Structure of the Hessian

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1. Introduction

We saw in earlier lectures that a core ingredient in quadratic constrained optimisation problems is the Hessian matrix $H$. So far we have simply given an “in principle” approach to the evaluation of this matrix. That is, for a system with state equations $x_{k+1} = Ax_k + Bu_k$, we can compute the Hessian as $H = \Gamma^T Q \Gamma + R$, where $\Gamma, Q, R$. 
This will be satisfactory for simple problems. However, for more complex problems (for example, high order systems or problems having mixed stable and unstable modes) this “brute force” approach may fail. A hint as to the source of the difficulties is that the direct way of computing the Hessian depends on powers of the system matrix $A$. Clearly, if the system has unstable modes, then some entries of $\Gamma$ will diverge as $N$ increases.
We will show in this lecture that this problem can be resolved by focusing attention on the stable and unstable parts of the system separately.
Our eventual goal in this lecture is to gain a better understanding of the structure of the Hessian matrix particularly for large optimisation horizons. However, as mentioned above, the straightforward approach to evaluating the Hessian will often meet difficulties for open loop unstable plants due to exponential divergence of the system impulse response.
One way of addressing this problem is to recognise that there is an intimate connection between “stability” and “causality.” In particular, a system having all modes unstable becomes stable if viewed in reverse time, that is, as an anti-causal system. This line of reasoning leads to an alternative viewpoint in which unstable modes are treated differently. We show below that this leads to an equivalent problem formulation with a different Hessian having different properties.
Consider a discrete time linear system and suppose that it has no eigenvalues on the unit circle. We can then partition the state vector \( x_k \in \mathbb{R}^n \) as

\[
  x_k = \begin{bmatrix}
    x_k^s \\
    x_k^u
  \end{bmatrix},
\]

where the states \( x_k^s \) and \( x_k^u \) are associated with the stable and unstable modes, respectively.
We can then factor the state equations into stable and unstable parts as follows:

\[
\begin{bmatrix}
x_{s,k+1}^s \\
x_{u,k+1}^u
\end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_s^s \\
x_u^u
\end{bmatrix} + \begin{bmatrix} B_s \\ B_u \end{bmatrix} u_k,
\]

(1)

\[
y_k = C x_k = \begin{bmatrix} C_s & C_u \end{bmatrix} \begin{bmatrix} x_s^s \\
x_u^u
\end{bmatrix},
\]

where \( u_k \in \mathbb{R}^m \) and \( y_k \in \mathbb{R}^p \) (\( p \geq m \)). The eigenvalues of \( A_s \) have moduli less than one, and the eigenvalues of \( A_u \) have moduli greater than one.
We can then express the solution of the system equations as

\[ x^s_k = A^s_k x^s_0 + \sum_{j=0}^{k-1} A^{s-1-j}_s B_s u_j \quad \text{for} \quad k = 1, \ldots, N, \]  
(2)

\[ x^u_k = A^{-(N-k)} u \mu - \sum_{j=k}^{N-1} A^{k-1-j}_u B_u u_j \quad \text{for} \quad k = 0, \ldots, N - 1. \]  
(3)

\[ x^u_N \triangleq \mu, \]
We then need an equality constraint that both $\mu$ and the sequence of control signals $\{u_0, \ldots, u_{N-1}\}$ need to satisfy in order to bring the unstable states back to their correct initial values, that is,

$$A_u^{-N}\mu - \sum_{j=0}^{N-1} A_u^{-j-1} B_u u_j = x_0^u.$$  \hspace{1cm} (4)
We are thus led to the following equivalent statement of the optimisation problem.

\[ \mathcal{P}_N(x) : \quad V_N^{\text{opt}}(x) \triangleq \min V_N(\{x_k\}, \{u_k\}, \mu), \]  

subject to:

\[ x_k = \begin{bmatrix} x_k^s \\ x_k^u \\ x_k \end{bmatrix} \quad \text{for} \quad k = 0, \ldots, N, \]

\[ x_k^s = A_s^k x_0^s + \sum_{j=0}^{k-1} A_s^{k-1-j} B_s u_j \quad \text{for} \quad k = 1, \ldots, N, \]

\[ x_k^u = A_u^{-(N-k)} \mu - \sum_{j=k}^{N-1} A_u^{k-1-j} B_u u_j \quad \text{for} \quad k = 0, \ldots, N - 1, \]  

