Limiting Performance of Optimal Linear Filters*

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Abstract

We study the lowest achievable mean square estimation error in two limiting optimal linear filtering problems. First, when the intensity of the process noise tends to zero, the lowest achievable mean square estimation error is a function of the unstable poles of the system. Second, when the intensity of the measurement noise tends to zero, the lowest achievable mean square estimation error is a function of the nonminimum phase zeros of the system. We link these results with Bode integral characterizations of performance limitations in linear filtering.

Keywords: Optimal Linear Filtering, Performance Limits, Nonminimum Phase Zeros, Unstable Poles, Bode Integrals, Singular Perturbations.

1 Introduction

The analysis of feedback limitations using analytic function theory was initiated by Bode in the 1940's and is of continuing interest (Bode 1945, Francis & Zames 1984, Freudenberg & Looze 1985, Middleton 1991, Chen 1995). In recent studies of performance limitations in filtering, Bode and Poisson integrals were established for functions representing sensitivity of the estimation error to process and measurement noise (Goodwin, Mayne & Shim 1995, Seron, Braslavsky & Goodwin 1997).

An alternative approach is to study the best achievable performance of optimal control when the cost of control tends to zero (cheap control) (Chang 1961, Kalman 1964, Kwakernaak & Sivan 1972, Francis 1979, Shaked 1980, Qiu & Davison 1993). This approach was recently linked to Bode integrals and extended to nonlinear systems (Seron, Braslavsky, Kokotović & Mayne 1997).

In this paper we derive performance limits in optimal filtering and establish their links with Bode integrals. By letting the intensity of either the process noise or the measurement noise tend to zero (as in Bryson & Johansen 1965, Krener 1986, Aganovic, Gajic & Shen 1995), we show that these performance limits are imposed by the unstable poles and nonminimum-phase (NMP) zeros of the transfer function from the process noise input to the measurements.

For vanishing process noise, the lowest achievable mean square (MS) estimation error must be nonzero if the system is unstable. In particular, when estimating the specific combination of states that form the measurements, we show that this error is an explicit function of the unstable poles, further characterized with a Bode integral.

For vanishing measurement noise, the lowest achievable MS estimation error must be nonzero if the system is NMP (Kwakernaak & Sivan 1972). We show that this error corresponds to that achieved by a reduced filtering problem for the unstable zero-dynamics of the system. In particular, for the problem of estimating a slowly-varying input to the system, the lowest achievable MS error is an explicit function of the NMP zeros, also characterized with a Bode integral.

Our filtering problem is the optimal MS estimation (Kalman & Bucy 1961) of the state x of the linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \quad x \in \mathbb{R}^n, w \in \mathbb{R}^m\,, \\ y(t) &= Cx(t) + v(t)\,, \qquad y \in \mathbb{R}^m, \end{aligned} \tag{1}$$

where B and C are full rank matrices and w and v are uncorrelated zero-mean Gaussian white noises with intensities Σ_w and Σ_v, that is,

$$\begin{aligned} E\{w(t)\} &= 0\,, \qquad & E\{w(t)w^T(\tau)\} &= \Sigma_w\,\delta(t-\tau)\,, \\ E\{v(t)\} &= 0\,, \qquad & E\{v(t)v^T(\tau)\} &= \Sigma_v\,\delta(t-\tau)\,. \end{aligned}$$

The matrices Σ_w and Σ_v are assumed constant and positive definite. The initial state $x(0)$ is a Gaussian random vector with mean $\bar{x}(0)$ and covariance Σ_0, uncorrelated with v and w.

The steady-state optimal filter for the system (1) is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + QC^T\Sigma_v^{-1}[y(t) - C\hat{x}(t)]\,, \qquad \hat{x}(0) = \bar{x}(0)\,, \tag{2}$$

where $Q = Q^T$ is a positive semidefinite solution of the algebraic Riccati equation (ARE)

$$AQ + QA^T + B\Sigma_w B^T - QC^T\Sigma_v^{-1}CQ = 0\,. \tag{3}$$

The matrix Q, which is unique if (1) is stabilizable and detectable, is the asymptotic variance of the estimation error $\tilde{x} = x - \hat{x}$, that is

$$\lim_{t\to\infty} E\{\tilde{x}(t)\tilde{x}^T(t)\} = Q\,.$$

The optimal filter (2) minimizes the MS estimation error

$$\lim_{t\to\infty} E\{\tilde{x}^T(t)\tilde{x}(t)\} = \operatorname{trace} Q\,.$$

In the remaining of the paper we use the notation of this section: for a state variable x, $\bar{x}(0)$ denotes the mean value of the initial state $x(0)$; $\hat{x}$ denotes the estimate of x and $\tilde{x}$ the corresponding estimation error.

2 Performance Limits Under Small Process Noise

2.1 Limiting Properties

We first investigate the properties of the optimal filter when the intensity of the process noise w becomes arbitrarily small.

Assumption 1.

(i) The system (1) is stabilizable and detectable.

(ii) The matrix A in (1) has no eigenvalues on the imaginary axis.

(iii) The intensities of the noises w and v are

$$\Sigma_w = \varepsilon^2 I\,, \quad \text{and} \quad \Sigma_v = I\,, \tag{4}$$

where $\varepsilon > 0$ is small. ∘

Conditions (i) and (ii) are necessary and sufficient for the filter (2) to be asymptotically stable in the limit as w vanishes. Because of (ii), a modal decomposition performed on (1) brings it to the form (Figure 1)

$$\begin{aligned} \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}B_1\\ B_2\end{bmatrix} w\,,\\ y &= \begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + v\,, \end{aligned} \tag{5}$$

where A_1 is Hurwitz and A_2 is *antistable* (that is $-A_2$ is Hurwitz). Condition (i) implies that (A_2, B_2, C_2) is controllable and observable. In the form (5), the eigenvalues of A_2 are the unstable poles of the system, that is, those of the transfer function from w to y.

Lemma 2.1 (Error Variance under No Process Noise). Under Assumption 1, the variance of the estimation error achieved by the optimal filter for (5) satisfies

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} E\left\{\begin{bmatrix}\tilde{x}_1(t)\\ \tilde{x}_2(t)\end{bmatrix}\begin{bmatrix}\tilde{x}_1(t)\\ \tilde{x}_2(t)\end{bmatrix}^T\right\} = \begin{bmatrix}0 & 0\\ 0 & Q_2\end{bmatrix}\,, \tag{6}$$

