Feedback Limitations in Nonlinear Systems: From Bode Integrals to Cheap Control *

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Abstract

In this paper we link the Bode integral characterization of performance limitations in linear control systems with the limiting cost of a particular linear cheap control. This link allows us to establish analogous feedback limitations in nonlinear systems. We show how unstable zero dynamics impose unavoidable obstructions to the closed-loop performance.

Keywords: Feedback limitations, Bode Integrals, Nonlinear cheap control, Zero Dynamics, Nonminimum phase systems.

1 Introduction

The frequency-domain analysis of limitations to the performance of linear feedback systems was initiated by Bode (1945). Since then, many authors have further clarified these limitations. In particular, Francis & Zames (1984) and Freudenberg & Looze (1985) made explicit the role of nonminimum phase (NMP) zeros and open-loop unstable poles in limiting the closed-loop performance. A time-domain counterpart of these results was given by Middleton (1991), who analyzed performance limitations in the step response. All these works rely on the analyticity of closed-loop transfer functions, typically the sensitivity function S and the complementary sensitivity function T, to obtain integral formulas that express feedback limitations. Sensitivity functions have also been used in input-output operator frameworks to study limitations in nonlinear systems (Shamma 1991, Seron & Goodwin 1996). These and other recent contributions to the frequency-domain/input-output operator analysis of feedback limitations are reviewed in Seron, Braslavsky & Goodwin (1997).

With the development of the linear quadratic optimal control design in the 1960's, a different approach to performance limitations was pursued by Chang (1961), Kalman (1964), Kwakernaak & Sivan (1972*a*, 1972*b*), Shaked (1980), and, most recently, Qiu & Davison (1993) and Chen, Qiu & Toker (1996). Letting the cost of control tend to zero ("cheap control"), these studies evaluated the best achievable performance. They showed that the closed-loop poles which do not converge to infinity, converge to the minimum phase zeros and to the *mirror image* (with respect to the imaginary axis) of the NMP zeros. The most detailed result in this direction is a formula by Qiu & Davison (1993) for the best achievable cost as function of the NMP zeros.

It is surprising that the relationship between the limiting cost of optimal linear control and the Bode integral characterization of performance limitations has not been investigated. Such an investigation is undertaken in this paper. For linear systems, we show in Section 2 that the "missing link" is a relationship between the limiting optimal cost for cheap control, obtained by Qiu & Davison (1993), and the Bode T-integral formula due to Middleton (1991). We also establish a dual relationship between the limiting optimal cost for minimum energy control and the Bode S-integral formula.

The linear analysis in Section 2 prepares the path for an investigation of feedback limitations in nonlinear systems, developed in Section 3. We present a novel singular perturbation analysis of a nonlinear cheap control problem and explain how the unstable nonlinear zero dynamics determine the limiting cost. Our analysis reveals that *the cheap control problem reduces to the minimum energy problem in which the system output is used to stabilize the nonlinear zero dynamics.* We also discuss a nonlinear analog of the fact that for linear systems some closed-loop poles converge to the mirror image of NMP zeros.

The results of a related investigation of the lowest achievable level of L_2 disturbance attenuation were reported by Schwartz, Isidori & Tarn (Schwartz, Isidori & Tarn 1997) after the submission of the original version of this paper. They show that the lowest level of attenuation is determined by the unstable zero dynamics of the system. The crucial role of the unstable zero dynamics is thus revealed in both disturbance attenuation and disturbance-free problems.

2 Feedback Limitations in Linear Systems

2.1 Cheap Control

For linear systems, it is known that NMP zeros and unstable poles impose limitations on the feedback loop that hold for *any* linear stabilizing controller, irrespective of the design technique. These limitations apply as well to optimal controllers, which, in addition to achieving closed-loop stability, minimize a quadratic cost functional.

Consider the linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= \tilde{A}x(t) + \tilde{B}u(t), \quad & x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ y(t) &= Cx(t)\ , & y \in \mathbb{R}^m, x(0) = x_0 \end{aligned} \tag{1}$$

and the cost functional

$$J_\varepsilon = \frac{1}{2}\int_0^\infty \left(y^T(t)y(t) + \varepsilon^2 u^T(t)u(t)\right) dt\ . \tag{2}$$

We are interested in the *cheap control problem*, that is, in finding a state feedback control u that minimizes J_ε when $\varepsilon > 0$ is small. It was shown by Chang (1961) and Kalman (1964) that, as $\varepsilon \to 0$, some closed-loop poles converge to the *mirror image* of the NMP zeros, while the remaining poles tend to infinity in Butterworth patterns. This fact reveals a fundamental limitation: the closed-loop poles that tend to the mirror image of the NMP zeros are not canceled. Instead, they remain in the observable part of the closed-loop dynamics.

