Proceedings of the 2002 International Conference
on Control and Automation
Xiamen, China, Jun. 2002

/SPL04-1.

Regional Stability and Performance Analysis for a class of Nonlinear Systems

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Abstract. This paper studies the problem of regional stability and performance analysis for a class of nonlinear uncertain systems. Both continuous-time and discrete-time systems are considered. Our approach is based on the use of polynomial Lyapunov functions. We show how these functions can be used to reduce the conservatism in analysis. The conditions for analysis are given in terms of inear matrix inequalities to allow feasible computational implementation. These inequalities are derived by introducing auxilary nonlinear algebraic equations and applying Finsler's lemma. It turns out that continuous-time systems and discrete-time systems require sharply different techniques in order to achieve simple linear matrix inequalities. A numerical example is used to illustrate the approach and show that the proposed method can lead to less conservative results when compared with results using quadratic Lyapunov functions.

1 Introduction

Robust control of nonlinear uncertain systems has been a very active research area over the last decade or so. A lot of recent research focuses on analysis and synthesis approaches in the framework of linear matrix inequalities (LMIs). Design approaches range from using quadratic Lyapunov functions ([1, 2, 3]) to those based on polynomial Lyapunov functions ([4, 5, 6, 7]). In general, non-quadratic Lyapunov functions are less conservative for dealing with uncertain and nonlinear systems than quadratic Lyapunov functions at the expense of extra computation.

In this paper, we study the problem of regional stability and performance analysis for a class of nonlinear uncertain systems. Both continuous-time and discrete-time systems are considered. We will use polynomial Lyapunov functions in conjunction with linear matrix inequalities. For continuous-time systems, two problems are studied: 1) Guaranteed L_2 performance and 2) Regional stability with bounded input disturbance. There is a common underlying idea in dealing with these problems, that is to decompose a given nonlinear system into a simpler form using auxiliary state variables. The decomposed system has system matrices dependent affinely in the state and uncertain parameters rather than a general nonlinear form. Such a decomposition allows us to convert various analysis problems into a set of LMIs, which can be tested numerically. For discrete-time systems, we study the problem of regional stability. We also use the decomposition idea mentioned above. However, it turns out that the detailed approaches for continuous-time and discrete-time systems are quite different due to the fact that the derivative of a Lyapunov function (for the continuous time) is linear in the system matrix whereas the difference of a Lyapunov function (for the discrete time) is quadratic in the system matrix. This distinction makes it much more difficult to derive LMIs conditions for discrete time systems when the system matrix involves nonlinearities and/or uncertain parameters. To get around this difficulty, we have generalised a recent robust stability result for discrete-time systems in [8] which uses a LMI not involving a quadratic term of the system matrix. An numerical example is used to illustrate the benefit of the use of polynomial Lyapunov functions.

2 Continuous-time Systems: Guaranteed L_2 Performance

Consider the uncertain nonlinear system

$$\begin{array}{rcl} \dot{x} & = & A(x,\delta)x, \quad x(0)=x_0 \\ z & = & C(x,\delta)x \end{array} \tag{1}$$

where $x \in \mathbf{R}^n$ is the state vector, $\delta \in \mathbf{R}^l$ is the uncertain paramater vector and $z \in \mathbf{R}^r$ is the output performance vector which belongs to a polytope $\mathcal{B}_\delta$.

Given a polytope $\mathcal{B}_x \subset \mathbf{R}^n$, which contains the origin, and a performance bound c, the problem of concern is to find the maximum invariant set $\mathcal{R}_c \subset \mathcal{B}_x$, such that the L_2-norm of the performance output signal is

$x(t) \in \mathcal{R}_c$ for all $t > 0$ and $\|z\|_2^2 < c$.

