Near-Optimal Cheap Control of Nonlinear Systems*

J.H. Braslavsky
julio@seidel.ece.ucsb.edu

M.M Seron
seron@seidel.ece.ucsb.edu

P.V. Kokotović
petar@ece.ucsb.edu

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Center for Control Engineering and Computations
University of California
Santa Barbara, CA 93106-9560

Phone: (805) 893-7066
Fax: (805) 893-3262
Email: yu@seidel.ece.ucsb.edu

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Abstract

For strict-feedback nonlinear systems with relative degree r, we show that there exists a coordinate transformation under which input-output feedback linearization with Butterworth pole placement is an $O(\varepsilon)$ approximation of the cheap control that minimizes a quadratic cost functional with control weighting ε^{2r}.

Keywords: Strict-feedback systems, Feedback linearization, Optimal cheap control, Asymptotic expansions, Minimum-energy control.

1 Introduction

Recent L_2-performance studies for nonlinear systems have renewed the interest in singular optimal control problems (Maas & van der Schaft 1994, Astolfi 1994, Marino, Respondek, van der Schaft & Tomei 1994, Isidori 1995, Isidori 1996, Schwartz, Isidori & Tarn 1997, Seron, Braslavsky, Kokotović & Mayne 1997). In the singular perturbation approach to these problems, a small parameter ε is introduced as a weight for the control cost ("cheap control"), and then the original singular problem is recovered in the limit as ε tends to zero. Jameson & O'Malley (1975) developed such an asymptotic method for the linear quadratic regulator.

We study here the cheap control of nonlinear systems in strict-feedback form (Krstić, Kanellakopoulos & Kokotović 1995). Exploiting the structure of these systems we derive an approximation of the optimal value function which is $O(\varepsilon)$-*near-optimal* in the sense that it is an asymptotic expansion of the optimal value function, and the $O(\varepsilon)$ truncation of the corresponding control asymptotically stabilizes the system incurring a bounded cost.

Using the cheap control Hamilton-Jacobi-Bellman (HJB) equation, we recursively construct an explicit expression for the $O(\varepsilon)$-near-optimal asymptotic expansion of the optimal value function. The form of the control law obtained from this expansion suggests a coordinate transformation that takes the system into an Isidori normal form. In the new coordinates, the $O(\varepsilon)$-near-optimal cheap control performs input-output feedback linearization and assigns the eigenvalues of the linearized part of the system in a Butterworth configuration. For linear systems, this fact agrees with the well-known asymptotic cheap control results of Kwakernaak & Sivan (1972).

2 Near-Optimal Cheap Control

The cheap optimal control for the strict-feedback system

$$\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)\xi_1 & x &\in \mathbb{R}^{n-r} \\
\dot{\xi}_1 &= f_1(x,\xi_1) + \xi_2\,, & \xi_i &\in \mathbb{R} \\
\dot{\xi}_2 &= f_2(x,\xi_1,\xi_2) + \xi_3\,, \\
&\vdots \\
\dot{\xi}_r &= f_r(x,\xi_1,\dots,\xi_r) + u\,, & u &\in \mathbb{R}
\end{aligned} \tag{1}$$

with an equilibrium at the origin, and where $f, g, f_i, i = 1, \dots, r$, are smooth functions, is the globally asymptotically stabilizing feedback that minimizes the cost functional

$$J_\varepsilon = \frac{1}{2}\int_0^\infty \left[\xi_1^2(t) + \varepsilon^{2r}u^2(t)\right] dt\,, \tag{2}$$

when $\varepsilon > 0$ is small. It is well known that, if there exists a smooth function $V(x,\xi_1,\dots,\xi_r,\varepsilon) \geq 0$, $V(0,\dots,0,\varepsilon) = 0$, which satisfies the HJB equation

$$\frac{\partial V}{\partial x}[f_0 + g_0\xi_1] + \frac{\partial V}{\partial \xi_1}[f_1 + \xi_2] + \cdots + \frac{\partial V}{\partial \xi_r} f_r + \frac{1}{2}\xi_1^2 - \frac{1}{2\varepsilon^{2r}}\left[\frac{\partial V}{\partial \xi_r}\right]^2 = 0 \tag{3}$$

and such that the feedback control

$$u = -\frac{1}{\varepsilon^{2r}} \frac{\partial V}{\partial \xi_r} \tag{4}$$

asymptotically stabilizes (1), then (4) is the optimal stabilizing control that minimizes J_ε and V is the optimal value function.

It has been shown recently that the limiting solution of the cheap control problem as $\varepsilon \to 0$ is the solution to a *minimum energy* problem for the x-subsystem of (1) (Seron, Braslavsky, Kokotović & Mayne 1997). We assume that the solution to this problem exists.