Kwakernaak & Sivan (1972*a*, 1972*b*) (see also Francis 1979, Scherzinger & Davison 1985) proved that the lowest achievable cost is zero *if and only if the system is minimum phase and right invertible*. The NMP case was analyzed in detail by Shaked (1980) (see also Grimble & Johnson 1988), who obtained an explicit expression for the lowest achievable cost as a function of the NMP zeros. Qiu & Davison (1993) provided a simpler formula by using inner-outer factorizations and balancing. A complete treatment of the linear cheap control problem can be found in Saberi & Sannuti (1987).

We will make the following assumption.

Assumption 1.

(i) System (1) is stabilizable and detectable.

(ii) System (1) is of relative degree one, that is, rank $C\tilde{B} = m$.

(iii) All zeros of $C(sI - \tilde{A})^{-1}\tilde{B}$ are nonminimum phase. ◦

Assumption (i) is necessary and sufficient for a meaningful linear optimal control problem. Assumptions (ii) and (iii), which represent the case with the maximum number of NMP zeros, are made only for simplicity of exposition and can be removed.

By a change of coordinates $\tilde{x} = [M^T\ M_0^T]^T x$, the full rank input matrix $\tilde{B}$ can be brought to the form $[B_1^T\ 0]^T$, where B_1 is nonsingular. Then, with the same M_0 and with the full rank matrix C, the change of coordinates

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} C \\ M_0 \end{bmatrix} x$$

transforms (1) to the form

$$\begin{aligned} \dot{y} &= A_1 y + A_2 z + B u \;, \qquad\qquad B = C\tilde{B} \;, \\ \dot{z} &= B_0 y + A_0 z \;. \end{aligned} \tag{3}$$

This form is of interest because it displays the zeros of $C(sI - \tilde{A})^{-1}\tilde{B}$ as the eigenvalues of A_0. We observe that (A_0, B_0) is controllable, because the system (3) is stabilizable and A_0 is *antistable* (that is $-A_0$ is Hurwitz) by (iii), Assumption 1.

The associated algebraic Riccati equation (ARE) for the optimal control of (3) with respect to the cost (2) is

$$\begin{bmatrix} A_1 & A_2 \\ B_0 & A_0 \end{bmatrix}^{T} P + P \begin{bmatrix} A_1 & A_2 \\ B_0 & A_0 \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{\varepsilon^2} P \begin{bmatrix} BB^{T} & 0 \\ 0 & 0 \end{bmatrix} P \;. \tag{4}$$

For this singularly perturbed equation Jameson & O'Malley (1975) (see also Young, Kokotović & Utkin 1977) provided an approximate solution in the form of a power series in ε. Its leading term is

$$P = \begin{bmatrix} \varepsilon P_1 & \varepsilon P_2 \\ \varepsilon P_2^{T} & P_0 + \varepsilon P_3 \end{bmatrix} + O(\varepsilon^2) \;. \tag{5}$$

The expressions for P_0, P_1 and P_2, which are independent of ε, are obtained explicitly from the substitution of (5) into (4):

$$I - P_1 BB^{T} P_1 + O(\varepsilon) = 0 \;, \tag{6}$$

$$B_0^{T} P_0 - P_1 BB^{T} P_2 + O(\varepsilon) = 0 \;, \tag{7}$$

$$A_0^{T} P_0 + P_0 A_0 - P_2^{T} BB^{T} P_2 + O(\varepsilon) = 0 \;. \tag{8}$$

Setting $\varepsilon = 0$, (6) gives $P_1 = (BB^{T})^{-1/2}$, and using this in (7) gives $P_2 = (BB^{T})^{-1/2} B_0^{T} P_0$. Substituting in (8) we find that P_0 is the unique positive definite solution of

$$A_0^{T} P_0 + P_0 A_0 = P_0 B_0 B_0^{T} P_0 \;. \tag{9}$$

Higher order coefficients of the entries of P are obtained by the same token. In particular, equating to zero the $O(\varepsilon)$ terms in (6)-(8) yields $P_3 = P_0 B_0 (BB^{T})^{-1/2} B_0^{T} P_0$.

