi}{\lambda}R_3(y,x)\right) \tag{5}$$

where r_y is the effective radius of the receive element r and

$$R_3(y,x) = \sqrt{R^2r_x^2 - 2Rr_x\delta_{r,y}\sin\beta_x}$$

To determine H_s, we consider the x^{th} scatterer at the transmitter side and the y^{th} scatterer at the receiver side. Define $R_2(y,x)$ as the path length between the two scatterers. Then, we have

$$R_2(y,x) \approx \sqrt{\begin{array}{l} D^2 + 2DR(t_x\cos\alpha_x - r_y\cos\beta_y) \\ \quad + R^2\left(t_x^2 + r_y^2 - 2t_xr_y\cos(\alpha_x - \beta_y)\right)\end{array}} \tag{6}$$

Therefore, the $(y,x)^{th}$ element of H_s is given by

$$H_s(y,x) \stackrel{\text{def}}{=} \sqrt{f\left(\frac{r_{s,y}}{R_2(y,x)}\right)}\exp\left(i\frac{2\pi}{\lambda}R_2(y,x)\right) \tag{7}$$

where $r_{s,y}$ is the effective radius of the *rx scatterer* y at the receive end.

IV. CONDITIONING

It is well-known [2] that the capacity of the channel is given by

$$C = \log_2\left(\det(I_M + \frac{\rho}{N}HH^*)\right) \tag{8}$$

assuming the receiver has the full knowledge of the channel, where ρ is the signal to noise ratio at the receiver elements.

Using singular value decomposition, we may rewrite

$$\frac{\rho}{N}HH^* = U\Sigma U^*$$

where U is a unitary matrix and Σ is a diagonal matrix containing all the singular values σ_i of $\frac{\rho}{N}HH^*$. Without loss of generality, we may assume that $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_M$. Therefore,

$$C = \log_2\prod_{i=1}^{M}(1+\sigma_i) = \sum_{i=1}^{M}\log_2(1+\sigma_i) \approx \sum_{i=1}^{\kappa}\log_2(1+\sigma_i)$$

where κ is the number of singular values which are not negligible.

Consequently, if H is poorly conditioned (i.e., κ is small), then the channel will have reduced capacity. We will show that the condition of the channel model (and hence its capacity) is largely determined by the parameter

$$\eta = \frac{2\pi R^2}{D\lambda} \tag{10}$$

We first analyze the conditioning of H_t and H_r. For H_t, we assume that

$$\delta_{t,N-1} \ll Rt_s$$

for every scatterer y near the transmit array. With this assumption, we have

$$H_t(y,x) \approx \sqrt{f\left(\frac{r_{t,y}}{Rt_y}\right)}\exp(i\frac{2\pi}{\lambda}Rt_y)\exp(-i\frac{2\pi}{\lambda}\delta_{t,x}\sin\alpha_y)$$

Therefore,

$$H_t \approx D_tH_{t1} \quad D_r = \text{diag}\left\{\sqrt{f\left(\frac{r_{t,y}}{Rt_y}\right)}\exp(i\frac{2\pi}{\lambda}Rt_y)\right\}$$
$$H_{t1}(y,x) = \exp(-i\frac{2\pi}{\lambda}\delta_{t,x}\sin\alpha_y)$$

It is obvious that D_t is well-conditioned, provided that the ratio of $r_{t,y}$ and Rt_y is similar for different scatterers. The matrix H_{t1} is also well-conditioned provided that the transmit elements are spaced at least $\lambda/2$ apart and that the angles of the scatterers are well-separated.