Assumption 1 (Minimum Energy Condition). There exists a C^{r+1} positive definite and proper function $V_0(x)$, $V_0(0) = 0$, which satisfies the HJB equation

$$\frac{\partial V_0}{\partial x} f_0(x) - \frac{1}{2} \frac{\partial V_0}{\partial x} g_0(x) g_0^T(x) \frac{\partial^T V_0}{\partial x} = 0 \ , \tag{5}$$

and such that

$$\xi_1 = \alpha_0(x) \triangleq -g_0^T(x) \frac{\partial^T V_0}{\partial x} \tag{6}$$

globally asymptotically stabilizes $\dot{x} = f_0(x) + g_0(x)\xi_1$. ∘

The assumption that V_0 is positive definite and proper rules out the case where $\dot{x} = f_0(x)$ is globally asymptotically stable (GAS), for, if $\dot{x} = f_0(x)$ is GAS, then $V_0(x) \equiv 0$. In this case, our results still hold (with $\alpha_0 = 0$ in (6)) provided that $\dot{x} = f_0(x) + g_0(x)\xi_1$ is input-to-state stable in the sense defined by Sontag (1989).

In this paper we construct a *near-optimal* solution to the cheap control problem. With V_0 as in (5), we find functions $V_1, V_2, \ldots, V_{2r}$, such that

$$V(x, \xi, \varepsilon) \quad = \quad V_0(x) \ + \ \sum_{i=1}^{r} \varepsilon^i V_i(x, \xi_1, \ldots, \xi_i) \ + \ \sum_{i=r+1}^{2r} \varepsilon^i V_i(x, \xi_1, \xi_2, \ldots, \xi_r) \tag{7}$$

satisfies the HJB equation (3) up to $O(\varepsilon^{2r})$. This means that, upon the substitution of (7) in (3),

$$\begin{aligned}
&\left\{ \frac{\partial V_0}{\partial x} [f_0 + g_0 \xi_1] + \frac{1}{2}\xi_1^2 - \frac{1}{2} \left[\frac{\partial V_r}{\partial \xi_r} \right]^2 \right\} \\
&\qquad + \varepsilon \left\{ \frac{\partial V_1}{\partial x} [f_0 + g_0 \xi_1] + \frac{\partial V_1}{\partial \xi_1} [f_1 + \xi_2] - \frac{\partial V_r}{\partial \xi_r} \frac{\partial V_{r+1}}{\partial \xi_r} \right\} \\
&+ \varepsilon^2 \left\{ \frac{\partial V_2}{\partial x} [f_0 + g_0 \xi_1] + \frac{\partial V_2}{\partial \xi_1} [f_1 + \xi_2] + \frac{\partial V_2}{\partial \xi_2} [f_2 + \xi_3] - \frac{\partial V_r}{\partial \xi_r} \frac{\partial V_{r+2}}{\partial \xi_r} - \frac{1}{2} \left[\frac{\partial V_{r+1}}{\partial \xi_r} \right]^2 \right\} \\
&\qquad\qquad + O(\varepsilon^3) = 0 \,,
\end{aligned} \tag{8}$$

each expression multiplying $\varepsilon^0, \varepsilon, \ldots, \varepsilon^{2r-1}$ is zero. The recursive procedure is as follows: Start with the substitution of V_0, the solution of (5), in the ε^0 term of (8); this yields

$\partial V_r/\partial \xi_r = \xi_1 - \alpha_0$. Use this $\partial V_r/\partial \xi_r$, with a candidate $V_1(x, \xi_1)$, in the $O(\varepsilon)$ term of (8); this gives $\partial V_{r+1}/\partial \xi_r = \dot{\xi}_1 - \dot{\alpha}_0$. Repeat up to the desired power of ε.

Using this procedure with the candidate set of functions

$$\begin{aligned} V_1 &= p_{11}\,(\xi_1 - \alpha_0)^2\,, \\ V_2 &= p_{12}\,(\xi_1 - \alpha_0)(\dot{\xi}_1 - \dot{\alpha}_0)\,, \\ V_3 &= p_{31}\,(\xi_1 - \alpha_0)(\ddot{\xi}_1 - \ddot{\alpha}_0) + p_{22}(\dot{\xi}_1 - \dot{\alpha}_0)^2\,, \\ &\ \vdots \end{aligned} \tag{9}$$

where $\{p_{11}, p_{12}, \ldots\}$ are some real coefficients, yields

$$\begin{aligned} \frac{\partial V_r}{\partial \xi_r} &= (\xi_1 - \alpha_0)\,, \\ \frac{\partial V_{r+1}}{\partial \xi_r} &= k_1\,(\dot{\xi}_1 - \dot{\alpha}_0)\,, \\ &\ \vdots \\ \frac{\partial V_{2r-1}}{\partial \xi_r} &= k_{r-1}\,(\xi_1^{(r-1)} - \alpha_0^{(r-1)})\,, \end{aligned} \tag{10}$$

where the coefficients $k_1, \ldots, k_{r-1}$ are to be determined, and

$$\frac{\partial V_{2r}}{\partial \xi_r} = f_r + \dot{f}_{r-1} + \cdots + f_1^{(r-1)} - \alpha_0^{(r)}\,. \tag{11}$$