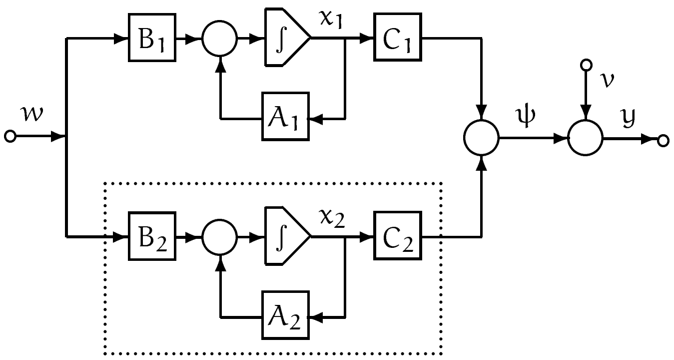

Figure 1: Modal decomposition of the system (5)

and the lowest achievable MS estimation error is

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} \mathrm{E}\left\{\begin{bmatrix}\tilde{x}_1(t)\\ \tilde{x}_2(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\tilde{x}_1(t)\\ \tilde{x}_2(t)\end{bmatrix}\right\} = \operatorname{trace} Q_2\,, \tag{7}$$

where Q_2 is the positive definite solution of the ARE

$$A_2Q_2 + Q_2A_2^{\mathrm{T}} = Q_2C_2^{\mathrm{T}}C_2Q_2\,. \tag{8}$$

Proof. Using (5) and (4) in (3) we obtain

$$\begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}Q + Q\begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}^{\mathrm{T}} + \varepsilon^2\begin{bmatrix}B_1B_1^{\mathrm{T}} & B_1B_2^{\mathrm{T}}\\ B_2B_1^{\mathrm{T}} & B_2B_2^{\mathrm{T}}\end{bmatrix} - Q\begin{bmatrix}C_1^{\mathrm{T}}C_1 & C_1^{\mathrm{T}}C_2\\ C_2^{\mathrm{T}}C_1 & C_2^{\mathrm{T}}C_2\end{bmatrix}Q = 0\,.$$

We seek its solution $Q = Q^{\mathrm{T}}$ in the form of the power series in ε

$$Q = \begin{bmatrix}\varepsilon^2Q_1 + O(\varepsilon^3) & \varepsilon^2Q_3 + O(\varepsilon^3)\\ * & Q_2 + O(\varepsilon)\end{bmatrix}\,.$$

A straightforward calculation shows that Q_1, Q_2 and Q_3 satisfy

$$\begin{aligned}\varepsilon^2\big(A_1Q_1 + Q_1A_1^{\mathrm{T}} + B_1B_1^{\mathrm{T}}\big) + O(\varepsilon^4) = 0\,,\\ \varepsilon^2\big(A_1Q_3 + Q_3A_2^{\mathrm{T}} + B_1B_2^{\mathrm{T}} - Q_1C_1^{\mathrm{T}}C_2Q_2 - Q_3C_2^{\mathrm{T}}C_2Q_2\big) + O(\varepsilon^4) = 0\,,\\ A_2Q_2 + Q_2A_2^{\mathrm{T}} - Q_2C_2^{\mathrm{T}}C_2Q_2 + O(\varepsilon^2) = 0\,.\end{aligned}$$

After setting $\varepsilon = 0$ we see that the only nonzero element of Q is Q_2, the positive definite solution of (8), which proves (6), (7), (8). □

The matrix Q_2 in (6) is the variance that would result from the filtering problem for

$$\begin{aligned}\dot{x}_2 &= A_2x_2\,,\\ y_2 &= C_2x_2 + \nu\,,\end{aligned}$$

the unstable subsystem in (5) (Figure 1). Hence, only the unstable modes of the system contribute to the MS estimation error (trace Q_2) when the process noise vanishes.

The optimal filter (2) takes the form

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{bmatrix} = \begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix}\varepsilon^2\big(Q_1C_1^T + Q_3C_2^T\big)\\ Q_2C_2^T + \varepsilon^2 Q_3^T C_1^T\end{bmatrix}\big(y - C_1\hat{x}_1 - C_2\hat{x}_2\big)\,, \qquad \begin{aligned}\hat{x}_1(0) &= \bar{x}_1(0)\,,\\ \hat{x}_2(0) &= \bar{x}_2(0)\,,\end{aligned}$$

and, in the limit as $\varepsilon \to 0$,

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{bmatrix} = \begin{bmatrix}A_1 & 0\\ -Q_2C_2^TC_1 & -Q_2A_2^TQ_2^{-1}\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix}0\\ Q_2C_2^T\end{bmatrix}y\,, \qquad \begin{aligned}\hat{x}_1(0) &= \bar{x}_1(0)\,,\\ \hat{x}_2(0) &= \bar{x}_2(0)\,,\end{aligned} \tag{9}$$

where we have used $A_2 - Q_2C_2^TC_2 = -Q_2A_2^TQ_2^{-1}$, which follows from (8). We see from (9) that the eigenvalues of the optimal filter converge, as $\varepsilon \to 0$, to the stable eigenvalues of (5) and to the *mirror image* (with respect to the imaginary axis) of the unstable poles of the system (Kwakernaak & Sivan 1972, Theorem 4.12).

As the process noise vanishes, the filtering problem becomes that of estimating the noise-free state of the system (5) using noisy measurements. Expression (9) shows that the filter for the stable subsystem $\dot{x}_1 = A_1x_1$ is disconnected from the measurements. However, the filter for the unstable subsystem $\dot{x}_2 = A_2x_2$ must continue to be driven by the noisy measurements due to the requirement of stability of the filter imposed by the optimization problem. Under this requirement, the lowest MS estimation error is achieved when the eigenvalues of the filter for the unstable subsystem are located at the mirror image of the unstable poles of the system (5).

2.2 A Bode Integral for the Estimation of the Noise-Free Output

For the problem of estimating $\psi = C_1x_1 + C_2x_2$, as the noise-free "output" of the system (5) (Figure 1), the optimal filter performance is determined by the system unstable poles.

Theorem 2.2 (Performance Limits Imposed by Unstable Poles). For the system (5) under Assumption 1, the lowest achievable MS error in the estimation of $\psi = C_1x_1 + C_2x_2$ is

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} E\left\{\tilde{\psi}(t)^T\tilde{\psi}(t)\right\} = 2\sum_{i=1}^{n_2}\beta_i\,, \tag{10}$$

where $\beta_1, \ldots, \beta_{n_2}$ are the system unstable poles.

Proof. Using (7) and (8) we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{t\to\infty} E\left\{\tilde{\psi}(t)^T\tilde{\psi}(t)\right\} &= \operatorname{trace} Q_2C_2^TC_2\\ &= \operatorname{trace} Q_2^{1/2}C_2^TC_2Q_2^{1/2}\\ &= 2\operatorname{trace} A_2 = 2\sum_{i=1}^{n_2}\beta_i\,.\end{aligned}$$

□

Theorem 2.2 shows that the lowest achievable MS estimation error in the problem of estimating the noise-free output of the system (5) is precisely quantified as twice the sum of the system unstable poles.

This theorem is dual to the feedback control result that characterizes the right hand side of (10) as the optimal cost achieved by the *minimum energy control* transferring the system (5) to rest from the initial condition originating from a unit impulse at the input (Seron, Braslavsky, Kokotović & Mayne 1997). The optimal cost for this minimum energy problem is equal to a feedback invariant quantity that involves the well-known Bode integral for the sensitivity function of a unitary feedback loop.

Bode integral formulae have also been obtained for transfer functions that represent sensitivity properties of filtering systems (Goodwin, Mayne & Shim 1995, Seron, Braslavsky & Goodwin 1997). We now connect one of these formulae with the lowest achievable MS error in the estimation of the noise-free output ψ of the system (5) in the single-input single-output (SISO) case.