A similar analysis applies to H_r, which is omitted. We may write $H_r \approx H_{r1}D_r$

The transfer matrix H_s is highly dependent on the separation of the antenna arrays and the radii of the scatterer rings. To analyze the conditioning of H_s we assume

$$\epsilon \stackrel{\text{def}}{=} \frac{R}{D} \ll 1 \quad \text{with} \quad d = i\frac{2\pi D}{\lambda} \tag{11}$$

With the above assumption, we have the following approximation:

$$f\left(\frac{r_{s,y}}{R_2(y,x)}\right) \approx f\left(\frac{r_{s,y}}{D}\right) \quad \text{and}$$
$$R_2(y,x) \approx D\left(1 + \epsilon a(y,x) + \frac{\epsilon^2}{2} b(y,x)\right)$$

$$a(y,x) = t_x \cos\alpha_x - r_y \cos\beta_y \tag{12}$$
$$\begin{aligned} b(y,x) &= t_x^2 + r_y^2 - 2t_x r_y \cos(\alpha_x - \beta_y) - a^2(y,x) \\ &= (t_x \sin\alpha_x - r_y \sin\beta_y)^2 \\ &= t_x^2 \sin^2\alpha_x + r_x^2 \sin^2\beta_y - 2t_x r_y \sin\alpha_x \sin\beta_y \end{aligned} \tag{13}$$

$$H_s \approx D_3 M D_2 D_1 \tag{14}$$

$$D_1 = \text{ diag}\left\{\exp(d\epsilon t_x \cos\alpha_x + d\epsilon^2 t_x^2 \sin^2\alpha_x)\right\} \tag{15}$$
$$D_2 = \text{ diag}\left\{\exp(d\epsilon r_y \cos\beta_y + d\epsilon^2 r_y^2 \sin^2\beta_y)\right\} \tag{16}$$
$$D_3 = \text{ diag}\left\{\sqrt{f\left(\frac{r_{s,y}}{D}\right)} \exp\left(i\frac{2\pi D}{\lambda}\right)\right\} \tag{17}$$

and M is a matrix with

$$\begin{aligned} M(y,x) &= \exp(d\epsilon^2 t_x r_y \sin\alpha_y \sin\beta_y) \\ &= \exp(i\eta t_x r_y \sin\alpha_x \sin\beta_y) \end{aligned} \tag{18}$$

It is clear that D_1, D_2, D_3 are well-conditioned, provided that the rx scatterers are of similar sizes. Hence, we need to analyze M in detail. To this end, we approximate M using κ order Taylor expansion, where κ is to be determined later:

$$M(y,x) \approx 1 + i\eta t_x r_y \sin\alpha_x \sin\beta_y + \cdots + (i\eta t_x r_y \sin\alpha_x \sin\beta_y)^\kappa$$

Defining

$$\begin{aligned} V_1 &\stackrel{\text{def}}{=} V(t_1 \sin\alpha_1, t_2 \sin\alpha_2, \cdots) \\ V_2 &\stackrel{\text{def}}{=} V(r_1 \sin\beta_1, r_2 \sin\beta_2, \cdots) \end{aligned} \tag{20}$$

where $V(x_1, x_2, \cdots x_n)$ denotes the $\kappa \times n$ Vandermonde matrix, i.e., the matrix with the $(i,j)^{th}$ element equal to x_j^i. Then, M can be written as

$$M = V_2^{'} \Gamma V_1 \tag{21}$$
$$\Gamma \stackrel{\text{def}}{=} \text{diag}\{1, i\eta, \cdots, (i\eta)^\kappa\} \tag{22}$$

Giving

$$H = aH_{r1} D_r D_3 V_1^* \Gamma V_2 D_2 D_1 D_t H_{t1} = aH_R \Gamma H_T \tag{23}$$
$$H_R \stackrel{\text{def}}{=} H_{r1} D_r D_3 V_1^* \quad ; \quad H_T \stackrel{\text{def}}{=} V_2 D_2 D_1 D_t H_{t1}$$

It can be seen from (23) that the magnitude of the singular values of H will be exponentially decreasing.