The value function for the cheap control problem is

$$V(y, z, \varepsilon) = \frac{1}{2} \begin{bmatrix} y^{T} & z^{T} \end{bmatrix} \begin{bmatrix} \varepsilon P_1 & \varepsilon P_2 \\ \varepsilon P_2^{T} & P_0 + \varepsilon P_3 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + O(\varepsilon^2) \;, \tag{10}$$

which tends in the limit, as $\varepsilon \to 0$, to

$$V(y, z, 0) = \frac{1}{2} z^{T} P_0 z \;. \tag{11}$$

This value is the best achievable performance for system (3) with respect to cost (2).

To interpret (11) we consider the *minimum energy problem* for the zero-dynamics subsystem: find y to stabilize the zero-dynamics subsystem $\dot{z} = A_0 z + B_0 y$ and to minimize

$$J = \frac{1}{2} \int_0^{\infty} y^{T}(t) y(t) \, dt. \tag{12}$$

Here the system output y acts as the control variable for the zero-dynamics subsystem. The value function for this problem, denoted by $V_0(z)$, is

$$V_0(z) = \frac{1}{2} z^T P_0 z \ ,$$

where P_0 is the unique, symmetric positive definite solution of (9). Thus we have established

$$V(y, z, 0) = V_0(z) = \frac{1}{2} z^T P_0 z \ , \tag{13}$$

that is, *the limiting cost for the cheap control problem is equal to the optimal cost for the minimum energy problem for the zero-dynamics subsystem.*

The optimal closed-loop system for the zero-dynamics subsystem satisfies

$$\dot{z} = -P_0^{-1} A_0^T P_0 z$$

and its eigenvalues are the mirror image of the NMP zeros of (3). Hence $n - m$ eigenvalues of the closed-loop system for the cheap control problem are the mirror image of the NMP zeros of (3), and the remaining m eigenvalues lie at infinity.

2.2 Singular Perturbation Analysis

The optimal value function (10) for the cheap control problem can be expressed as

$$V(y, z, \varepsilon) = V_0(z) + \varepsilon V_1(y, z) + O(\varepsilon^2) \ , \tag{14}$$

where $V_0(z)$ is defined in (13), and $V_1(y, z) = \frac{1}{2}(y + B_0^T P_0 z)^T (BB^T)^{-1/2}(y + B_0^T P_0 z)$. When the optimal feedback control

$$\begin{aligned} u^* &= -\frac{1}{\varepsilon^2} B^T \frac{\partial^T V}{\partial y} \\ &= -\frac{1}{\varepsilon} B^T \left[\frac{\partial^T V_1}{\partial y} + O(\varepsilon) \right] \\ &= -\frac{1}{\varepsilon} B^T \left[(BB^T)^{-1/2}(y + B_0^T P_0 z) + O(\varepsilon) \right] \end{aligned} \tag{15}$$

is applied to (3), it induces a two-time-scale decomposition in the closed-loop system

$$\begin{aligned} \varepsilon \dot{y} &= -(BB^T)^{1/2}(y + B_0^T P_0 z) + O(\varepsilon) \ , \\ \dot{z} &= B_0 y + A_0 z \ . \end{aligned} \tag{16}$$

It is well known (see Kokotović, Khalil & O'Reilly 1986) that the slow subsystem of (16) is approximated by $\dot{z} = -P_0^{-1} A_0^T P_0 z$, resulting from the substitution of $y = -B_0^T P_0 z$ into the z-equation. The slow dynamics evolve in the *slow invariant subspace* $y = \phi_\varepsilon z$ of (16), where $\phi_\varepsilon = -B_0^T P_0 + O(\varepsilon)$. This means that $y = -B_0^T P_0 z$ is an $O(\varepsilon)$ approximation of the slow invariant subspace, and $y - \phi_\varepsilon z = y + B_0^T P_0 z + O(\varepsilon)$ is the fast variable which rapidly converges to zero. The change of coordinates $\eta = y - \phi_0 z = y + B_0^T P_0 z$ decouples the fast and slow parts of the system:

$$\begin{aligned} \varepsilon\dot{\eta} &= -(BB^T)^{1/2}\eta + O(\varepsilon) \ , \\ \dot{z} &= -P_0^{-1}A_0^T P_0 z + B_0\eta \ . \end{aligned} \tag{17}$$

An important observation is that $y = \phi_0 z$ ($\eta = 0$) is the subspace where

$$\frac{\partial V_1(y,z)}{\partial y} = 0 \ . \tag{18}$$

In our nonlinear analysis, the slow dynamics will evolve in the *slow invariant manifold* $y = \phi_\varepsilon(z)$, the nonlinear analog of $y = \phi_\varepsilon z$. To define this slow manifold we will use the fact that its $O(\varepsilon)$ approximation $y = \phi_0(z)$ is given by $\partial V_1(y,z)/\partial y = 0$.