We suppose that system (1) can be decomposed as:

$$\begin{array}{rcl} \dot{x} & = & A_1(x,\delta)x + A_2(x,\delta)\pi \\ z & = & C_1(x,\delta)x + C_2(x,\delta)\pi \\ 0 & = & \Omega_1(x,\delta)x + \Omega_2(x,\delta)\pi \end{array} \qquad (2)$$

where $\pi \in \mathbf{R}^m$ is an auxiliary state vector which is a nonlinear function of (x, δ) and the matrices $A_i(x,\delta), C_i(x,\delta)$ and $\Omega_i(x,\delta)$, $i = 1, 2$, are affine functions of (x, δ). It turns out that a large class of nonlinear systems can be decomposed into the form (2). For more details, see [5].

We choose the Lyapunov function to be of the form:

$$\begin{array}{rcl} v(x,\delta) & = & x'\mathcal{P}(x,\delta)x \\ \mathcal{P}(x,\delta) & = & \begin{bmatrix} \Theta(x,\delta) \\ I_n \end{bmatrix}' P \begin{bmatrix} \Theta(x,\delta) \\ I_n \end{bmatrix} \end{array} \qquad (3)$$

where $\Theta(x,\delta)$ is an affine function of (x,δ) and P is a symmetric matrix to be determined. Defining

$$\xi = \begin{bmatrix} \Theta(x,\delta) \\ I_n \end{bmatrix} x \qquad (4)$$

We can rewrite $v(x) = \xi' P \xi$. Then,

$$\dot{v} = \dot{\xi}' P \xi + \xi' P \dot{\xi}$$

Observe that

$$\frac{d}{dt}(\Theta(x,\delta)x) = \dot{\Theta}(x,\delta)x + \Theta(x,\delta)\dot{x}$$

Since $\Theta(x,\delta)$ is an affine matrix of (x,δ), we may write

$$\dot{\Theta}(x,\delta)x = \tilde{\Theta}(x,\delta)\dot{x} \qquad (5)$$

for some $\tilde{\Theta}(x,\delta)$ which is affine in (x,δ).

To upper-bound the L_2-norm of z, we require

$$\begin{array}{rcl} \epsilon_1 x'x & \leq & v(x,\delta) \leq \epsilon_2 x'x \\ \dot{v} & < & -z'z \end{array}, \ \forall x \in \mathcal{B}_x, \delta \in \mathcal{B}_\delta \qquad (6)$$

where $\epsilon_i > 0$ are constants. Also define

$$\mathcal{R}_c = \{x : v(x,\delta) \leq c, \ \forall \delta \in \mathcal{B}_\delta\} \qquad (7)$$

It follows that $\mathcal{R}_c$ is an invariant set and

$$\|z\|_2^2 \leq v(x_0,\delta) < c \ \ \forall x_0 \in \mathcal{R}_c \qquad (8)$$

provided c is such that

$$\mathcal{R}_c \subset \mathcal{B}_x \qquad (9)$$

To deal with (9), we represent the polytope $\mathcal{B}_x$ by its vertices or by using a set of inequalities, i.e.,

$$\mathcal{B}_x = \{x : a_k' x \leq 1, \ k = 1, 2, \cdots, n_e\} \qquad (10)$$

of $\mathcal{B}_x$. Then, the condition (9) can be rewritten as

$$1 - a_k' x \geq 0, \ \forall x : v(x,\delta) < c, k = 1, 2, \cdots, n_e \qquad (11)$$

Applying the well-known S-procedure, we get

$$\begin{bmatrix} 1 \\ \xi \end{bmatrix}' \begin{bmatrix} 2-c & [0 \ \ -a_k'] \\ \begin{bmatrix} 0 \\ -a_k \end{bmatrix} & P \end{bmatrix} \begin{bmatrix} 1 \\ \xi \end{bmatrix} \geq 0, \ \forall k \qquad (12)$$

Applying the Finsler's lemma, it turns out that the testing of (6) and (12) can be done using LMIs, as given in the result below.