By replacing (7) with the selected functions (9), (10) in (4) and truncating, we obtain

$$\begin{aligned} \hat{u} &= -\frac{1}{\varepsilon^{2r}}\left[\varepsilon\frac{\partial V_1}{\partial \xi_r} + \cdots + \varepsilon^{2r}\frac{\partial V_{2r}}{\partial \xi_r}\right] \\ &= -\frac{1}{\varepsilon^r}\left[\xi_1 - \alpha_0\right] - \frac{k_1}{\varepsilon^{r-1}}\left[\dot{\xi}_1 - \dot{\alpha}_0\right] - \cdots - \frac{k_{r-1}}{\varepsilon}\left[\xi_1^{(r-1)} - \alpha_0^{(r-1)}\right] \\ &\quad - f_r - \dot{f}_{r-1} - \cdots - f_2^{(r-2)} - f_1^{(r-1)} + \alpha_0^{(r)}\,. \end{aligned} \tag{12}$$

We will say that the control (12) is $O(\varepsilon)$-*near-optimal* if there exists $\varepsilon^* > 0$ such that $\hat{u}$ globally asymptotically stabilizes (1) and gives a bounded cost (2) for all $\varepsilon \in (0, \varepsilon^*]$.

As we will see, the control $\hat{u}$ induces in the closed-loop system the invariant manifold $\xi_1(t) \equiv \alpha_0(x(t))$. In order to ensure that $\hat{u}$ incurs a bounded cost, V_{2r}, which satisfies the constraint (11), must satisfy *on this manifold* a HJB equation corresponding to the minimum energy problem for the system (1).

Assumption 2. (Constrained Minimum Energy Condition). There exists a C^2 function $V_{2r}(x, \xi)$ satisfying (11), with $V_{2r}(0, 0) = 0$, such that on the manifold $\xi_1(t) \equiv \alpha_0(x(t))\ \forall t \geq 0$ satisfies the HJB equation

$$\begin{aligned} \frac{\partial V_{2r}}{\partial x}[f_0(x) + g_0(x)\xi_1] + \frac{\partial V_{2r}}{\partial \xi_1}[f_1(x, \xi_1) + \xi_2] + \cdots + \frac{\partial V_{2r}}{\partial \xi_{r-1}}[f_{r-1}(x) + \xi_r] \\ + \frac{\partial V_{2r}}{\partial \xi_r}f_r(x, \xi) - \frac{1}{2}\left[\frac{\partial V_{2r}}{\partial \xi_r}\right]^2 = 0\,. \end{aligned} \tag{13}$$

∘

We are now ready to state our main result, which we prove at the very end of the paper.

Theorem 2.1. (Near-Optimal Cheap Control for Strict-Feedback Systems). Under Assumptions 1 and 2, the control (12) with α_0 given by (6) and the coefficients $\{k_1, k_2, \ldots, k_{r-1}\}$ such that

$$\beta_r(s) = 1 + k_1 s + k_2 s^2 + \cdots + k_{r-1} s^{r-1} + s^r$$

is a Butterworth polynomial, is $O(\varepsilon)$-near-optimal for the system (1) and the cost (2). $\circ$

We observe that the control law (12) linearly combines $\xi_1 - \alpha_0$ and its derivatives up to the order $r-1$. This observation suggests a transformation of coordinates to the Isidori normal form with $z_1 = \xi_1 - \alpha_0$ as the output. Using this transformation, in the following section we explore the connections between near-optimality and input-output feedback linearization.