The analysis above shows clearly the role the parameter η plays in the condition of the channel model. Since the scatterers are arranged uniformly at random, we have $E\left\{\|t_x r_y \sin\alpha_x \sin\beta_y\|\right\} = \left(\frac{1}{2}\right)^4 \leq 1$, the contribution of the κ^{th} term diminishes exponentially as a function of κ when η is small. Small η implies the channel model is poorly conditioned. That is, the "effective rank" of the channel model is small. In this case, increasing the number of transmit or receive elements has little effect on the capacity of the channel once the number of elements reaches certain limit. Similarly, increasing the number of local scatterers has little effect on the effective rank either.

On the other hand, the transfer matrix H can be well-conditioned if η is large. Only in this case, is it possible to assume that the transfer from the t^{th} transmit element to the r^{th} receive element to be independent, leading to linear growth in channel capacity as promised by [1] and [2].

V. Simulation

We use Monte-Carlo simulations to analyze the model predictions. A wavelength of $\lambda = 0.15m$, corresponding to approximately 2GHz, is used. We use 15 receive and 15 transmit elements, and positioned the elements in uniform linear arrays, with inter-element spacings of $\frac{\lambda}{2}$. Scatterers are placed uniformly at random, within circular regions surrounding each array. We found that 50 scatterers are sufficient to ensure H_r and H_t were well behaved. The regions have a fixed radius of $10m$. Scatterers are given random gains selected uniformly over an interval $G \in [0,1]$. These gains mimic the varying sizes of the scatterers. Monte-Carlo iterations are used to account for different random placements of scatterers with the rings.

It is found from the simulations that the expected roll-off in capacity is independent of the physical characteristics of the scatterers. Figure 4 shows the condition number of the transfer matrix H as the function of η. Note that the sharp transition from small condition number to large condition number occurs at around $\eta \approx 10$ corresponding to $D \approx 400m = 4R^2$.

Figure 5 plots the magnitude of successive singular values of H and shows that the poor conditioning is not confined to a few extreme singular values. Rather, the singular values of H decrease geometrically. Plotted is the singular value decomposition at different values of η from Monte-Carlo simulation.

In both figure 4 and figure 5 the smallest singular values are limited by the machine precision ($\approx 10^{-15}$).

Figure 6 shows the channel capacity as a function of η, using (8). It is assumed in the simulation that $\rho = 10dB$. It can be seen that for sufficiently large η the capacity of the channel approaches the expected capacity. However, for small η the capacity is dominated by η and not $\min(N, M)$.

VI. Conclusion

In this paper, we have developed an NLOS channel model for multi-antenna wireless systems which involve local scatter-

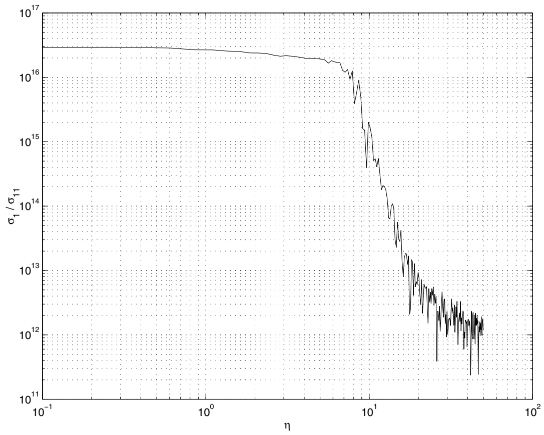

Fig. 4. Condition number $\frac{\sigma_1}{\sigma_{11}}$ of MIMO channel H with increasing η. Roll-off at small η due to machine precision

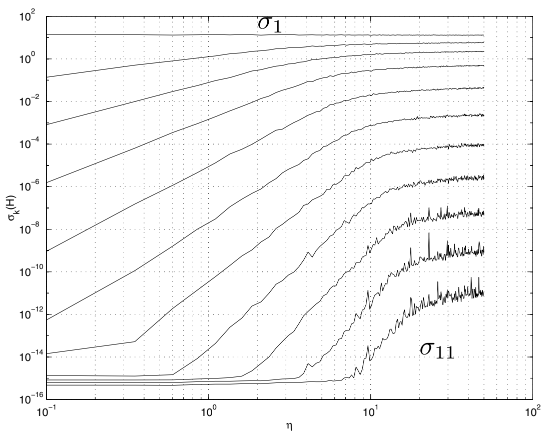

Fig. 5. Singular values of MIMO channel H with increasing η

ers at both the transmit and receive ends. We have identified a key parameter, η, which plays a crucial role in determining the condition of the channel and thus the channel capacity of the system.