Theorem 1 *Consider the system (2). Let $\Theta(x,\delta)$ be a given affine matrix of (x,δ). Define*

$$\begin{array}{rcl} G & = & [0 \ \ \Omega_1(x,\delta)] \\ E & = & \begin{bmatrix} I & -\tilde{\Theta}(x,\delta) + \Theta(x,\delta) \\ 0 & I_n \end{bmatrix} \\ H & = & \begin{bmatrix} 0 \\ A_2(x,\delta) \end{bmatrix} \\ M & = & \begin{bmatrix} x_2 & -x_1 & 0 & \cdots & 0 \\ 0 & x_3 & -x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_n & -x_{n-1} \end{bmatrix} \\ N & = & \begin{bmatrix} 0 & M \\ I & -\Theta \end{bmatrix}, \ F = \begin{bmatrix} 0 & 0 \\ 0 & A_1 \end{bmatrix} \end{array} \qquad (13)$$

Suppose there exist matrices $P = P'$, R and L_{ij}, $i, j = 1, 2, 3$ that solve the following optimization problem, where the LMIs are satisfied at all vertices of $\mathcal{B}_x \times \mathcal{B}_\delta$:

$$\begin{array}{l} \max c \ \textit{subject to:} \\ \begin{bmatrix} 2-c & [0 \ \ -a_k'] \\ \begin{bmatrix} 0 \\ -a_k \end{bmatrix} & P + RN + N'R' \end{bmatrix} \geq 0, \ \forall k \\ He \begin{bmatrix} -L_{12}E & \Delta_{12} & \Delta_{13} & 0 \\ P - L_{22}E & \Delta_{22} & \Delta_{23} & 0 \\ -L_{32}E & \Delta_{32} & \Delta_{33} & 0 \\ 0 & [0 \ C_1] & C_2 & -I_r/2 \end{bmatrix} < 0 \end{array} \qquad (14)$$

where

$$\Delta_{i2} = L_{i1}G + L_{i2}F + L_{i3}N; \ \Delta_{i3} = L_{i1}\Omega_2(x,\delta) + L_{i2}H$$

Then, $v(x,\delta)$ is a Lyapunov function for the system (2) and (6) and (9) are satified.

See [5] for proof.

3 Continuous-time Systems: Robust Stability with Bounded Input Disturbance

Consider the following uncertain nonlinear system:

$$\dot{x} = A(x,\delta)x + B_w(x,\delta)w, \ x(0) = 0 \qquad (15)$$

where $x, \delta, A(x,\delta)$ are the same as in (1), and w is an input disturbance. We assume that $w \in \mathcal{W}$ with

$$\mathcal{W} = \left\{ w(t) : \mu^{-1} \int_0^\infty w(t)'w(t)dt \leq 1 \right\} \qquad (16)$$

where $\mu > 0$ controls the "size" of $\mathcal{W}$.

Given a polytope $\mathcal{B}_x \subset \mathbf{R}^n$ as in (10), which contains the origin, and $\mathcal{W}$, we want to determine the regional stability of the system (15) which is defined below.

Definition 1 *The system (15) is called regionally stable (with respect to $\mathcal{W}$ and $\mathcal{B}_x$) if $x(t) \in \mathcal{B}_x$ for all $t \geq 0$ and all $w \in \mathcal{W}$. The corresponding set $\mathcal{W}$ is called a set of admissible input disturbances (with respect to $\mathcal{B}_x$).*