(5)
\[ x_0 = \begin{bmatrix} x_0^s \\ x_0^u \\ x_0 \end{bmatrix} = x, \]
\[ u_k \in \mathbb{U} \quad \text{for } k = 0, \ldots, N - 1, \]
\[ x_k \in \mathbb{X} \quad \text{for } k = 0, \ldots, N - 1, \]
\[ x_N = \begin{bmatrix} x_N^s \\ \mu \end{bmatrix} \in \mathbb{X}_f \subset \mathbb{X}, \]

where

\[ V_N(\{x_k\}, \{u_k\}, \mu) \overset{\Delta}{=} \frac{1}{2} \begin{bmatrix} x_N^s \\ \mu \end{bmatrix}^T P \begin{bmatrix} x_N^s \\ \mu \end{bmatrix} + \frac{1}{2} \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k). \quad (7) \]
The above formulation of the problem, at the expense of the introduction of the additional optimisation variable $\mu$, avoids exponentially diverging terms in the computation of the Hessian matrix. Thus, at least intuitively, it would seem to be more apposite for studying the structure of the problem for large horizons.
We represent the time evolution of the system output using the usual vector notation. Thus, let

\[
y = \begin{bmatrix} y_1^T & y_2^T & \cdots & y_N^T \end{bmatrix}^T,
\]

\[
u = \begin{bmatrix} u_0^T & u_1^T & \cdots & u_{N-1}^T \end{bmatrix}^T.
\]
\[ y = (\Gamma_{S} + \Gamma_{U}) u + \Omega_{S} x_{0}^{S} + \Omega_{U} \mu, \quad (9) \]

where

\[
\Omega_{S} = \begin{bmatrix}
C_{S}A_{S} \\
C_{S}A_{S}^{2} \\
\vdots \\
C_{S}A_{S}^{N}
\end{bmatrix}, \quad \Gamma_{S} = \begin{bmatrix}
C_{S}B_{S} & 0 & \cdots & 0 \\
C_{S}A_{S}B_{S} & C_{S}B_{S} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
C_{S}A_{S}^{N-1}B_{S} & C_{S}A_{S}^{N-2}B_{S} & \cdots & C_{S}B_{S}
\end{bmatrix},
\]
\[
\Omega_u = \begin{bmatrix}
C_u A_u^{-(N-1)} \\
C_u A_u^{-(N-2)} \\
\vdots \\
C_u A_u^{-1} \\
C_u
\end{bmatrix}, \quad \Gamma_u = -\begin{bmatrix}
0 & C_u A_u^{-1} B_u & C_u A_u^{-2} B_u & \ldots & C_u A_u^{-(N-1)} B_u \\
0 & 0 & C_u A_u^{-1} B_u & \ldots & C_u A_u^{-(N-2)} B_u \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & C_u A_u^{-1} B_u \\
0 & 0 & \ldots & 0 & 0
\end{bmatrix}
\]
The matrix $\bar{\Gamma} \triangleq \Gamma_s + \Gamma_u$ has the form

$$\bar{\Gamma} \triangleq \Gamma_s + \Gamma_u = \begin{bmatrix}
h_0 & h_{-1} & h_{-2} & \ldots & \ldots & h_{-(N-1)} \\
h_{-1} & h_0 & h_{-1} & \ldots & \ldots & h_{-(N-2)} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
h_{N-1} & h_{N-2} & \ldots & \ldots & h_0 & h_{-1}
\end{bmatrix}, \quad (10)$$
where \( \bar{h}_k : k = -(N-1), \ldots, N-1 \), is a finite subsequence of the infinite sequence \( \{ \bar{h}_k : k = -\infty, \ldots, \infty \} \) defined by

\[
\{ \bar{h}_k : k = 0, \ldots, \infty \} \triangleq \{ C_S B_s, C_S A_S B_s, C_S A_S^2 B_s, \ldots \},
\]

(11)

\[
\{ \bar{h}_k : k = -1, \ldots, -\infty \} \triangleq \{-C_u A_u^{-1} B_u, -C_u A_u^{-2} B_u, -C_u A_u^{-3} B_u, \ldots \}.
\]