2.3 Cheap Control and the Bode Integral for T

Let us now consider the problem of regulating the output y of (3) to a constant setpoint r. We employ a feedforward term $\bar{u}$ to guarantee that the equilibrium of (3) is at $y = r$. Solving for this equilibrium gives $\bar{u} = -B^{-1}(A_1 - A_2 A_0^{-1} B_0)r$. Introducing the error variables $e = y - r$, $\tilde{z} = z + A_0^{-1}B_0 r$, and $\tilde{u} = u - \bar{u}$, we rewrite (3) as

$$\begin{aligned} \dot{e} &= A_1 e + A_2\tilde{z} + B\tilde{u} \ , \\ \dot{\tilde{z}} &= A_0\tilde{z} + B_0 e \ . \end{aligned} \tag{19}$$

The optimal control $\tilde{u}$ that minimizes

$$\tilde{J}_\varepsilon = \frac{1}{2}\int_0^\infty \left(e^T(t)e(t) + \varepsilon^2\tilde{u}^T(t)\tilde{u}(t)\right) dt \ , \tag{20}$$

is of the same form as (15), and the value function is of the same form as (10), but with $(e, \tilde{z})$ replacing (y, z). From (13) we know that the limit of the value function $V(e(0), \tilde{z}(0), \varepsilon)$ as $\varepsilon \to 0$ is $V(e(0), \tilde{z}(0), 0) = V_0(\tilde{z}(0))$. What is of interest here is the value of this limit for the particular initial conditions $e(0) = r$ and $\tilde{z}(0) = A_0^{-1}B_0 r$, which correspond to the state y being transferred from 0 to r. The answer is given by a simple formula due to Qiu & Davison (1993).

Lemma 2.1 (Qiu-Davison Formula). For the initial condition $e(0) = r$ and $\tilde{z}(0) = A_0^{-1}B_0 r$, the limit of the value function $V(e(0), \tilde{z}(0), \varepsilon)$ for the cheap control problem as $\varepsilon \to 0$ satisfies

$$V_0(\tilde{z}(0)) = \frac{1}{2}r^T H r \ , \qquad \text{where} \qquad \operatorname{trace} H = 2\sum_{i=1}^{k}\frac{1}{\alpha_i} \ , \tag{21}$$

and $\alpha_1, \ldots, \alpha_k$, are the NMP zeros of system (3).

Proof. From (13) we have

$$V_0(\tilde{z}(0)) = \frac{1}{2}\tilde{z}^T(0)P_0\tilde{z}(0) = \frac{1}{2}r^T B_0^T (A_0^T)^{-1} P_0 A_0^{-1} B_0 r = \frac{1}{2}r^T H r \ ,$$

where $H \triangleq B_0^T (A_0^T)^{-1} P_0 A_0^{-1} B_0$. Then

$$\begin{aligned}\operatorname{trace}(H) &= \operatorname{trace}\left(P_0^{1/2} A_0^{-1} B_0 B_0^T (A_0^T)^{-1} P_0^{1/2}\right)\\ &= \operatorname{trace}\left(P_0^{1/2} A_0^{-1} P_0^{-1/2} + P_0^{-1/2} (A_0^T)^{-1} P_0^{1/2}\right)\\ &= 2 \operatorname{trace} A_0^{-1} = 2 \sum_{i=1}^{k} \frac{1}{\alpha_i},\end{aligned}$$

where the second identity follows from the ARE (9). □

Thus, the lowest achievable cost to steer the system from rest to the setpoint $y = r$ is an explicit function of the NMP zeros, independent of the system realization. This limiting cost will be larger when the NMP zeros are closer to the imaginary axis.

How is this result related to the performance limitations defined by well-known Bode and Poisson integrals (Bode 1945, Freudenberg & Looze 1985)? These hold for the sensitivity and complementary sensitivity functions of the unity feedback loop in Figure 1, where L is the open-loop transfer function, formed by the series connection of the plant and the controller; y is the output; and r is a reference signal. We are particularly interested in the complementary sensitivity function $T = L\,(I + L)^{-1}$ which maps r to y.