The key idea involved in the study of admissible input disturbances is to overbound the state trajectory generated by a disturbance input using a level set of a Lyapunov function, which in turn is overbounded by $\mathcal{B}_x$. More precisely, we consider a Lyapunov function of the type $v(x,\delta) = x'\mathcal{P}(x,\delta)x$ as in (3) and the overbounding level set is given by

$$\mathcal{R} = \{\ x : v(x,\delta) \leq 1\} \qquad (17)$$

In addition, suppose $v(x,\delta)$ satisfies the following conditions for ϵ_1 and ϵ_2:

$$\epsilon_1 x'x \leq v(x,\delta) \leq \epsilon_2 x'x \qquad (18)$$

$$\dot{v}(x,\delta) \leq \mu^{-1} w'(t)w(t), \ \forall t \geq 0 \qquad (19)$$

$$\mathcal{R} \subset \mathcal{B}_x \qquad (20)$$

Integrating both sides of (19) from 0 to T yields

$$v(x(T),\delta) \leq \mu^{-1} \int_0^T w'(t)w(t)dt \leq 1, \ \forall \delta \in \mathcal{B}_\delta, w \in \mathcal{W}$$

If the condition (20) is satisfied, then $x(t) \in \mathcal{R} \subset \mathcal{B}_x$ for all $t \geq 0$ and $w \in \mathcal{W}$.

Using (3), it can be shown that

$$\dot{v}(x,\delta) = 2 \begin{bmatrix} (\Theta + \tilde{\Theta})Ax + (\Theta + \tilde{\Theta})B_w w \\ Ax + B_w w \end{bmatrix}' P \begin{bmatrix} \Theta x \\ x \end{bmatrix}$$

Similar to (2), we also suppose that (15) can be decomposed into the following form:

$$\begin{aligned} \dot{x} &= A_1(x,\delta)x + A_2(x,\delta)\pi + B_1(x,\delta)w + B_2(x,\delta)\zeta \\ 0 &= \Omega_1(x,\delta)x + \Omega_2(x,\delta)\pi \\ 0 &= \Xi_1(x,\delta)w + \Xi_2(x,\delta)\zeta \end{aligned} \qquad (21)$$

where π and ζ are nonlinear vector functions of (x,δ) and $A_i(x,\delta), B_i(x,\delta), \Omega_i(x,\delta)$ and $\Xi_i(x,\delta)$ are affine matrix functions of (x,δ).

Again, by applying Finsler's lemma, we can convert (18), (19) and (12) into a set of LMIs. To express our result, we define M and N as in (13) and

$$A_{a1} = \begin{bmatrix} 0_{n_\theta} & (\Theta + \tilde{\Theta})A_1 \\ 0 & A_1 \end{bmatrix}, \ A_{a2} = \begin{bmatrix} (\Theta + \tilde{\Theta})A_2 \\ A_2 \end{bmatrix},$$
$$B_{a1} = \begin{bmatrix} (\Theta + \tilde{\Theta})B_1 \\ B_1 \end{bmatrix}, B_{a2} = \begin{bmatrix} (\Theta + \tilde{\Theta})B_2 \\ B_2 \end{bmatrix},$$
$$\Psi = \begin{bmatrix} N & 0 & 0 & 0 \\ [0 \ \ \Omega_1] & \Omega_2 & 0 & 0 \\ 0 & 0 & \Xi_1 & \Xi_2 \end{bmatrix} \qquad (22)$$

Theorem 2 *Consider the system (15) and its nonlinear decomposition (21). Let $\mathcal{B}_x, \mathcal{B}_\delta$ and $\mathcal{W}$ be given. Suppose there exist $P = P'$, R and $L_k, k = 1, \cdots, n_e$ solving the following LMI problem, where the LMIs are constructed at all vertices of $\mathcal{B}_x \times \mathcal{B}_\delta$:*

$$\begin{bmatrix} 1 & [\ 0 \ \ a_k'\ ] \\ \begin{bmatrix} 0 \\ a_k \end{bmatrix} & (P + L_k N + N' L_k') \end{bmatrix} > 0,$$
$$k = 1, \ldots, n_e \qquad (23)$$

$$\begin{bmatrix} (A_{a1}'P + PA_{a1}) & PA_{a2} & PB_{a1} & PB_{a2} \\ A_{a2}'P & 0 & 0 & 0 \\ B_{a1}'P & 0 & -\mu^{-1}I & 0 \\ B_{a2}'P & 0 & 0 & 0 \end{bmatrix}$$
$$+ R\Psi + \Psi' R' < 0 \qquad (24)$$

Then, the system (15) is regionally stable with respect to $\mathcal{B}_x$ and $\mathcal{W}$.