Theorem 2.3 (Bode Invariant for the Estimation of the Noise-Free Output). Let G be the transfer function from w to $\psi = C_1x_1 + C_2x_2$ in (5), and F be the transfer function of any stable, strictly proper filter mapping the system output y to the estimate $\hat{\psi} = C_1\hat{x}_1 + C_2\hat{x}_2$. Assume that F is such that the estimation error transfer function $(1 - F)G$ is stable and minimum phase. Then

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \log|1 - F(j\omega)|\, d\omega + \lim_{s\to\infty} sF(s) = \lim_{\varepsilon\to 0}\lim_{t\to\infty} E\left\{\tilde{\psi}^2(t)\right\}\,, \tag{11}$$

where $\tilde{\psi}$ is the error in the optimal estimation of ψ in (5), (4).

Proof. Since $(1 - F)G$ is stable and minimum phase, the set of NMP zeros of $1 - F$ is equal to the set of unstable poles of G. The result then follows from (10) and Theorem 8.2.4 of Seron, Braslavsky & Goodwin (1997). □

We observe that $1 - F$ is the transfer function from ψ to $\tilde{\psi}$ (Figure 2). Hence, $|1 - F(j\omega)|$ quantifies the performance of the filter: the smaller this magnitude is over the frequency range where the spectrum of ψ is concentrated, the better the estimate $\hat{\psi}$ "tracks" the signal ψ. In this sense, $1-F$ is a *filtering sensitivity function* dual to the sensitivity function S of the unitary feedback control loop (Seron, Braslavsky & Goodwin 1997, § 7.2).

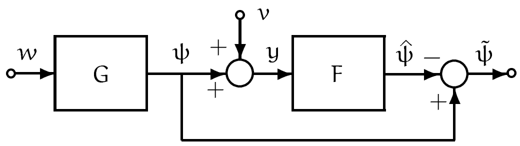

Figure 2: Estimation of the noise-free output

The first term on the left-hand side (LHS) of (11) is a *Bode sensitivity integral* which represents the area subtended by the plot of $|1 - F(j\omega)|$ on a logarithmic scale. A negative value for this integral indicates that the contribution to this area of the frequency ranges where there is filtering sensitivity attenuation ($|1 - F(j\omega)| < 1$) must be greater than that of the ranges where there is sensitivity amplification ($|1 - F(j\omega)| > 1$), and vice versa.

The second term on the LHS of (11) is equal to the slope of the step response of the filter F at time $t = 0$, which is finite and nonzero if and only if F has relative degree one.

Any filter F satisfying the conditions of Theorem 2.3 must satisfy (11), which equals twice the sum of the unstable poles of the system. We may thus view (11) as a filtering *sensitivity invariant* in the sense that the terms on its LHS must be traded-off against each other to add to a quantity fixed by the system and *independent of the filter parameters*. Theorem 2.3 states that this filtering invariant is the MS estimation error achieved by the optimal filter when there is no process noise.

When F is the *optimal* filter estimating $\psi = C_1x_1 + C_2x_2$ in (5), (4), the sensitivity invariant (11) gives the following closed-form expression for the MS estimation error.

Corollary 2.4 (Optimal Mean-Square Error in the Estimation of the Noise-Free Output). The MS error in the estimation of $\psi = C_1x_1 + C_2x_2$ in (5), (4) is

$$\lim_{t\to\infty} E\left\{\tilde{\psi}^2(t)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \log\left[1 + \varepsilon^2|G(j\omega)|^2\right]\,d\omega + 2\sum_{i=1}^{n_2} \beta_i\,. \tag{12}$$

Proof. From (2), the transfer function of the optimal filter for the estimation of ψ in (5), (4) is $F(s) = C(sI - A + QC^TC)^{-1}QC^T$. Hence,

$$\lim_{s\to\infty} sF(s) = CQC^T = \lim_{t\to\infty} E\left\{\tilde{\psi}^2(t)\right\} \tag{13}$$

and

$$\begin{aligned} |1 - F(j\omega)|^2 &= |1 + C(j\omega I - A)^{-1}QC^T|^{-2} \\ &= (1 + \varepsilon^2|G(j\omega)|^2)^{-1}\,, \end{aligned} \tag{14}$$

where (14) follows from the application of the Return Difference Identity $1 + \varepsilon^2|G(j\omega)|^2 = \left|1 + C(j\omega I - A)^{-1}QC^T\right|^2$ (Anderson & Moore 1990, § 5.2). Using (10), (13) and (14) in (11) yields (12). □

For $\varepsilon = 1$, the expression (12) was obtained in its control dual form by Anderson & Mingori (1985), who extended to unstable G a formula originally derived in Wiener filtering by Yovits & Jackson (1955).

Since, from (14),

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \log\left[1 + \varepsilon^2|G(j\omega)|^2\right]\,d\omega = -\frac{1}{\pi}\int_{-\infty}^{\infty} \log|1 - F(j\omega)|\,d\omega\,,$$

the formula (12) shows that, for each fixed $\varepsilon > 0$, the Bode sensitivity integral represents the degradation in performance (as measured by the MS estimation error) with respect to its best achievable given by (10).

3 Performance Limits Under Small Measurement Noise

3.1 Limiting Properties

We now study the properties of the optimal filter when the intensity of the measurement noise is arbitrarily small, under the following assumption.

Assumption 2.

(i) The system (1) is stabilizable and detectable.

(ii) The system (1) has relative-degree one: rank $CB = m$.

(iii) All the zeros of $C(sI - A)^{-1}B$ are nonminimum phase.

(iv) The intensities of the noises w and ν are

$$\Sigma_w = I\,, \quad \text{and} \quad \Sigma_\nu = \varepsilon^2 I\,, \tag{15}$$

where $\varepsilon > 0$ is small. ◦

Assumptions (ii) and (iii), which represent the case with the maximum number of NMP zeros, are made only for simplicity of exposition and may be removed.

Let M_0 be a basis of the left null-space of the input matrix B (that is, $M_0 B = 0$). Then, because rank $CB = m$, the change of coordinates $\xi = Cx$ and $z = M_0 x$ is a similarity transformation which brings (1) to the form

$$\begin{aligned} \begin{bmatrix} \dot{\xi} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} A_1 & C_0 \\ B_0 & A_0 \end{bmatrix} \begin{bmatrix} \xi \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w\,, \\ y &= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \xi \\ z \end{bmatrix} + \nu\,, \end{aligned} \tag{16}$$

This form is of interest because it displays the zeros of $C(sI - A)^{-1}B$ (the transfer function from w to y in (1)) as the eigenvalues of A_0. When $\nu = 0$ (so $y = \xi$), (16) can be seen to consist of two systems in feedback interconnection: the zero dynamics subsystem described by $\dot{z} = A_0 z + B_0 y$, with input y and output $C_0 z$, and the output subsystem $\dot{y} = A_1 y + C_0 z + B_1 w$, with input $C_0 z + B_1 w$ and output y (Figure 3). The triple (A_0, B_0, C_0) is controllable and observable because (16) is stabilizable and detectable and A_0 is antistable by (iii), Assumption 2.