3 Near-Optimality and Input-Output Linearization

In the new coordinates

$$\begin{aligned} x &= x\,, \\ z_1 &= \xi_1 - \alpha_0(x)\,, \\ z_2 &= \dot{\xi}_1 - \dot{\alpha}_0(x)\,, \\ &\vdots \\ z_r &= \xi_1^{(r-1)} - \alpha_0^{(r-1)}\,, \end{aligned} \tag{14}$$

the Isidori normal form of system (1) is

$$\begin{aligned} \dot{x} &= f_0(x) + g_0(x)\alpha_0(x) + g_0(x)y\,, \\ \dot{z} &= Az + B\left[u + \tilde{f}(x,z)\right]\,, \\ y &= Cz\,, \end{aligned} \tag{15}$$

where $z = [z_1, z_2, \ldots, z_r]^T$, $\tilde{f} = f_r + \dot{f}_{r-1} + \cdots + f_2^{(r-2)} + f_1^{(r-1)} - \alpha_0^{(r)}$, and

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}. \tag{16}$$

We observe that, for the output y, the zero dynamics system is $\dot{x} = f_0(x) + g_0(x)\alpha_0(x)$, that is the x-subsystem of (1) stabilized with the minimum energy control α_0.

The cost (2) and the HJB equation (3) become

$$J_\varepsilon = \frac{1}{2}\int_0^\infty \left[(y + \alpha_0)^2 + \varepsilon^{2r} u^2\right] dt\,, \tag{17}$$

$$\frac{\partial V}{\partial x}\left[f_0 + g_0[\alpha_0 + y]\right] + \frac{\partial V}{\partial z}\left[Az + B\tilde{f}\right] + \frac{1}{2}\left(y + \alpha_0\right)^2 - \frac{1}{2\varepsilon^{2r}}\frac{\partial V}{\partial z}BB^T\frac{\partial^T V}{\partial z} = 0, \quad (18)$$

and the optimal control is

$$u = -\frac{1}{\varepsilon^{2r}}\frac{\partial^T V}{\partial z_r}. \quad (19)$$

In the new coordinates, a candidate set of functions V_i's as in (9) is

$$\sum_{i=1}^{2r-1} V_i(z)\varepsilon^i = \frac{1}{2}z^T P_\varepsilon z, \quad (20)$$

$$\text{where}\begin{cases} P_\varepsilon = \varepsilon EPE, \quad P = P^T \in \mathbb{R}^{r\times r} \\ E = \mathrm{diag}\{[1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{r-1}]\} \end{cases}$$

$$V_{2r}(x,z) \quad \text{satisfies } \frac{\partial V_{2r}}{\partial z_r} = \tilde{f}. \quad (21)$$

The following result shows that there exists a matrix P for which the candidate functions (20)-(21) satisfy (18) up to $O(\varepsilon^{2r})$.

Lemma 3.1 ($O(\varepsilon^{2r})$ HJB Condition). Under Assumption 1 the candidate functions (20)-(21) satisfy (18) up to $O(\varepsilon^{2r})$ if and only if $P = P^T$ is a solution of the algebraic Riccati equation

$$PA + A^T P + C^T C - PBB^T P = 0, \quad (22)$$

where A, B, C are defined in (16).

Proof. Using $V = V_0 + \frac{1}{2}z^T P_\varepsilon z + \varepsilon^{2r} V_{2r}$ and

$$\frac{\partial V_0}{\partial x}\left[f_0 + g_0[\alpha_0 + y]\right] + \frac{1}{2}\alpha_0^2 + \alpha_0 y = 0$$

(which follows from Assumption 1) in (18) gives, after rearranging,

$$\frac{1}{2}z^T\left[P_\varepsilon A + A^T P_\varepsilon + C^T C - \frac{1}{\varepsilon^{2r}}P_\varepsilon BB^T P_\varepsilon\right] z + z^T P_\varepsilon B\left[\tilde{f} - \frac{\partial^T V_{2r}}{\partial z_r}\right] + O(\varepsilon^{2r}) = 0. \quad (23)$$

The second term in (23) is zeroed with the choice of $\partial V_{2r}/\partial z_r = \tilde{f}$ in (20)-(21). For the first term we have

$$\begin{aligned} &P_\varepsilon A + A^T P_\varepsilon + C^T C - \frac{1}{\varepsilon^{2r}}P_\varepsilon BB^T P_\varepsilon \\ &\qquad = E\left[\varepsilon P(EAE^{-1}) + \varepsilon(E^{-1}A^T E)P + E^{-1}C^T C E^{-1} - \frac{1}{\varepsilon^{2r-2}}P(EBB^T E)\right]E \\ &\qquad = E\left[PA + A^T P + C^T C - PBB^T P\right]E, \end{aligned}$$

where, in the last step, we make use of $EAE^{-1} = A/\varepsilon$, $(EBB^T E) = BB^T\varepsilon^{2r-2}$ and $E^{-1}C^T C E^{-1} = C^T C$. This proves that the first term in (23) has powers of ε no greater than $2r-2$ and therefore it must be zero, which yields (22). □