This analysis gives a quantitative measure of the system configurations which can possibly achieve high channel capacity. In particular, large separation between the transmit array and receive array leads to poor condition for the channel transfer matrix and thus the poor channel capacity. The simulation results clearly indicate the significance of the parameter η.

It has been shown that η may be used as a measure, of the gain in capacity achieved through the use of additional receive and/or transmit elements. For large η we can expect the channel capacity to grow linearly, proportional to the minimum number of transmit and receive elements as expected. How-

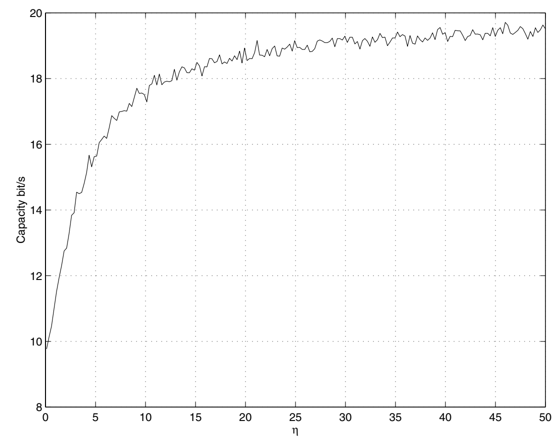

Fig. 6. Capacity of MIMO channel with increasing η

ever below a given threshold $\eta < 10$ the benefit provided by additional elements is significantly reduced. It has also been shown that arrangement of the transmit and receive elements do not play a significant role in ensuring that the linear increase in capacity is possible.

This suggests that for certain arrangements of scatterers, particularly for transmission over long distances in outdoor environments, the predicted linear increases in capacity of spatially diverse systems, for increasing numbers of antennas, may not be realistic.

REFERENCES

[1] G. J. Foschini and M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Personal Communications*, vol. 6, pp. 311–335, 1998.

[2] I. E. Telatar, "Capacity of multi-antenna gaussian channels," *European Transactions on Telecommunications*, vol. 10, no. 6, pp. 585–595, Nov. 1999.

[3] P. W. Wolniansky, G. D. Golden, G. J. Foschini, and R. A. Valenzuela, "V-BLAST: An architecture for realizing very high data rates over the rich-scattering wireless channel," in *Proc. ISSSE-98*, 1998.

[4] A. F. Naguib, N. Seshadri, and A. R. Calderbank, "Increasing data rate over wireless channels," *IEEE Signal Processing Magazine*, vol. 17, no. 3, pp. 76–92, May 2000.

[5] I. E. Telatar and D. N. C. Tse, "Capacity and mutual information of wideband multipath fading channels," *IEEE Transactions on Information Theory*, vol. 46, no. 4, pp. 1384–1400, July 2000.

[6] C-N. Chuah, G. J. Foschini, R. A. Valenzuela, D. Chizhik, J. Ling, and J. M. Kahn, "Capacity growth of multi-element arrays in indoor and outdoor wireless channels," in *IEEE Proc. Globecom00*, 2000.

[7] D. Tse, C. Chauh, and J. Kahn, "Capacity scaling in dual-antenna-array wireless systems," in *2000 IEEE Wireless Communications and Networking Conference*, sep 2000.

[8] S. Sandhu and A. Paulraj, "Unified design of linear space-time block codes," in *IEEE Proc. Globecom01*, 2001.

[9] D. Gesbert, H. Bölcskei, D. Gore, and A. Paulraj, "MIMO Wireless channels: Capacity and performance prediction," in *IEEE Proc. Globecom00*, 2000.