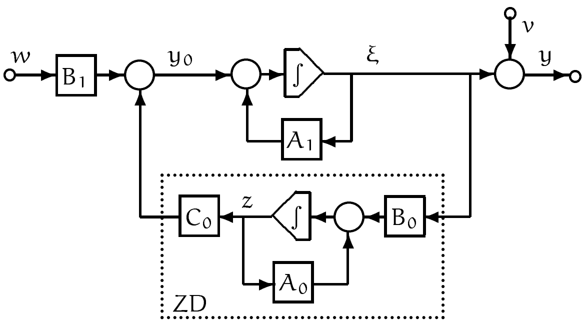

Figure 3: Output/zero-dynamics subsystem interaction in the system (16)

Lemma 3.1 (Error Variance under No Measurement Noise). Under Assumption 2, the variance of the estimation error achieved by the optimal filter for (16) satisfies

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} \mathrm{E}\left\{\begin{bmatrix}\tilde{\xi}(t)\\ \tilde{z}(t)\end{bmatrix}\begin{bmatrix}\tilde{\xi}(t)\\ \tilde{z}(t)\end{bmatrix}^{\mathsf{T}}\right\} = \begin{bmatrix}0 & 0\\ 0 & Q_0\end{bmatrix}, \tag{17}$$

and the lowest achievable MS estimation error is

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} \mathrm{E}\left\{\begin{bmatrix}\tilde{\xi}(t)\\ \tilde{z}(t)\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\tilde{\xi}(t)\\ \tilde{z}(t)\end{bmatrix}\right\} = \operatorname{trace} Q_0\,, \tag{18}$$

where Q_0 is the positive definite solution of

$$A_0Q_0 + Q_0A_0^{\mathsf{T}} = Q_0C_0^{\mathsf{T}}(B_1B_1^{\mathsf{T}})^{-1}C_0Q_0\,. \tag{19}$$

Proof. Using (16) and (15) in (3) we obtain the ARE

$$\begin{bmatrix}A_1 & C_0\\ B_0 & A_0\end{bmatrix}Q + Q\begin{bmatrix}A_1 & C_0\\ B_0 & A_0\end{bmatrix}^{\mathsf{T}} + \begin{bmatrix}B_1B_1^{\mathsf{T}} & 0\\ 0 & 0\end{bmatrix} - \frac{1}{\varepsilon^2}Q\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}Q = 0\,. \tag{20}$$

We seek its solution $Q = Q^{\mathsf{T}}$ in the form of the power series

$$Q = \begin{bmatrix}\varepsilon Q_1 + \varepsilon^2\check{Q}_1 + O(\varepsilon^3) & *\\ \varepsilon Q_2 + \varepsilon^2\check{Q}_2 + O(\varepsilon^3) & Q_0 + O(\varepsilon)\end{bmatrix}. \tag{21}$$

Its leading entries Q_0, Q_1, Q_2 are obtained explicitly from the substitution of (21) into (20):

$$B_1B_1^{\mathsf{T}} + O(\varepsilon) = Q_1^2\,, \tag{22}$$
$$C_0Q_0 + O(\varepsilon) = Q_1Q_2^{\mathsf{T}}\,, \tag{23}$$
$$A_0Q_0 + Q_0A_0^{\mathsf{T}} + O(\varepsilon) = Q_2Q_2^{\mathsf{T}}\,. \tag{24}$$

When $\varepsilon = 0$, (22) gives $Q_1 = (B_1B_1^{\mathsf{T}})^{1/2}$ and (23) gives $Q_2 = Q_0C_0^{\mathsf{T}}(B_1B_1^{\mathsf{T}})^{-1/2}$. The substitution of Q_1 and Q_2 in (24) shows that Q_0 is a solution of (19); since $(A_0, B_1^{-1}C_0)$ is observable, this solution is unique and positive definite. The expressions for $\check{Q}_1$ and $\check{Q}_2$ are found similarly by equating $O(\varepsilon)$ terms, not shown in (22), (23), (24). We conclude that Q is given by

$$Q = \begin{bmatrix}\varepsilon(B_1B_1^{\mathsf{T}})^{1/2} + \varepsilon^2\check{Q}_1 + O(\varepsilon^3) & *\\ \varepsilon Q_0C_0^{\mathsf{T}}(B_1B_1^{\mathsf{T}})^{-1/2} + \varepsilon^2\check{Q}_2 + O(\varepsilon^3) & Q_0 + O(\varepsilon)\end{bmatrix}. \tag{25}$$

Identities (17) and (18) then follow from (25) by letting $\varepsilon \to 0$. □

We observe that if A_0 were Hurwitz then $Q_0 = 0$ would be the nonnegative definite solution of (19), and the error variance achieved by the optimal filter as $\varepsilon \to 0$ would be zero. In our case A_0 is antistable, and hence the error variance is positive definite.

In the limit when the measurement noise vanishes, $\nu = 0$, the state $\xi = y$ is exactly measured. Therefore, in the output system $\dot{\xi} - A_1\xi = C_0z + B_1w$ we can treat $y_0 = \dot{\xi} - A_1\xi$

as the noisy measurement $y_0 = C_0 z + B_1 w$ of the state of the unstable zero dynamics subsystem (Figure 3). This is equivalent to the filtering problem for the system

$$\begin{aligned} \dot{z} &= A_0 z\,, \\ y_0 &= C_0 z + B_1 w\,. \end{aligned} \tag{26}$$

The variance of the estimation error corresponding to this filtering problem is Q_0, the positive definite solution of (19). We see that the limiting estimation problem (26) is characterized by the output pair (A_0, C_0) of the zero-dynamics subsystem with the driving term $B_0\xi$ treated as a known input.

Hence, only the unstable zero dynamics of the system contribute to the MS estimation error (trace Q_0) when the measurement noise vanishes. In other words, the problem reduces to that of estimating the state z of the zero-dynamics subsystem given the noisy measurement y_0 obtained from the output subsystem.

This result is dual to the following limiting property for the *cheap control problem* for (16): the lowest achievable cost for the cheap control is the optimal cost achieved by the minimum energy problem for the input pair (A_0, B_0) of the zero-dynamics subsystem (Seron, Braslavsky, Kokotović & Mayne 1997).

3.2 Optimal Filter

By dualizing the cheap control results, Kwakernaak & Sivan (1972, Theorem 4.12) showed that, in the limit as $\varepsilon \to 0$, the eigenvalues of the optimal filter which do not converge to infinity converge to the system minimum phase zeros and to the mirror image of the NMP zeros. We will recover this result using singular perturbation analysis.