(12)
Consider, $P = Q$ and

$$Q = C^T C \text{ and } R = \rho I_m > 0.$$ 

$V_N$ as follows$^1$:

$$V_N(x, u, y, \mu) = \frac{1}{2} (x^T Q x + y^T y + \rho u^T u).$$

$^1$We keep the function $V_N$ but change its arguments as appropriate.
\[ V_N(x, u, \mu) = \frac{1}{2} \left[ x^T Q x + (\Gamma u + \Omega_s x_0^s + \Omega_u \mu)^T (\Gamma u + \Omega_s x_0^s + \Omega_u \mu) + \rho u^T u \right] \]

\[ = \frac{1}{2} u^T [\Gamma^T \Gamma + \rho I] u + u^T \Gamma^T (\Omega_s x_0^s + \Omega_u \mu) \]

\[ + \frac{1}{2} (\Omega_s x_0^s + \Omega_u \mu)^T (\Omega_s x_0^s + \Omega_u \mu). \]
With respect to the new optimisation variables \((u, \mu)\), the modified Hessian of the quadratic objective function is

\[
H' \triangleq \begin{bmatrix}
\tilde{\Gamma}^T \tilde{\Gamma} + \rho l & \tilde{\Gamma}^T \Omega_u \\
\Omega_u^T \tilde{\Gamma} & \Omega_u^T \Omega_u
\end{bmatrix}.
\] (14)
For future use, we extract the left-upper submatrix, which we call the “regularised sub-Hessian”:

$$\bar{H}_N \triangleq \bar{\Gamma}^T \bar{\Gamma} + \rho I.$$  (15)
The above problem formulation has the ability to ameliorate the numerical difficulties encountered when dealing with unstable plants. Indeed, we see that all terms depend only on exponentially decaying quantities.
Consider a single input-single output system with stable and unstable modes defined via the following matrices:

\[
A_s = \begin{bmatrix} 1.442 & -0.64 \\ 1 & 0 \end{bmatrix}, \quad B_s = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_s = \begin{bmatrix} 0.721 & -0.64 \end{bmatrix},
\]

and

\[
A_u = 2, \quad B_u = 1, \quad C_u = -1.
\]
we take

\[ Q = \begin{bmatrix} C_s & C_u \end{bmatrix}^T \begin{bmatrix} C_s & C_u \end{bmatrix}, \quad R = 0.1, \quad P = Q. \]

We consider input constraints of the form \(|u_k| \leq 1\) and no state constraints.
Figure: Condition number of the modified Hessian (14) (circle-dashed line) and that of the standard Hessian (plus-solid line).
Figure: Comparison of the objective function value for different prediction horizons: values achieved using the modified Hessian (circle-dashed line) and using the standard Hessian (plus-solid line).
4. Revision of System Frequency Response

Consider again the system split into stable and unstable parts. The system transfer function is clearly

\[ G(z) = G_s(z) + G_u(z), \quad (16) \]

where

\[ G_s(z) = C_s(zI - A_s)^{-1} B_s, \quad (17) \]
\[ G_u(z) = C_u(zI - A_u)^{-1} B_u, \quad (18) \]
We will study properties of the frequency response of, $G(e^{j\omega})$, via its singular value decomposition [SVD], which has the form

$$G(e^{j\omega}) = U(e^{j\omega})\Sigma(\omega)V^H(e^{j\omega}).$$

(19)

where, $U$ and $V$ are matrices containing the left and right singular vectors of $G$, satisfying $U \in \mathbb{C}^{p \times p}$, $U^H U = UU^H = I_p$, $V \in \mathbb{C}^{m \times m}$, $V^H V = VV^H = I_m$, and

$$\Sigma(\omega) = \text{diag}\{\sigma_1(\omega), \ldots, \sigma_m(\omega)\},$$

(20)
Lemma

Let \( \tilde{G}(z) \) be the two-sided \( Z \)-transform of the sequence \( \{\tilde{h}_k : k = -\infty, \ldots, \infty\} \) embedded in \( \tilde{\Gamma} \). Then \( \tilde{G}(z) \) is given by

\[
\tilde{G}(z) = zG(z),
\]

(21)

where \( G(z) \) is the system transfer function. Moreover, the region of convergence of \( \tilde{G}(z) \) is given by

\[
\max\{||\lambda_i(A_s)||\} < |z| < \min\{||\lambda_i(A_u)||\},
\]

where \( \{\lambda_i(\cdot)\} \) is the set of eigenvalues of the corresponding matrix.