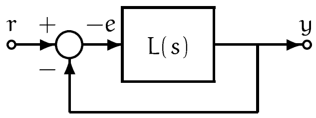

Figure 1: Unitary feedback loop.

A Bode integral for T was derived by Middleton (1991). We review this formula next. Consider single-input single-output (SISO) systems described by rational proper transfer functions. Let $\{\alpha_1, \ldots, \alpha_k\}$ be the set of open right half plane (ORHP) zeros of the open-loop system L. Then, if the closed-loop system in Figure 1 is stable and $T(0) = 1$, we have

$$\frac{1}{\pi} \int_0^\infty \log |T(j\omega)| \, \frac{d\omega}{\omega^2} + \frac{1}{2K_\nu} = \sum_{i=1}^{k} \frac{1}{\alpha_i}, \tag{22}$$

where $K_\nu = \lim_{s\to 0} sL(s)$ is the *velocity constant* (Truxal 1955, p. 286). When the controller is minimum phase, then $\{\alpha_1, \ldots, \alpha_k\}$ is the set of ORHP zeros of the plant. In this case, we will refer to the left hand side of (22) as the *complementary sensitivity invariant*, briefly, the T-*invariant*.

The connection between the Middleton formula (22) and the Qiu & Davison formula (21) for cheap control has, apparently, remained unnoticed. We make this connection explicit.

Lemma 2.2 (Cheap Control and T-Invariant). Let L be a SISO open-loop system formed by the series connection of a plant and a linear, minimum phase controller. Suppose that the corresponding complementary sensitivity function $T = L(1+L)^{-1}$ is stable and such that $T(0) = 1$. Then

$$\frac{1}{\pi}\int_0^\infty \log|T(j\omega)| \frac{d\omega}{\omega^2} + \frac{1}{2K_v} = \lim_{\varepsilon\to 0} \frac{1}{2}\int_0^\infty e^2(t)\,dt\ , \tag{23}$$

where $e(t) = y(t) - r$ is the error in the cheap control problem transferring the system from rest to the setpoint $r = 1$.

Proof. Immediate from (22), Lemma 2.1 specialized to the scalar case, and

$$\lim_{\varepsilon\to 0} \frac{1}{2}\int_0^\infty e^2(t)\,dt = V_0(\tilde{z}(0))\ .$$

□

Thus the T-invariant is the lowest achievable cost in the cheap control problem transferring the system from rest to a unit setpoint.

In the problem of setpoint regulation described before, let T be the transfer function from r to y. The analysis in Section 2.1 shows that T has poles located at the mirror image of its NMP zeros, and the feedforward part of the controller guarantees that $T(0) = 1$. Hence in the limit, as $\varepsilon \to 0$, T satisfies $|T(j\omega)| = 1$ for all ω (see also Kwakernaak & Sivan 1972*a*, Theorem 3.13), which in turn implies that $\int_0^\infty \log|T(j\omega)|\, d\omega/\omega^2 = 0$. Therefore, an implication of Lemma 2.2 is that the cheap control yields the velocity constant

$$\lim_{\varepsilon\to 0} \frac{1}{K_v} = 2\sum_{i=1}^{n-1} \frac{1}{\alpha_i}\ . \tag{24}$$

On the other hand, for any relative-degree-one T, we have

$$\frac{1}{K_v} = \sum_{i=1}^{n-1} \frac{1}{\alpha_i} - \sum_{i=1}^{n} \frac{1}{\beta_i}\ ,$$

where $\alpha_1, \ldots, \alpha_{n-1}$, and $\beta_1, \ldots, \beta_n$, are respectively the zeros and poles of T (Truxal 1955, p. 282). When T is the result of the above cheap control problem, then, as $\varepsilon \to 0$, one of the poles goes to $-\infty$, while the remaining $n-1$ poles converge to the mirror image of the NMP zeros, which again gives (24).