With the asymptotic expansion

$$V = V_0 + \frac{1}{2} z^{\mathsf{T}} P_\varepsilon z + \varepsilon^{2r} V_{2r} + O(\varepsilon^{2r+1}) \tag{24}$$

with V_0 from Assumption 1, and P_ε and V_{2r} from (20)-(21), (22), the optimal control (19) takes the form

$$\begin{aligned} u &= -\frac{1}{\varepsilon^{2r}} \left[\varepsilon B^{\mathsf{T}} E P E z + \varepsilon^{2r} \tilde{f} \right] + O(\varepsilon) \\ &= -\frac{1}{\varepsilon^{r}} B^{\mathsf{T}} P E z - \tilde{f} + O(\varepsilon)\,. \end{aligned}$$

Its $O(\varepsilon)$-truncation

$$\hat{u} = -\frac{1}{\varepsilon^{r}} B^{\mathsf{T}} P E z - \tilde{f} \tag{25}$$

input-output linearizes (15) producing the cascade closed-loop system

$$\dot{x} = f_0(x) + g_0(x)\alpha_0(x) + g_0(x) y\,, \tag{26}$$

$$\begin{aligned} \dot{z} &= (A - \frac{1}{\varepsilon^{r}} B B^{\mathsf{T}} P E) z\,, \\ y &= C z\,, \end{aligned} \tag{27}$$

which is GAS provided that P in (20) is chosen as the unique positive definite solution of (22).

Lemma 3.2 (Global Asymptotic Stability). Under Assumption 1, the control (25), where P is the unique positive definite solution of (22), renders the system (15) GAS by input-output feedback linearization, placing the eigenvalues of the linear part of the closed-loop system in a Butterworth pattern of radius $1/\varepsilon$.

Proof. The characteristic polynomial of the linear z-subsystem (27) is given by

$$B^{\mathsf{T}} P \left[\frac{1}{\varepsilon^{r}}, \frac{s}{\varepsilon^{r-1}}, \ldots, \frac{s^{r-1}}{\varepsilon}, s^{r} \right]^{\mathsf{T}} = \frac{1}{\varepsilon^{r}} \beta_r(\varepsilon s), \tag{28}$$

where $\beta_r(s)$ is the characteristic polynomial of the matrix $(A - BB^{\mathsf{T}}P)$. Since P is the unique positive definite solution of the ARE (22), the polynomial $\beta_r(s)$ is the stable factor of the characteristic polynomial of its associated Hamiltonian matrix,

$$\begin{aligned} \beta_r(s)\beta_r(-s) &= \det \begin{bmatrix} sI - A & BB^{\mathsf{T}} \\ C^{\mathsf{T}} C & sI + A^{\mathsf{T}} \end{bmatrix} \\ &= 1 + (-1)^r s^{2r}\,, \end{aligned}$$

which is the Butterworth polynomial of degree r. Hence, from (28), the eigenvalues of the z-subsystem (27) lie on a Butterworth pattern of radius $1/\varepsilon$.

Because the z-subsystem (27) is asymptotically stable (since β_r is Hurwitz), and the x-subsystem (26) is GAS (by Assumption 1), it follows from Proposition 4.1 of Sepulchre, Janković & Kokotović (1997) that every solution of the cascade (26), (27) either converges to zero or is unbounded. It is then sufficient for GAS to show that x remains bounded.

From (5) we have

$$\dot{V}_0 = \frac{\partial V_0}{\partial x}[f_0 + g_0\alpha_0] + \frac{\partial V_0}{\partial x} g_0 z_1 = -\frac{1}{2}\alpha_0^2 - \alpha_0 z_1 \,.$$

Using Young's inequality and integrating yields

$$\begin{aligned} V_0(x(t)) &\le V_0(x(0)) - \frac{1}{4}\int_0^t [\alpha_0(x(\tau))]^2 \, d\tau + \int_0^t z_1^2(\tau)\, d\tau \,, \\ &\le V_0(x(0)) + \|z_1\|^2 \,, \end{aligned} \tag{29}$$

where $\|z_1\|^2 = \int_0^\infty z_1^2(\tau)\, d\tau < \infty$ since the z-subsystem is exponentially stable. Hence, $V_0(x(t))$ is bounded for all $t \ge 0$ and so is x because V_0 is positive definite and proper, which proves that (26), (27) is GAS. □