See [6] for proof.

4 Discrete-time Systems

Consider the following discrete-time nonlinear system:

$$x_+ = A(x(k),\delta)x(k), \ x_+ = x(k+1) \qquad (25)$$

where $x(k)$, δ and $A(x,\delta)$ are similar to the continuous-time case.

The problem of concern in this paper is to determine a region in the state space in which robust stability and performance of system (25) is guaranteed.

We have the following basic result:

Lemma 1 *Consider system (25). Let $v(x,\delta) = x'\mathcal{P}(x,\delta)x$ be a given Lyapunov function candidate, where $\mathcal{P}(x,\delta)$ is a matrix function of (x,δ). Define a region in the state space as follows:*

$$\mathcal{R} = \{x : x \in \mathbf{R}^n, x'\mathcal{P}(x,\delta)x \leq 1, \ \forall \ \delta \in \mathcal{B}_\delta\} \qquad (26)$$

Suppose there exist $\epsilon_i > 0$, $i = 1, 2, 3$, such that

$$\epsilon_1 x' x \le x' \mathcal{P}(x,\delta) x \le \epsilon_2 x' x \quad (27)$$

$$x' \left(A(x,\delta)' \mathcal{P}(x_+,\delta) A(x,\delta) - \mathcal{P}(x,\delta) \right) x \le -\epsilon_3 x' x \quad (28)$$
$$\forall\, x \in \mathcal{R},\ \delta \in \mathcal{B}_\delta$$

where x_+ is as defined in (25). Then, $v(x,\delta)$ is a Lyapunov function in $\mathcal{R}$ and $\mathcal{R}$ is a domain of attraction for system (25).

The conditions (27) and (28) are difficult to check because the coupling of $A(x,\delta)$ and $P(x_+,\delta)$ gives high nonlinearity. To get around this problem, we introduce the following technical result:

Lemma 2 *Consider system (25) and $v(x,\delta)$ and $\mathcal{R}$ as defined in Lemma 1. Suppose (27) and the following inequality holds for some auxilary matrix function $\mathcal{G}(x,\delta)$:*

$$\begin{bmatrix} x \\ y \end{bmatrix}' \begin{bmatrix} -\mathcal{P}(x,\delta) & A'(x,\delta)\mathcal{G}'(x,\delta) \\ \mathcal{G}(x,\delta)A(x,\delta) & \mathcal{P}^+ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\le -\epsilon_3 x' x, \quad \forall x \in \mathcal{R}, y \in \mathbf{R}^n, \delta \in \mathcal{B}_\delta \quad (29)$$

where

$$\mathcal{P}^+ = \mathcal{P}(x_+,\delta) - \mathcal{G}(x,\delta) - \mathcal{G}'(x,\delta)$$

Then, $v(x,\delta)$ is a Lyapunov function in $\mathcal{R}$ and $\mathcal{R}$ is a domain of attraction for system (25).

See [7] for proof.

We note that when $A(x,\delta), \mathcal{P}(x,\delta)$ and $\mathcal{X}(x,\delta)$ do not depend on x, the result above reduces to a result in [8].

In order to proceed further, we also need to decompose the system (25) using auxiliary state variables. But differently from the continuous-time case, we suppose the following decomposition:

$$\begin{array}{l} x_+ = A(x,\delta)x = \mathcal{A}\Pi(x,\delta)x \\ \Omega(x,\delta)\Pi(x,\delta) = 0 \\ Q\Pi(x,\delta) = I_n \end{array} \quad (30)$$

where $\mathcal{A}$ and Q are constant matrices and $\Omega(x,\delta)$ and $\Pi(x,\delta)$ are matrix functions which are affine in (x,δ).