2.4 Minimum Energy Control and the Bode Integral for S

We now turn to the well-known Bode integral for the sensitivity function $S = (I+L)^{-1}$, which maps r to $-e$ in the feedback configuration of Figure 1. Assume that the open-loop system L is a SISO strictly proper rational function, and let $\{\beta_1, \ldots, \beta_n\}$ be the set of ORHP poles of L. When the closed-loop system in Figure 1 is stable, we have

$$\frac{1}{\pi}\int_0^\infty \log|S(j\omega)|\, d\omega + \frac{K_\infty}{2} = \sum_{i=1}^{n} \beta_i\ , \tag{25}$$

where $K_\infty = \lim_{s\to\infty} sL(s)$ (Seron et al. 1997). If L is stable and has relative degree greater than one, then (25) reduces to the well-known expression $\int_0^\infty \log|S(j\omega)|\, d\omega = 0$. When the controller is stable, then $\{\beta_1, \ldots, \beta_n\}$ is the set of ORHP poles of the plant. In this case, we will refer to the left hand side of (25) as the *sensitivity invariant*, briefly, the *S-invariant*.

The S-integral formula (25) is connected with the following minimum energy problem. For the single-input plant $\dot{x} = Ax + Bu$, where A is antistable and (A, B) is controllable, find a stabilizing feedback control that minimizes

$$J = \frac{1}{2}\int_0^\infty u^2(t)\, dt$$

for any initial state x_0. The optimal control for this problem is $u^* = -B^T P x$, and the optimal cost is $V(x_0) = (1/2)x_0^T P x_0$, where P is the unique symmetric positive definite solution of

$$A^T P + PA = PBB^T P\ . \tag{26}$$

The optimal closed-loop system is

$$\dot{x} = P^{-1}(-A^T)Px\ . \tag{27}$$

Lemma 2.3 (Minimum Energy Control and S-Invariant). Let L be a SISO strictly proper open-loop system formed by the series connection of a plant and a linear, stable, controller. Suppose that the corresponding sensitivity function $S = (1 + L)^{-1}$ is stable. Then

$$\frac{1}{\pi}\int_0^\infty \log|S(j\omega)|\, d\omega + \frac{K_\infty}{2} = \frac{1}{2}\int_0^\infty {u^*}^2(t)\, dt\ , \tag{28}$$

where $u^*(t)$ is the minimum energy control transferring the system to rest from the initial condition originating from a unit impulse at the input.

Proof. Let $\dot{x} = Ax+Bu$ be a controllable realization of the plant, and assume (for simplicity) that A is antistable. A unit impulse at the input sets the initial condition $x_0 = B$. Then, the optimal cost for the minimum energy problem is $V(x_0) = (1/2)x_0^T P x_0 = (1/2)B^T PB$, where P satisfies the ARE (26). Hence,

$$\begin{aligned} V(x_0) &= \frac{1}{2}\Big(B^T P^{1/2} P^{1/2} B\Big) \\ &= \frac{1}{2}\,\mathrm{trace}\Big(P^{1/2} BB^T P^{1/2}\Big) \\ &= \frac{1}{2}\,\mathrm{trace}\Big(P^{-1/2} A^T P^{1/2} + P^{1/2} A P^{-1/2}\Big) \\ &= \mathrm{trace}\, A = \sum_{i=1}^{n} \beta_i\ . \end{aligned}$$

Equation (28) then follows from (25). □

Thus the S-invariant is the minimum energy required to transfer the system to rest from the initial condition originating from a unit impulse at the input.

For a sensitivity interpretation of the above minimum energy problem we let L be the transfer function from u to u^* in

$$\begin{aligned} \dot{x} &= Ax + Bu \\ u^* &= -B^T P x \ . \end{aligned}$$

Since this system is optimal, the relative degree of L is one (e.g., Anderson & Moore 1990, p. 121). Moreover, it follows from (27) that the closed-loop poles are located at the mirror image of the unstable open-loop poles. This implies that the optimal $S = (1+L)^{-1}$ satisfies $|S(j\omega)| = 1$ and, hence, $\int_0^\infty \log|S(j\omega)|\, d\omega = 0$. Therefore, the minimum energy control yields

$$\lim_{\varepsilon \to 0} K_\infty = 2 \sum_{i=1}^{n} \beta_i \ . \tag{29}$$

Alternatively, since L is of relative degree one,

$$K_\infty = \sum_{i=1}^{n} (\beta_i - \gamma_i) \ ,$$

where β_i, γ_i, $i = 1, \ldots, n$, are the open- and closed-loop poles of the system, respectively (see Truxal 1955, p. 302). The mirroring property of the closed-loop poles again yields (29).