The feedback linearization of the system (15) is attained by virtue of the condition (21) imposed on V_{2r}, which cancels potentially destabilizing nonlinearities in $\tilde{f}$. Furthermore, this cancellation induces the ε-independent invariant manifold $z \equiv 0$ in the closed-loop system (26), (27). As anticipated in Section 2, for the control (25) to have a bounded cost, we further require V_{2r} to satisfy (13) on the manifold $\xi_1(t) \equiv \alpha_0(x(t))$. In the z-coordinates this manifold is just $z \equiv 0$, and (13) becomes

$$\begin{aligned} &\frac{\partial V_{2r}}{\partial x}[f_0(x) + g_0(x)\alpha_0(x) + g_0(x)Cz] + \frac{\partial V_{2r}}{\partial z}[Az + B\tilde{f}(x,z)] \\ &\qquad - \frac{1}{2}\frac{\partial V_{2r}}{\partial z} BB^T \frac{\partial^T V_{2r}}{\partial z} = 0 \,. \end{aligned} \tag{30}$$

The left hand side of (30) is the expression multiplying ε^{2r} in the HJB equation (18) after substitution of the asymptotic expansion for V. Assumption 2 requires that this term be zero on the manifold $z \equiv 0$.

Lemma 3.3. (Boundedness of the Cost). Under the conditions of Lemma 3.2 and Assumption 2, and for any initial conditions $x(0)$, $z(0)$, and $\varepsilon < 1$, the cost incurred by the control (25) is

$$J_\varepsilon = V_0(x(0)) + O(\varepsilon) \,. \tag{31}$$

Proof. The substitution of (25) in (17) yields

$$J_\varepsilon = \frac{1}{2}\int_0^\infty \left[(y + \alpha_0)^2 + \left(B^T P E z + \varepsilon^r \tilde{f}\right)^2 \right] \, dt \,. \tag{32}$$

The first term in the integrand of (32) can be expressed from (5) and (6) as

$$\begin{aligned} (y + \alpha_0)^2 &= y^2 + 2\alpha_0 y + \alpha_0^2 \\ &= y^2 - 2\frac{\partial V_0}{\partial x}\,[f_0 + g_0\alpha_0 + g_0 y] \\ &= y^2 - 2\dot{V}_0 \,, \end{aligned} \tag{33}$$

where in the last step we have substituted $\dot{x} = f_0 + g_0\alpha_0 + g_0 y$ from (27). The second term in the integrand of (32) can be written as

$$(B^T PEz + \varepsilon^r \tilde{f})^2 = z^T EPBB^T PEz + 2\varepsilon^r \tilde{f} B^T PEz + \varepsilon^{2r}\tilde{f}^2 \quad (34)$$

where

$$\begin{aligned} z^T EPBB^T PEz &= z^T E(PBB^T P + C^T C)Ez - y^2 \\ &= z^T E(2PBB^T P - PA - A^T P)Ez - y^2 \\ &= 2z^T E(PBB^T P - PA)Ez - y^2 \\ &= 2\varepsilon z^T EPE(\frac{1}{\varepsilon^r} BB^T PE - A)z - y^2 \\ &= -\frac{d(z^T P_\varepsilon z)}{dt} - y^2 \end{aligned} \quad (35)$$

(the last step used $\dot{z} = (A - \frac{1}{\varepsilon^r} BB^T PE)z$ from (27)).

Because the closed-loop system (26), (27) is GAS and $V_0(0) = 0$, (33), (34) and (35) in (32) give

$$J_\varepsilon = V_0(x(0)) + \frac{\varepsilon}{2} z(0)^T EPEz(0) + \frac{1}{2}\int_0^\infty \left[2\varepsilon^r \tilde{f} B^T PEz + \varepsilon^{2r}\tilde{f}^2\right]. \quad (36)$$

We complete the proof by showing that the integral in (36) is $O(\varepsilon)$. Using (26), (27) and (21), rewrite (30) as

$$\begin{aligned} &\frac{\partial V_{2r}}{\partial x}\dot{x} + \frac{\partial V_{2r}}{\partial z}\dot{z} + \frac{\partial V_{2r}}{\partial z}\left(\frac{1}{\varepsilon^r} BB^T PEz + B\tilde{f}\right) - \frac{1}{2}\frac{\partial V_{2r}}{\partial z} BB^T \frac{\partial^T V_{2r}}{\partial z} \\ &\qquad = \dot{V}_{2r} + \frac{1}{\varepsilon^r}\tilde{f} B^T PEz + \frac{\tilde{f}^2}{2}. \end{aligned} \quad (37)$$

Since (37) equals zero when $z = 0$, and V_{2r} and $\tilde{f}$ are at least C^2 by Assumption 2, there exist a continuous function $\rho(x)$ independent of ε such that for $\varepsilon < 1$

$$\dot{V}_{2r} + \frac{1}{\varepsilon^r}\tilde{f} B^T PEz + \frac{\tilde{f}^2}{2} \le \frac{\rho(x)}{\varepsilon^r}\|z\|. \quad (38)$$