Note first that, since $\Sigma_v = \varepsilon^2 I$, we can express y in (16) as

$$y = \xi + \varepsilon v_1\,, \tag{27}$$

where v_1 is a zero-mean Gaussian white noise with intensity $E\{v_1(t)v_1{}^T(\tau)\} = I\,\delta(t-\tau)$. With Q from (25), the optimal filter (2) is a singularly perturbed system

$$\begin{aligned} \begin{bmatrix} \dot{\hat{\xi}} \\ \dot{\hat{z}} \end{bmatrix} &= \begin{bmatrix} A_1 & C_0 \\ B_0 & A_0 \end{bmatrix} \begin{bmatrix} \hat{\xi} \\ \hat{z} \end{bmatrix} + \frac{1}{\varepsilon} \begin{bmatrix} (B_1 B_1^T)^{1/2} + \varepsilon \check{Q}_1 + O(\varepsilon^2) \\ Q_0 C_0^T (B_1 B_1^T)^{-1/2} + \varepsilon \check{Q}_2 + O(\varepsilon^2) \end{bmatrix} \left(\xi + \varepsilon v_1 - \hat{\xi}\right), \\ \hat{\xi}(0) &= \bar{\xi}(0)\,, \qquad \hat{z}(0) = \bar{z}(0)\,, \end{aligned} \tag{28}$$

but, because the matrix

$$\begin{bmatrix} (B_1 B_1^T)^{1/2} \\ Q_0 C_0^T (B_1 B_1^T)^{-1/2} \end{bmatrix}$$

is singular, it is not in the standard form (Kokotović, Khalil & O'Reilly 1986). Noting that $B_1 B_1^T$ is nonsingular, we bring (28) into the standard form via the change of variable $\eta = \hat{z} - Q_0 C_0^T (B_1 B_1^T)^{-1} \hat{\xi}$ with $\eta(0) = \bar{\eta}(0) \triangleq \bar{z}(0) - Q_0 C_0^T (B_1 B_1^T)^{-1} \bar{\xi}(0)$. This yields

$$\begin{aligned} \begin{bmatrix} \varepsilon \dot{\hat{\xi}} \\ \dot{\eta} \end{bmatrix} &= \begin{bmatrix} -(B_1 B_1^T)^{1/2} & 0 \\ B_\eta - K_\eta & -Q_0 A_0^T Q_0^{-1} \end{bmatrix} \begin{bmatrix} \hat{\xi} \\ \eta \end{bmatrix} + \begin{bmatrix} (B_1 B_1^T)^{1/2} \\ K_\eta \end{bmatrix} \left(\xi + \varepsilon v_1\right) + O(\varepsilon)\,, \\ \hat{\xi}(0) &= \bar{\xi}(0)\,, \qquad \eta(0) = \bar{\eta}(0)\,, \end{aligned} \tag{29}$$

where $B_\eta = B_0 - Q_0[C_0^T(B_1B_1^T)^{-1}A_1 + A_0^TC_0^T(B_1B_1^T)^{-1}]$, $K_\eta = \check{Q}_2 - Q_0C_0^T(B_1B_1^T)^{-1}\check{Q}_1$, and where we have used $A_0 - Q_0C_0^T(B_1B_1^T)^{-1}C_0 = -Q_0A_0^TQ_0^{-1}$, which follows from (19). The matrices $-(B_1B_1^T)^{1/2}$ and $-Q_0A_0^TQ_0^{-1}$ are Hurwitz, and hence (29) satisfies the assumptions of Theorem 1 in Khalil & Gajic (1984), which shows that the solutions of the system

$$\begin{bmatrix} \varepsilon\dot{\hat{\xi}}_0 \\ \dot{\eta}_0 \end{bmatrix} = \begin{bmatrix} -(B_1B_1^T)^{1/2} & 0 \\ B_\eta - K_\eta & -Q_0A_0^TQ_0^{-1} \end{bmatrix} \begin{bmatrix} \hat{\xi}_0 \\ \eta_0 \end{bmatrix} + \begin{bmatrix} (B_1B_1^T)^{1/2} \\ K_\eta \end{bmatrix} \xi\,, \qquad \begin{aligned} \hat{\xi}_0(0) &= \bar{\xi}(0)\,, \\ \eta_0(0) &= \bar{\eta}(0)\,, \end{aligned} \tag{30}$$

approximate the exact solutions of (29) in the sense that the variance of $\hat{\xi} - \hat{\xi}_0$ is $O(\varepsilon)$ and the variance of $\eta - \eta_0$ is $O(\varepsilon^2)$. In the noise-free system (30), the stability of the fast subsystem matrix $-(B_1B_1^T)^{1/2}$ guarantees that, during a fast ("boundary layer") transient, $\hat{\xi}_0$ converges towards its quasi-steady state $\hat{\xi}_0 = \xi$. Substitution of $\hat{\xi}_0 = \xi$ in the η_0-subsystem of (30) shows that the quasi-steady state of the slow variable η_0 is governed by $\dot{\eta}_0 = -Q_0A_0^TQ_0^{-1}\eta_0 + B_\eta\xi$. Thus, as the noise $\varepsilon\nu_1$ vanishes, the filter (28) reduces to

$$\begin{aligned} \hat{\xi} &= \xi\,, \\ \dot{\eta} &= -Q_0A_0^TQ_0^{-1}\eta + B_\eta\xi\,, \\ \hat{z} &= \eta + Q_0C_0^T(B_1B_1^T)^{-1}\xi\,. \end{aligned} \tag{31}$$

The eigenvalues of $-Q_0A_0^TQ_0^{-1}$ are the *mirror image* of the eigenvalues of A_0, which are the NMP zeros of (16). It follows that the corresponding $n - m$ eigenvalues of the optimal filter converge, as $\varepsilon \to 0$, to the mirror image of the NMP zeros of the system and the remaining m eigenvalues tend to infinity.

The optimal filter for (16) with zero measurement noise was originally derived by Bryson & Johansen (1965) without a recourse to the above limiting argument. By differentiating the output y they obtain a new measurement $y_d = \dot{y}$ and then construct the optimal filter for the system

$$\begin{aligned} \begin{bmatrix} \dot{\xi} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} A_1 & C_0 \\ B_0 & A_0 \end{bmatrix} \begin{bmatrix} \xi \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w\,, \\ y_d &= \begin{bmatrix} A_1 & C_0 \end{bmatrix} \begin{bmatrix} \xi \\ z \end{bmatrix} + \nu_d\,, \end{aligned} \tag{32}$$

where ν_d is zero-mean Gaussian white noise of intensity $B_1B_1^T$. Noting that w and ν_d in (32) are correlated with cross-covariance B_1^T, one can verify that the error variance achieved by filtering (32) is the right hand side of (17).

The optimal filter for (32), with state $(\hat{\xi}, \hat{z})$, is driven by $y_d = \dot{y}$. Differentiation of the output can be avoided in the following way (Bryson & Johansen 1965). First, since ξ is measured, take $\hat{\xi} = \xi$. Then the change of coordinates $\eta = \hat{z} - Q_0C_0^T(B_1B_1^T)^{-1}\xi$ again yields the filter (31).

Hence, the filters obtained from the two approaches coincide. Note, however, that when the filter (31) is obtained from (32) its states are continuous functions of time for $t \geq 0$ (Moylan 1974). On the other hand, when (31) is obtained from the small-noise optimal filter (28), singular perturbation theory shows that the state $\hat{\xi}$ has a boundary-layer jump at $t = 0$ to accommodate different initial conditions of $\hat{\xi}$ and ξ. As suggested by Moylan (1974), this jump can be avoided if, instead of taking $\hat{\xi}(0)$ equal to $\bar{\xi}(0)$, we select it to be $O(\varepsilon)$-close to $\xi(0)$, which is physically possible since ξ is measured with $O(\varepsilon)$ noise.