3 Feedback Limitations in Nonlinear Systems

3.1 Cheap Control

We now analyze the cheap control problem to determine feedback limitations for nonlinear systems of the form

$$\begin{aligned} \dot{y} &= f(y,z) + g(y,z)u, \qquad && y,\ u \in \mathbb{R}^m,\ y(0) = y_0 \\ \dot{z} &= f_0(z) + g_0(z)y, && z \in \mathbb{R}^{n-m},\ z(0) = z_0 \end{aligned} \tag{30}$$

where f, g, f_0, g_0 are C^1 vector fields such that $f(0,0) = 0$, $f_0(0) = 0$ and $g(y,z)$ is nonsingular everywhere, that is, there exists a constant $b > 0$ such that

$$\|g(y,z)\| \geq b \ , \qquad \text{for all } (y,z) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \ . \tag{31}$$

The system (30) is the nonlinear analog of (3). It is in the *Isidori normal form* (Isidori 1995), in which the zero-dynamics subsystem $\dot{z} = f_0(z)$ is obtained by setting $y(t) \equiv 0$ in the z-equation. The control that makes z unobservable from y is $u = -g^{-1}(y,z)f(y,z)$. This control cannot be used when $\dot{z} = f_0(z)$ is unstable, which we assume to be the case.

We are interested in a feedback control u that asymptotically stabilizes system (30) in a domain $\mathcal{D} \subset \mathbb{R}^n$, containing the origin, and minimizes the cost functional

$$J_\varepsilon = \frac{1}{2} \int_0^\infty \left(y^T(t)y(t) + \varepsilon^2 u^T(t)u(t) \right) dt \tag{32}$$

in the limiting case when $\varepsilon \to 0$.

It is well-known (e.g., Sepulchre, Janković & Kokotović 1997, p. 91) that this problem has a solution for $\varepsilon > 0$ if there exists a C^1 positive semidefinite function $V(y, z, \varepsilon)$ which satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\frac{\partial V}{\partial y} f(y,z) + \frac{\partial V}{\partial z}\left(f_0(z) + g_0(z)y\right) + \frac{1}{2} y^T y - \frac{1}{2\varepsilon^2}\frac{\partial V}{\partial y} g(y,z) g^T(y,z)\frac{\partial^T V}{\partial y} = 0\ , \quad V(0,0,\varepsilon) = 0\ , \tag{33}$$

and such that the feedback control

$$u(y,z,\varepsilon) = -\frac{1}{\varepsilon^2} g^T(y,z)\frac{\partial^T V}{\partial y} \tag{34}$$

asymptotically stabilizes (30) in $\mathcal{D}$. Then (34) is the optimal stabilizing control that minimizes J_ε and $V(y,z,\varepsilon)$ is the optimal value function.

As we will show, the solution of this problem when $\varepsilon \to 0$ hinges upon the solution to the following minimum energy problem, where the system output is treated as the control variable: Find y to asymptotically stabilize the zero dynamics

$$\dot{z} = f_0(z) + g_0(z)y \tag{35}$$

in $\mathcal{D}$ and to minimize the cost

$$J = \frac{1}{2}\int_0^\infty y^T(t) y(t)\, dt. \tag{36}$$

If there exists a C^1 positive semidefinite function $V_0(z)$ which satisfies the HJB equation

$$\frac{\partial V_0}{\partial z} f_0(z) - \frac{1}{2}\frac{\partial V_0}{\partial z} g_0(z) g_0^T(z)\frac{\partial^T V_0}{\partial z} = 0\ , \quad V_0(0) = 0\ , \tag{37}$$

and such that

$$y = \phi_0(z) \triangleq -g_0^T(z)\frac{\partial^T V_0}{\partial z} \tag{38}$$

asymptotically stabilizes (35) in $\mathcal{D}$, then (38) is the optimal control and $V_0(z)$ the optimal value function. If $\dot{z} = f_0(z)$ were asymptotically stable, then the optimal solution of (37) would be $V_0(z) \equiv 0$. But otherwise, $V_0(z)$ must be greater than zero for some values of z in order to asymptotically stabilize the zero dynamics (35) .

Our analysis requires the existence of the cheap and minimum energy value functions satisfying the above optimality conditions.

Assumption 2.

(i) There exists $\varepsilon^* > 0$ such that for each $\varepsilon \in (0, \varepsilon^*]$ there is a C^1 positive semidefinite function $V(y,z,\varepsilon)$ which satisfies (33) and such that $u(y,z,\varepsilon)$ given by (34) asymptotically stabilizes (30) in $\mathcal{D}$.