Following steps similar to those in Sussmann & Kokotović (1991, Appendix D) it is possible to show that the solutions of the linear z-subsystem (27) satisfy the bound

$$\|z(t)\| \le \frac{1}{\varepsilon^{r-1}}\gamma\|z(0)\| e^{-\sigma t/\varepsilon}, \qquad \varepsilon < 1, \quad (39)$$

where γ and σ are positive constants independent of ε. On the other hand, the boundedness of the closed-loop trajectories shown by (29) guarantees that for some continuous function λ

$$\rho(x(t)) \le \lambda(x(0), z_1(0)), \qquad \forall t \ge 0. \quad (40)$$

We note that λ is as well independent of ε because z_1, which drives the x-subsystem, does not peak with ε (Sepulchre, Janković & Kokotović 1997, Proposition 4.32). It then follows from (38), (39) and (40) that the integral

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty \left[2\varepsilon^r \tilde{f} B^T P E z + \varepsilon^{2r}\tilde{f}^2\right]\, dt \\
&\quad \le -\varepsilon^{2r}\int_0^\infty \dot{V}_{2r}(x,z)\, dt + \varepsilon\gamma\lambda(x(0), z_1(0))\, \|z(0)\| \int_0^\infty e^{-\sigma t/\varepsilon}\, dt \\
&\quad \le \varepsilon^{2r} V_{2r}(x(0), z_1(0)) + \varepsilon^2 \frac{\gamma}{\sigma}\lambda(x(0), z_1(0))\, \|z(0)\|
\end{aligned}$$

is $O(\varepsilon^2)$. Thus (36) has the form (31). □

We conclude from Lemmas 3.1, 3.2 and 3.3 that the $O(\varepsilon)$-truncation of the control obtained from the asymptotic expansion (24) is $O(\varepsilon)$-near-optimal.

Theorem 3.4. (Near-Optimal Feedback Linearization). Let $\beta_r(s) = 1 + k_1 s + k_2 s^2 + \cdots + k_{r-1} s^{r-1} + s^r$ be the Butterworth polynomial of degree r. Then, under Assumptions 1 and 2, the feedback linearizing control

$$u = -\frac{1}{\varepsilon^r} z_1 - \frac{k_1}{\varepsilon^{r-1}} z_2 \cdots - \frac{k_{r-1}}{\varepsilon} z_r - \tilde{f}, \tag{41}$$

is $O(\varepsilon)$-near-optimal for the system (15) and the cost (17).

An immediate consequence of Theorem 3.4 is that if $\tilde{f} = 0$ in system (15), the $O(\varepsilon)$-near-optimal cheap control *is in fact optimal*; the optimal value function has no terms in powers of ε greater than $2r - 1$.

Corollary 3.5. (Optimal Cheap Control for a Chain of Integrators). Under Assumption 1, the optimal value for the system

$$\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)\alpha_0(x) + g_0(x) z_1, \\
\dot{z_1} &= z_2 \\
&\;\vdots \\
\dot{z_r} &= u
\end{aligned} \tag{42}$$

and the cost (17) is

$$V(z) = V_0(x) + \frac{1}{2}\varepsilon \begin{bmatrix} z_1 \\ \varepsilon z_2 \\ \vdots \\ \varepsilon^{r-1} z_r \end{bmatrix}^T P \begin{bmatrix} z_1 \\ \varepsilon z_2 \\ \vdots \\ \varepsilon^{r-1} z_r \end{bmatrix}, \tag{43}$$

where P is the unique symmetric positive definite solution of the Riccati equation (22).

Example 3.1. (Closed-Form Cheap Optimal Value). The optimal value for the system

$$\begin{aligned}
\dot{x} &= x^3 + \xi_1 \\
\dot{\xi}_1 &= 6x^5 + \xi_2 \\
\dot{\xi}_2 &= -12x\xi_1^2 - 72\xi_1 x^4 - 96x^7 - 6\xi_2 x^2 + u \\
y &= \xi_1
\end{aligned} \tag{44}$$

with cost (2) is

$$V(x,\xi_1,\xi_2,\varepsilon) = \frac{x^4}{2} + \frac{\sqrt{2}}{2}\varepsilon(\xi_1 + 2x^3)^2 + \varepsilon^2(\xi_1 + 2x^3)(6\xi_1 x^2 + 12x^5 + \xi_2) + \frac{\sqrt{2}}{2}\varepsilon^3(6\xi_1 x^2 + 12x^5 + \xi_2)^2 .$$

Proof. The minimum energy problem for the zero dynamics of system (44) gives $V_0(x) = x^4/2$ and $\alpha_0(x) = -2x^3$. The change of coordinates $z_1 = \xi_1 + 2x^3$, $z_2 = 6\xi_1 x^2 + 12x^5 + \xi_2$ takes the system to the form (42). The value V is then obtained from Corollary 3.5 by substituting the original coordinates. □

For linear systems, Theorem 3.4 has the following consequence.