3.3 Estimation of a Random-Walk Input

A filtering problem of practical relevance is that of estimating a slowly-varying input disturbance. Such a disturbance may be modeled as integrated white noise (Kwakernaak & Sivan 1972, p. 347), that is "random walk". The system (16) with a random-walk input disturbance ψ is

$$\begin{aligned}\begin{bmatrix}\dot{\xi}\\ \dot{\psi}\\ \dot{z}\end{bmatrix} &= \begin{bmatrix}A_1 & B_1 & C_0\\ 0 & 0 & 0\\ B_0 & 0 & A_0\end{bmatrix}\begin{bmatrix}\xi\\ \psi\\ z\end{bmatrix} + \begin{bmatrix}0\\ I\\ 0\end{bmatrix} w\,,\\ y &= \begin{bmatrix}I & 0 & 0\end{bmatrix}\begin{bmatrix}\xi\\ \psi\\ z\end{bmatrix} + v\,,\end{aligned} \tag{33}$$

where w and v are zero-mean Gaussian white noises of intensities given by (15). Defining a new variable $\zeta = B_1\psi + C_0 z$, we rewrite (33) as

$$\begin{aligned}\begin{bmatrix}\dot{\xi}\\ \dot{\zeta}\\ \dot{z}\end{bmatrix} &= \begin{bmatrix}A_1 & I & 0\\ C_0B_0 & 0 & C_0A_0\\ B_0 & 0 & A_0\end{bmatrix}\begin{bmatrix}\xi\\ \zeta\\ z\end{bmatrix} + \begin{bmatrix}0\\ B_1\\ 0\end{bmatrix} w\,,\\ y &= \begin{bmatrix}I & 0 & 0\end{bmatrix}\begin{bmatrix}\xi\\ \psi\\ z\end{bmatrix} + v\,,\end{aligned} \tag{34}$$

From the optimal filter for (34), (15), an estimate for the disturbance ψ is computed as $\hat{\psi} = B_1^{-1}(\hat{\zeta} - C_0\hat{z})$.

Theorem 3.2 (Performance Limits Imposed by NMP Zeros). Assume that the system (34) is stabilizable and detectable, with B_1 invertible and A_0 antistable. Then the filtering problem for (34), (15) has the following limiting properties:

(i) The variance of the optimal estimation error satisfies

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} E\left\{\begin{bmatrix}\tilde{\xi}(t)\\ \tilde{\zeta}(t)\\ \tilde{z}(t)\end{bmatrix}\begin{bmatrix}\tilde{\xi}(t)\\ \tilde{\zeta}(t)\\ \tilde{z}(t)\end{bmatrix}^T\right\} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & Q_0\end{bmatrix}\,, \tag{35}$$

where Q_0 is the positive definite solution of

$$A_0Q_0 + Q_0A_0^T = Q_0A_0^TC_0^T(B_1B_1^T)^{-1}C_0A_0Q_0\,. \tag{36}$$

(ii) The lowest achievable MS error in the estimation of the input disturbance ψ is

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} E\left\{\tilde{\psi}(t)^T\tilde{\psi}(t)\right\} = 2\sum_{i=1}^{n-m}\frac{1}{\alpha_i}\,, \tag{37}$$

where $\alpha_1, \ldots, \alpha_{n-m}$ are the NMP zeros of the system.

Proof. The ARE corresponding to the optimal filter for (34) is

$$\begin{bmatrix} A_1 & I & 0 \\ C_0B_0 & 0 & C_0A_0 \\ B_0 & 0 & A_0 \end{bmatrix} Q + Q \begin{bmatrix} A_1 & I & 0 \\ C_0B_0 & 0 & C_0A_0 \\ B_0 & 0 & A_0 \end{bmatrix}^{\mathsf T} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & B_1B_1^{\mathsf T} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{\varepsilon^2} Q \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q\,,$$

whose solution

$$Q = \begin{bmatrix} \varepsilon^{3/2}\sqrt{2}(B_1B_1^{\mathsf T})^{1/4} + O(\varepsilon^2) & * & * \\ \varepsilon\,(B_1B_1^{\mathsf T})^{1/2} + \varepsilon^{3/2}\check{Q}_2 + O(\varepsilon^2) & \sqrt{2\varepsilon}\,(B_1B_1^{\mathsf T})^{3/4} + O(\varepsilon) & * \\ \varepsilon\,K(B_1B_1^{\mathsf T})^{1/2} + \varepsilon^{3/2}\check{Q}_3 + O(\varepsilon^2) & \sqrt{2\varepsilon}\,K(B_1B_1^{\mathsf T})^{3/4} + O(\varepsilon) & Q_0 + O(\sqrt{\varepsilon}) \end{bmatrix}, \tag{38}$$

where $K = Q_0A_0^{\mathsf T}C_0^{\mathsf T}(B_1B_1^{\mathsf T})^{-1}$ and Q_0 is the positive definite solution of (36), is found following the same steps as in the proof of Lemma 3.1. Then (35) follows from (38) by taking the limit as $\varepsilon \to 0$. To derive (37) we rewrite (36) as

$$Q_0^{1/2}A_0^{\mathsf T^{-1}}Q_0^{-1/2} + Q_0^{-1/2}A_0^{-1}Q_0^{1/2} = Q_0^{1/2}C_0^{\mathsf T}(B_1B_1^{\mathsf T})^{-1}C_0Q_0^{1/2}\,. \tag{39}$$

Then

$$\begin{aligned} \lim_{\varepsilon\to 0}\lim_{t\to\infty} \mathrm{E}\left\{\tilde{\psi}(t)^{\mathsf T}\tilde{\psi}(t)\right\} &= \operatorname{trace} Q_0C_0^{\mathsf T}B_1^{\mathsf T^{-1}}B_1^{-1}C_0 \\ &= \operatorname{trace} Q_0^{1/2}C_0^{\mathsf T}B_1^{\mathsf T^{-1}}B_1^{-1}C_0Q_0^{1/2} \\ &= 2\operatorname{trace} A_0^{-1} = 2\sum_{i=1}^{n-m}\frac{1}{\alpha_i}\,. \end{aligned}$$

□

Theorem 3.2 shows that there exists a nonzero MS error in the estimation of a random-walk input to a NMP system. This error is an explicit function of the NMP zeros, independent of the system realization. This result is dual to the Qiu-Davison formula (Qiu & Davison 1993, Theorem 3), which states that the lowest achievable cost for the cheap control problem of regulating the output y of (16) to a constant setpoint r is $\frac{1}{2}r^{\mathsf T}Hr$, where H is an Hermitian matrix whose trace equals (37).

We next derive the optimal filter estimating the disturbance $\psi = B_1^{-1}(\zeta - C_0z)$ in (34). We will see that, since the transfer function from w to y in (34) has relative-degree two, the optimal filter becomes *improper* in the limit when $\varepsilon \to 0$ (Francis 1979).