The Lyapunov function will be the same as in (2). Observe from lemma 2 that we need to compute

$$\mathcal{P}(x_+,\delta) = \begin{bmatrix} \Theta(x_+) \\ I_{n_x} \end{bmatrix}' P(\delta) \begin{bmatrix} \Theta(x_+) \\ I_{n_x} \end{bmatrix}.$$

To this end, we require the following constraint:

$$\begin{array}{l} \begin{bmatrix} \Theta(x) \\ I_{n_x} \end{bmatrix} = F\Pi(x,\delta) = \begin{bmatrix} F_1 \\ Q \end{bmatrix} \Pi(x,\delta) \\ \begin{bmatrix} \Theta(x_+) \\ I_{n_x} \end{bmatrix} = H\Pi(x,\delta) = \begin{bmatrix} H_1 \\ Q \end{bmatrix} \Pi(x,\delta) \end{array} \quad (31)$$

where F_1, H_1 are constant matrices.

We choose the auxilary matrix function $\mathcal{G}(x,\delta)$ to be of the following form:

$$\mathcal{G}(x,\delta) = \Pi'(x,\delta)G(\delta) \quad (32)$$

where $G(\delta)$ is affine in δ but to be determined.

We also need a polytopic bounding set $\mathcal{B}_x$ for $\mathcal{R}$. We will require (29) to hold for all $x \in \mathcal{B}_x$ instead of $\mathcal{R}$. This bounding set is also described by (10).

In order to ensure that the Lyapunov matrix function $\mathcal{P}(x,\delta)$ is positive definite for all $x \in \mathcal{B}_x$, we can apply the Finsler's lemma and obtain the following condition:

$$\begin{array}{c} P(\delta) + L\Psi_1(x) + L'\Psi_1'(x) > 0 \\ \forall x \in \mathcal{B}_x, \delta \in \mathcal{B}_\delta \end{array} \quad (33)$$

where L is a free matrix to be determined and

$$\Psi_1(x) = \begin{bmatrix} I & -\Theta(x) \end{bmatrix} \quad (34)$$

In order to maximize the volume of $\mathcal{R}$, we normally approximate it by minimising the trace of the Lyapunov matrix. However, $\mathcal{P}(x,\delta)$ is a nonlinear function of (x,δ) that leads to a nonconvex condition. To overcome this problem, we will approximate the volume maximisation by

$$\min \max_{x \in \mathcal{B}_x, \delta \in \mathcal{B}_\delta} \mathsf{trace}\, (P(\delta) + L\Psi_1(x) + L'\Psi_1'(x)) \quad (35)$$

Here is the main result for discrete-time systems:

Theorem 3 *Consider the system (25) as decomposed in (30). Let $\Theta(x)$ be a given affine matrix function of x satisfying (31) and the Lyapunov matrix function $\mathcal{P}(x,\delta)$ be in the form of (2). Let $\mathcal{B}_x$ be a given bounding set as in (10). Define $\Psi_1(x)$ as in (34) and*

$$\Psi_2(x,\delta) = \begin{bmatrix} \Omega(x,\delta) & 0 \\ 0 & \Omega(x,\delta) \end{bmatrix}. \quad (36)$$

Suppose there exist affine matrices $G(\delta)$ and $P(\delta)$ and constant matrices L, R and $M_j, j = 1, \cdots, n_e$ solving the following linear matrix inequalities at all vertices of $\hat{\mathcal{X}} \times \Delta$:

$\min \eta$ *subject to:*

$$\eta - \mathsf{trace}\left(P(\delta) + L\Psi_1(x) + \Psi_1'(x)L'\right) \ge 0 \quad (37)$$

$$P(\delta) + L\Psi_1(x) + \Psi_1'(x)L' > 0 \quad (38)$$

$$\begin{bmatrix} 1 & \begin{bmatrix} 0 & a_j' \end{bmatrix} \\ \begin{bmatrix} 0 \\ a_j \end{bmatrix} & \left(P(\delta) + M_j\Psi_1(x) + \Psi_1'(x)M_j'\right) \end{bmatrix} \ge 0 \quad (39)$$
$$\forall\, j = 1, \cdots, n_e \quad (40)$$

$$\begin{bmatrix} -F'P(\delta)F & \mathcal{A}'G(\delta)' \\ G(\delta)\mathcal{A} & H'P(\delta)H - G(\delta)Q - Q'G(\delta)' \end{bmatrix} \quad (41)$$
$$+R\Psi_2(x,\delta) + \Psi_2'(x,\delta)R' < 0$$

Then, $v(x,\delta) = x'\mathcal{P}(x,\delta)x$ is a Lyapunov function in $\mathcal{R}$ and $\mathcal{R}$ is a domain of attraction for system (25).

See [7] for proof.

5 Illustrative Example

To illustrate the benifit of using polynomial Lyapunov functions, we analyze a continuous-time example with input disturbance and we compute an upper bound on the L_2-gain of the system.

Consider the following time-invariant and uncertain system which is based on the Van der Pol equation:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & -1 \\ 1 & (0.8+0.2\delta)(x_1^2-1) \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \\ z &= x_1 + w,\ x(0)=0,\ \delta \in [-1,1] \end{aligned} \tag{42}$$

where $w \in \mathcal{W}$ for some $\mu > 0$. We want to find the maximum μ for regional stability. To this end, we carry out the decomposition of (42) as in (21) with

$$A_1 = \begin{bmatrix} 0 & -1 \\ 1 & -0.8 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -0.2 & 0.8 & 0.2 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, D_1 = 1$$

$$\Omega_1 = \begin{bmatrix} x_2 & 0 \\ 0 & \delta \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ \Omega_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ x_1 & 0 & -1 & 0 \\ 0 & 0 & \delta & -1 \end{bmatrix}$$

$$\pi = [x_1x_2\ \delta x_2\ x_1^2x_2\ \delta x_1^2 x_2]'$$

Let $\mathcal{B}_x$ be the polytope defined by the vertices

$$\left\{ \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}, \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}, \begin{bmatrix} -\alpha \\ \alpha \end{bmatrix}, \begin{bmatrix} -\alpha \\ -\alpha \end{bmatrix} \right\} \tag{43}$$

where α is a given scalar.

Define the Lyapunov function by choosing

$$\Theta(x,\delta) = \begin{bmatrix} x_1 I_2 \\ [0\ \ x_2\ ] \\ \delta I_2 \end{bmatrix} \Rightarrow \tilde{\Theta} = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \\ 0_{2\times1} & 0_{2\times1} \end{bmatrix}$$

Also, consider that the matrix partition for P below:

$$P = \begin{bmatrix} P_2 & P_1' \\ P_1 & P_0 \end{bmatrix}$$

With the above partition, we can obtain the following types of Lyapunov matrices:

1. $\mathcal{P}(x,\delta)$ is quadratic in (x,δ);
2. $P_2 = 0$: $\mathcal{P}(x,\delta)$ is affine in (x,δ);
3. $P_1 = 0$, $P_2 = 0$: $\mathcal{P}(x,\delta) = P_0$ is constant characterizing a quadratic Lyapunov function.

Table 1 shows the estimated admissible sets $\mathcal{W}$ (defined in size by μ) using theorem 2 with the above Lyapunov matrices. For all solutions $\alpha = 0.7$ is used. As expected, the polynomial Lyapunov function (quadratic Lyapunov matrix) achieved the best estimates, thus justifying the required extra computation.

Upper-bounds	Lyapunov Matrix		
	Constant	Affine	Quadratic
μ	0.298	0.300	0.486

Table 1: Estimated sizes of input disturbance.

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