Corollary 3.6. (Linear Near-Optimal Cheap Control). For a linear scalar system with $n-r$ zeros and r poles, the $O(\varepsilon)$-near-optimal cheap control places $n-r$ closed-loop poles at the minimum-phase open-loop zeros and the mirror image of the nonminimum-phase zeros, and the remaining r poles in a Butterworth pattern with radius $1/\varepsilon$.

Proof. Let (15) be the linear system

$$\begin{aligned}\dot{x} &= Fx + G\alpha_0(x)Gz_1\\ \dot{z}_1 &= z_2\,,\\ &\vdots\\ \dot{z}_r &= \phi_0 x + \phi_1 z_1 + \phi_2 z_2 + \cdots + \phi_r z_r + u\,,\end{aligned}$$

where $\alpha_0(x) = -G^T Px$, with $P \geq 0$ satisfying the equation $PF + F^T P = PGG^T P$, is the minimum energy control. The linear minimum energy control places the poles of $F - GG^T P$ at the stable poles of F and at the mirror image of its unstable poles (Kwakernaak & Sivan 1972, Theorem 3.11). The result is then completed by using Theorem 3.4 for the case $\tilde{f} = \phi_0 x + \phi_1 z_1 + \phi_2 z_2 + \cdots + \phi_r z_r$. □

A well-known cheap control result of Kwakernaak & Sivan (1972, Theorem 3.11) shows that the closed-loop pole locations specified in Corollary 3.6 are approached *asymptotically* as $\varepsilon \to 0$. Corollary 3.6 establishes that the $O(\varepsilon)$-near-optimal cheap control achieves these locations *exactly*. In fact, the control (41) in the linear case makes this pole placement for *any* value of ε, not necessarily small.

As a final remark, note that Theorem 3.4 and Corollary 3.6 imply that for the nonlinear system (15), feedback linearization with near-optimal linear cheap control is near optimal nonlinear control with respect to the cost (17).

The proof of Theorem 2.1 is now straightforward.

Proof. Proof of Theorem 2.1 Follows from the fact that in the original coordinates, (41) has the form (12), which is thus $O(\varepsilon)$-near-optimal since the coordinate transformation (14) is smooth and has a smooth inverse. □

References

Astolfi, A. (1994), Singular H_∞ control, *in* 'Proc. of the 33th IEEE CDC', Lake Buena Vista, USA, pp. 2543–2548.

Isidori, A. (1995), *Nonlinear control systems*, 3rd edn, Springer.

Isidori, A. (1996), 'Global almost disturbance decoupling with stability for non minimum-phase single-input single-output nonlinear systems', *Systems and Control Letters* **28**(2), 115–22.

Jameson, A. & O'Malley, Jr., R. (1975), 'Cheap control of the time-invariant regulator', *Applied Mathematics and Optimization* **1**(4), 337–354.

Krstić, M., Kanellakopoulos, I. & Kokotović, P. (1995), *Nonlinear and adaptive control design*, John Wiley & Sons.

Kwakernaak, H. & Sivan, R. (1972), *Linear Optimal Control Systems*, Wiley-Interscience.

Maas, W. & van der Schaft, A. (1994), Singular nonlinear H_∞ optimal control by state feedback, *in* 'Proc. of the 33th IEEE CDC', Lake Buena Vista, USA, pp. 1415–1420.

Marino, R., Respondek, W., van der Schaft, A. & Tomei, P. (1994), 'Nonlinear H_∞ almost disturbance decoupling', *Systems and Control Letters* **23**(3), 159–68.

Schwartz, B., Isidori, A. & Tarn, T. (1997), Performance bounds for disturbance attenuation in nonlinear nonminimum-phase systems, *in* 'Proceedings of the European Control Conference', Brussels, Belgium.

Sepulchre, R., Janković, M. & Kokotović, P. (1997), *Constructive Nonlinear Control*, Springer.

Seron, M., Braslavsky, J., Kokotović, P. & Mayne, D. (1997), Feedback limitations in nonlinear systems: From Bode integrals to cheap control, Technical Report 97-0304, CCEC, University of California, Santa Barbara. <http://www-ccec.ece.ucsb.edu/techrpts/index.html#1997>.

Sontag, E. (1989), 'Smooth stabilization implies coprime factorization', *IEEE Trans. on Automatic Control* **34**, 435–443.

Sussmann, H. & Kokotović, P. (1991), 'The peaking phenomenon and the global stabilization of nonlinear systems', *IEEE Trans. on Automatic Control* **36**(4), 424–440.