For $\varepsilon > 0$, the optimal filter of the form (2) for the system (34) is

$$\begin{aligned} \begin{bmatrix} \dot{\hat{\xi}} \\ \dot{\hat{\zeta}} \\ \dot{\hat{z}} \end{bmatrix} &= \begin{bmatrix} A_1 & I & 0 \\ C_0B_0 & 0 & C_0A_0 \\ B_0 & 0 & A_0 \end{bmatrix} \begin{bmatrix} \hat{\xi} \\ \hat{\zeta} \\ \hat{z} \end{bmatrix} + \frac{1}{\varepsilon} \begin{bmatrix} \sqrt{2\varepsilon}\,(B_1B_1^{\mathsf T})^{1/4} + O(\varepsilon) \\ (B_1B_1^{\mathsf T})^{1/2} + \check{Q}_2\sqrt{\varepsilon} + O(\varepsilon) \\ K(B_1B_1^{\mathsf T})^{1/2} + \check{Q}_3\sqrt{\varepsilon} + O(\varepsilon) \end{bmatrix} \left(\xi + \varepsilon\nu_1 - \hat{\xi}\right), \\ \hat{\xi}(0) &= \xi(0) + O(\varepsilon)\,, \qquad \hat{\zeta}(0) = \bar{\zeta}(0)\,, \qquad \hat{z}(0) = \bar{z}(0)\,, \end{aligned} \tag{40}$$

obtained using (27) and (38), and selecting the initial condition $\hat{\xi}(0)$ to be $O(\varepsilon)$-close to $\xi(0)$, as pointed out in Section 3.2. To study (40) in the limit as $\varepsilon \to 0$, it is convenient to introduce the error variables $\sigma = (\xi - \hat{\xi})/\sqrt{\varepsilon} = \tilde{\xi}/\sqrt{\varepsilon}$ and $\eta = (\tilde{z} - K\tilde{\zeta})$. We observe that σ is well-defined because $\sigma(0)$ is $O(\sqrt{\varepsilon})$. Using (34) and (40), and noting that $\tilde{z}(0) = z(0) - \hat{z}(0)$, we find that the dynamics of σ, $\tilde{\zeta}$ and η are governed by

$$\begin{bmatrix} \sqrt{\varepsilon}\dot{\sigma} \\ \sqrt{\varepsilon}\dot{\tilde{\zeta}} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} -\sqrt{2}\left(B_1 B_1^T\right)^{1/4} & I & 0 \\ -\left(B_1 B_1^T\right)^{1/2} & 0 & 0 \\ -\check{Q}_3 + K\check{Q}_2 & (I - KC_0)A_0 K & (I - KC_0)A_0 \end{bmatrix} \begin{bmatrix} \sigma \\ \tilde{\zeta} \\ \eta \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ KB_1 \end{bmatrix} w + O(\sqrt{\varepsilon}),$$
$$\sigma(0) = O(\sqrt{\varepsilon})\,, \qquad \tilde{\zeta}(0) = \zeta(0) - \hat{\zeta}(0)\,, \qquad \eta(0) = \tilde{z}(0) - K\tilde{\zeta}(0)\,, \tag{41}$$

which is a singularly perturbed system — in standard form — driven by white noise. Denote by $(\sigma_0, \tilde{\zeta}_0, \eta_0)$ the solutions of the system obtained by setting $\varepsilon = 0$ in the right-hand side of (41) and in the initial conditions. Then, because the matrices

$$\begin{bmatrix} -\sqrt{2}\left(B_1 B_1^T\right)^{1/4} & I \\ -\left(B_1 B_1^T\right)^{1/2} & 0 \end{bmatrix}$$

of the fast $(\sigma, \tilde{\zeta})$-subsystem, and $(I - KC_0)A_0 = A_0 - Q_0 A_0^T Q_0^{-1}$ of the slow η-subsystem of (41) are Hurwitz, Theorem 1 in Khalil & Gajic (1984) guarantees that $(\sigma_0, \tilde{\zeta}_0, \eta_0)$ approximate the exact solutions of (41) in the sense that the variance of $(\sigma - \sigma_0, \tilde{\zeta} - \tilde{\zeta}_0)$ is $O(\sqrt{\varepsilon})$ and the variance of $\eta - \eta_0$ is $O(\varepsilon)$. A fast-slow analysis as in Section 3.2 yields the optimal filter

$$\begin{aligned} \hat{\xi} &= \xi\,, \\ \hat{\zeta} &= \dot{\xi} - A_1 \xi\,, \\ \dot{\eta} &= \left(I - KC_0\right)A_0 \eta + \left(I - KC_0\right)B_0 \xi + \left(I - KC_0\right)A_0 K\left(\dot{\xi} - A_1 \xi\right)\,, \\ \hat{z} &= \eta + K\hat{\zeta}\,, \\ \hat{\psi} &= B_1^{-1}\left[\left(I - C_0 K\right)\left(\dot{\xi} - A_1 \xi\right) - C_0 \eta\right]\,, \end{aligned} \tag{42}$$

which estimates the random-walk input disturbance $\psi = B_1^{-1}(\zeta - C_0 z)$ in the limit as $\varepsilon \to 0$, that is, when there is no measurement noise. Again, the eigenvalues of the filter (42) are the mirror image of the NMP zeros of the system. Here, however, the filter is driven by both the system output $y = \xi$ and its derivative $\dot{y} = \dot{\xi}$. Moreover, it has a feedthrough from $\dot{y}$ to the estimates, and, hence, its relative degree is minus one.

3.4 A Bode Integral for the Estimation of a Random-Walk Input

We now present a SISO result, complementary to Theorem 2.3, which connects Theorem 3.2 with a Bode integral.

Theorem 3.3 (Bode Invariant for the Estimation of a Random-Walk Input). Let G be the transfer function from ψ to y in (33), and F be the transfer function of any stable filter mapping the system output y to the estimate $\hat{\psi}$. Assume that F is such that

(i) the estimation error transfer function $1 - FG$ is stable and proper,

(ii) the NMP zeros of FG are equal to the NMP zeros of G.

Then

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \log|F(j\omega)G(j\omega)| \frac{d\omega}{\omega^2} - \lim_{s\to 0}\frac{d(FG)}{ds} = \lim_{\varepsilon\to 0}\lim_{t\to\infty} E\left\{\tilde{\psi}^2(t)\right\}, \tag{43}$$

where $\tilde{\psi}(t)$ is the error in the optimal estimation of the random-walk input ψ in (33), (15).

Proof. The result follows from (37) and Theorem 8.2.5 of Seron, Braslavsky & Goodwin (1997). □

Note that FG is the transfer function from ψ to $\hat{\psi}$ (Figure 4), and thus $|F(j\omega)G(j\omega)|$ quantifies the performance of the filter. Indeed, the closer $|F(j\omega)G(j\omega)|$ is to one over the frequency range where the spectrum of ψ is concentrated, the better the estimate $\hat{\psi}$ "tracks" the disturbance ψ. In this sense, FG is a *filtering complementary sensitivity function* dual to the complementary sensitivity function T of the unitary feedback control loop (Seron, Braslavsky & Goodwin 1997, § 7.2).