Proof: By direct calculation.
5. Connecting the Singular Values of the Regularised Sub-Hessian to the System Frequency Response

We show that there is a direct connection between the singular values of $\bar{H}_N$, for large $N$, and the system frequency response.
The result of the previous Lemma ensures that given any $\varepsilon > 0$ there exists $k_0 > 0$ such that

$$\left\| \bar{G}(e^{j\omega}) \right\|_2^2 - \left\| \sum_{k=-k_0}^{k_0} \bar{h}_k e^{-j\omega k} \right\|_2^2 < \varepsilon \quad \text{for all } w \in [-\pi, \pi],$$

(22)

since the sequence $\{\bar{h}_k\}$ contains only exponentially decaying terms.
Recall the structure of $\bar{\Gamma}$ and using the definition of $\Psi_\ell$ given above, we see that the regularised sub-Hessian can be written as

$$
\bar{H}_N = \begin{bmatrix}
X_1 & | & 0 & \ldots & 0 \\
\Psi_{-2k_0} & \ldots & \Psi_0 & \ldots & \Psi_{2k_0} & 0 & \ldots & 0 \\
0 & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \Psi_{-2k_0} & \ldots & \Psi_0 & \ldots & \Psi_{2k_0} \\
0 & \ldots & 0 & | & 0 & \ldots & 0 \\
\end{bmatrix} + \rho I,
$$

provided $N \geq (4k_0 + 1)m$. Also, $X_1$ and $X_2$ are appropriate submatrices.
Consider the $Nm \times m$ matrix

$$E_{N,\omega} \triangleq \begin{bmatrix} e_{N,\omega}^1 & \cdots & e_{N,\omega}^m \end{bmatrix} \triangleq \frac{1}{\sqrt{N}} \bar{E}_{N,\omega} V(e^{j\omega}), \quad (24)$$

where

$$\bar{E}_{N,\omega} \triangleq \begin{bmatrix} I_m \\ e^{-j\omega I_m} \\ \vdots \\ e^{-j(N-1)\omega I_m} \end{bmatrix}, \quad \omega = \frac{2\pi}{N} \ell \quad \text{for } \ell \in \{0, \ldots, N-1\}.$$
Consider the discrete time linear system, and the SVD of its frequency response. Let \( k_0 > 0 \) be such that the tails of the impulse response are negligible. Consider \( \tilde{H}_N \) for \( N \geq 4k_0 + 1 \) and \( \rho = 0 \), which is the regularised sub-Hessian of the quadratic objective function \( V_N \) with \( \rho = 0 \). Then, for every given frequency \( \omega_0 = \frac{2\pi}{N_0} \ell_0, \ell_0 \in \{0, \ldots, N_0 - 1\} \), we have that

\[
\lim_{\frac{N}{N_0} \to \infty} \| \tilde{H}_N \mathbf{e}_N^{i,\omega_0} \|_2 = \sigma_i^2(\omega_0) \quad \text{for } i = 1, \ldots, m,
\]

where \( \mathbf{e}_N^{i,\omega_0} \) is the \( i \)th column of \( E_{N,\omega_0} \) defined in (24), and \( \frac{N}{N_0} \in \{1, 2, \ldots\} \).
The key idea is to note that the inner sub-matrix of $\bar{H}_N$ has a simple shift structure which allows one to easily apply Fourier methods. We then simply have to bound the end effects due to the other terms.
Corollary

Consider the same conditions of the above Theorem and choose $\rho > 0$. Then

$$
\lim_{\frac{N}{N_0} \to \infty} \left\| \bar{H}_N \mathbf{e}_{i,N,\omega_0} \right\|_2 = \sigma_i^2(\omega_0) + \rho \quad \text{for } i = 1, \ldots, m, \quad (25)
$$

where $N/N_0 \in \{1, 2, \ldots \}$. 

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Consider a 2 input-2 output system with stable and unstable modes defined via the following matrices.

\[
A_s = \begin{bmatrix} 1.442 & -0.64 \\ 1 & 0 \end{bmatrix}, \quad B_s = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_s = \begin{bmatrix} 0.721 & -0.36 \\ -0.64 & 0.32 \end{bmatrix},
\]

and

\[
A_u = 2, \quad B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_u = \begin{bmatrix} -0.1 \\ -0.1 \end{bmatrix}.
\]
Consider the finite horizon optimal control problem (5)–(7) and select

\[ Q = \begin{bmatrix} C_s & C_u \end{bmatrix}^T \begin{bmatrix} C_s & C_u \end{bmatrix}, \quad R = 0, \quad P = Q. \]
Figure: Singular values of the regularised sub-Hessian $\bar{H}_N$ (circles) with $N = 61$. The continuous lines represent the square of the two principal gains of the system.
Figure: Singular values of the regularised sub-Hessian $\bar{H}_N$ (circles) with $N = 401$. The continuous lines represent the square of the two principal gains of the system.
In this lecture, we have seen:

(i) Numerical conditioning of the Hessian can be improved by solving the stable part in forward time and the unstable part in reverse time. This exploits the connection between “stability” and “causality”.

(ii) That the singular values of $\bar{H}_N$ converge to the Frequency Domain Principle gains of the system.