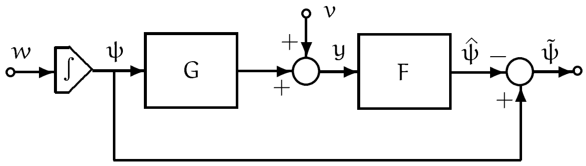

Figure 4: Estimation of a random-walk input disturbance

The integral in (43) can be used to measure the filter performance since its value gives the difference between areas of complementary sensitivity amplification and attenuation. The second term on the LHS of (43) is equal to the steady-state error $\tilde{\psi}$ that would result if ψ were a ramp signal. When F is *any* filter satisfying the conditions of Theorem 3.3, then (43) is a fixed quantity equal to twice the sum of the reciprocal of the NMP zeros of the system. Thus, similar to (11), the LHS of (43) is a filtering *complementary sensitivity invariant* in the sense that its terms must be traded-off against each other to add to twice the sum of the reciprocal of the NMP zeros of the system. Theorem 3.3 then states that this filtering invariant is equal to the MS estimation error achieved by the optimal filter for this problem under no measurement noise.

When F is the optimal filter the filtering complementary sensitivity FG is all-pass.

Lemma 3.4. Let G be the transfer function from ψ to y in (33), and let F be the optimal filter transfer function from $y = \xi$ to the estimate $\hat{\psi}$ in (42). Then $|F(j\omega)G(j\omega)| = 1$ for all ω. ∘

Hence, the absence of measurement noise allows the optimal filter to "invert" the system G (with its NMP zeros mirrored with respect to the imaginary axis) by using derivatives of the system output. The all-pass property induced by optimality implies that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \log|F(j\omega)G(j\omega)| \frac{d\omega}{\omega^2} = 0,$$

and therefore

$$-\lim_{s\to 0}\frac{d(FG)}{ds} = 2\sum_{i=1}^{n-m}\frac{1}{\alpha_i}.$$

The following fact will be required to prove Lemma 3.4.

Fact 3.5. Given the observable pair (A_0, C_0), $C_0 \in \mathbb{R}^{1\times k}$, $A_0 \in \mathbb{R}^{k\times k}$, where A_0 is anti-stable, and a scalar $b > 0$, let Q_0 be the positive definite solution of

$$A_0 Q_0 + Q_0 A_0^T = Q_0 A_0^T C_0^T b C_0 A_0 Q_0 \tag{44}$$

and $K = bQ_0 A_0^T C_0^T$. Then

$$C_0 K = \begin{cases} 0 & \text{if } k \text{ is even,} \\ 2 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Introduce $L(s) = C_0 A_0 (sI - A_0)^{-1} K$ and

$$S(s) = 1 - \frac{L(s)}{1 + L(s)} = 1 - C_0 A_0 \big[sI - (A_0 - KC_0 A_0)\big]^{-1} K\,.$$

Since, from (44), $A_0 - KC_0A_0 = -Q_0 A_0^T Q_0^{-1}$, the poles of S are the mirror image of the eigenvalues of A_0, while its zeros are the poles of its inverse system, that is, the eigenvalues of $(A_0 - KC_0A_0) + KC_0A_0 = A_0$. Hence $|S(j\omega)| = 1 = |S^{-1}(j\omega)| = |L(j\omega) + 1|$, which implies that the Nyquist plot of L is the unit circle centered at -1. Noting that $L(0) = -C_0K$ and $L(\infty) = 0$, the result follows from the Nyquist stability criterion and the fact that S is stable. □

Proof of Lemma 3.4. Noting that, in the SISO case, $B_1 > 0$ is scalar, the use of Fact 3.5 with $b = (B_1 B_1^T)^{-1}$ implies that $1 - C_0K$ in (42) is 1 if the number of NMP zeros of G is even, and -1 if it is odd. Hence, F is an improper transfer function of relative degree -1. Since G has relative-degree 1, FG has relative degree 0. From (33) and (42), and noting that $\dot{\xi} - A_1\xi = C_0 z + B_1\psi$, a realization for FG is

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_1 & 0 & C_0 \\ (I - KC_0)B_0 & (I - KC_0)A_0 & (I - KC_0)A_0KC_0 \\ B_0 & 0 & A_0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ (I - KC_0)A_0KB_1 \\ 0 \end{bmatrix} \psi\,,$$
$$\hat{\psi} = \begin{bmatrix} 0 & -B_1^{-1}C_0 & B_1^{-1}C_0(I - KC_0) \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ z \end{bmatrix} + \big(1 - B_1^{-1}C_0KB_1\big)\psi\,. \tag{45}$$

Let $\{A_\psi, B_\psi, C_\psi, D_\psi\}$ denote the state-space matrices in (45). Interchange of rows and columns shows that the poles of (45) are those of G, plus the eigenvalues of $(I - KC_0)A_0 = -Q_0A_0^TQ_0^{-1}$, which are the mirror image of the NMP zeros of G. The zeros of (45) are the roots of its inverse system characteristic polynomial $\left|sI - A_\psi + B_\psi D_\psi^{-1} C_\psi\right|$, where $|\cdot|$ denotes determinant. For $1 - C_0K = \pm 1$, we have that $D_\psi = \pm 1$, and

$$\left|sI - A_\psi + B_\psi D_\psi^{-1} C_\psi\right| = \begin{vmatrix} sI - A_1 & \mp C_0 & 0 \\ -(I - KC_0)B_0 & sI - (I - KC_0)A_0(I \pm KC_0) & 0 \\ -B_0 & 0 & sI - A_0 \end{vmatrix}\,. \tag{46}$$

The determinant (46) is the product of $|sI - A_0|$, which yields the zeros of G, and

$$\begin{vmatrix} sI - A_1 & \mp C_0 \\ -\big(I - KC_0\big)B_0 & sI - \big(I - KC_0\big)A_0\big(I + KC_0\big) \end{vmatrix} =$$

$$\begin{vmatrix} I & 0 \\ 0 & I - KC_0 \end{vmatrix} \begin{vmatrix} sI - A_1 & -C_0 \\ -B_0 & sI - A_0 \end{vmatrix} \begin{vmatrix} I & 0 \\ 0 & I \pm KC_0 \end{vmatrix}, \quad (47)$$

which yields the zeros of F — at the *poles* of G — because $(I - KC_0)(I \pm KC_0) = I$. □

4 Conclusions

We have shown that, when the intensity of the process noise tends to zero, the optimal MS estimation error is determined by the system unstable poles. In particular, the lowest achievable MS error in the estimation of the noise-free output of the system equals twice the sum of its unstable poles, and is further characterized as a filtering Bode sensitivity invariant that expresses tradeoffs in design for a general class of filters. For the optimal filter corresponding to process and measurement noises of unit-intensity variance, this Bode invariant gives the MS error in a closed-form formula known from Wiener filtering.

When the intensity of the measurement noise tends to zero, the optimal MS estimation error is determined by the system NMP zeros. In the case of estimation of a random-walk input, the lowest achievable MS error equals twice the sum of the reciprocal of its NMP zeros. We have characterized this error as a filtering Bode complementary sensitivity invariant that holds for the class of filters that solve this estimation problem.

For the greatest clarity of the presentation, the results obtained in Section 3 are restricted to multivariable systems with all the zeros in the open right half plane and having relative-degree one. The extension to systems having zeros also in the open left half plane is immediate by using a modal decomposition as in Section 2. The extension to systems with relative degree greater than one will require a more intricate derivation as in Saberi & Sannuti (1987).

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