(ii) There exists a C^1 positive semidefinite function $V_0(z)$ which satisfies (37) and such that y given by (38) globally asymptotically stabilizes (35). ◦

3.2 Singular Perturbation Analysis

From the linear analysis in Section 2.2, we expect that the value function for the cheap control problem (30), (32) has the form (14). Therefore, we postulate as an $O(\varepsilon^2)$ approximation of $V(y, z, \varepsilon)$ the expansion in powers of ε

$$\hat{V}(y, z, \varepsilon) = V_0(z) + \varepsilon V_1(y, z) + O(\varepsilon^2) \ , \tag{39}$$

where V_0 is the optimal solution of (37). We obtain $\partial V_1/\partial y$ as follows.

Substitute (39) in (33) to obtain

$$\frac{\partial V_0}{\partial z}\left[f_0(z) + g_0(z)y\right] + \frac{1}{2}y^{\mathsf{T}}y - \frac{1}{2}\frac{\partial V_1}{\partial y}g(y,z)g(y,z)^{\mathsf{T}}\frac{\partial^{\mathsf{T}} V_1}{\partial y} + O(\varepsilon) = 0 \ . \tag{40}$$

Then, completing squares, after adding and subtracting the term $\frac{1}{2}\|\phi_0(z)\|^2 = \frac{1}{2}\|g_0^{\mathsf{T}}\,\partial^{\mathsf{T}} V_0/\partial z\|^2$ to the LHS of (40), gives, with the use of (37),

$$\frac{1}{2}\left[y - \phi_0(z)\right]^{\mathsf{T}}\left[y - \phi_0(z)\right] - \frac{1}{2}\frac{\partial V_1}{\partial y}g(y,z)g(y,z)^{\mathsf{T}}\frac{\partial^{\mathsf{T}} V_1}{\partial y} + O(\varepsilon) = 0 \ ,$$

Setting $\varepsilon = 0$ in (3.2) yields

$$\frac{\partial^{\mathsf{T}} V_1}{\partial y} = g^{\mathsf{T}}(y,z)^{-1}\left[y - \phi_0(z)\right] \ . \tag{41}$$

After substituting (39), (41) in (34), and truncating the optimal control up to $O(\varepsilon)$, we obtain the closed-loop system (30), (34) in the standard singular perturbation form

$$\begin{aligned} \varepsilon\dot{y} &= -g(y,z)\left[y - \phi_0(z)\right] + \varepsilon f(y,z) \ , \qquad y(0) = y_0 \ , \\ \dot{z} &= f_0(z) + g_0(z)y \ , \qquad z(0) = z_0 \ , \end{aligned} \tag{42}$$

which is the nonlinear analog of (16). The change of coordinates $\eta = y - \phi_0(z)$ brings (16) to

$$\begin{aligned} \varepsilon\dot{\eta} &= -\tilde{g}(\eta,z)\eta + \varepsilon\tilde{f}(\eta,z) \ , \qquad \eta(0) = \eta_0 \ , \\ \dot{z} &= \tilde{f}_0(z) + g_0(z)\eta \ , \qquad z(0) = z_0 \ , \end{aligned} \tag{43}$$

where $\tilde{g}(\eta,z) = g(\eta + \phi_0(z), z)$, $\tilde{f}_0(z) = f_0(z) + g_0(z)\phi_0(z)$, and $\tilde{f}(\eta,z) = f(\eta + \phi_0(z), z) - (\partial\phi_0/\partial z)\left[\tilde{f}_0(z) + g_0(z)\eta\right]$.

The slow invariant manifold of (70) is $\eta = 0$, which follows after setting $\varepsilon = 0$ in the η-equation of (70). The *reduced* (slow) system

$$\begin{aligned} \eta &= 0 \ , \\ \dot{z} &= \tilde{f}_0(z) \ , \qquad z(0) = z_0 \end{aligned} \tag{44}$$

is obtained from the substitution of $\eta = 0$ into the z-equation of (70). The solution of the z-equation in (44) is denoted by $z_s(t)$.

The *boundary-layer* (fast) system is defined by making the change of time scale $\tau = t/\varepsilon$ in (70) and then setting $\varepsilon = 0$, which yields

$$\begin{aligned} \frac{d\eta}{d\tau} &= -\tilde{g}(\eta,z)\eta \ , \qquad \eta(0) = \eta_0 \ , \\ z &= \text{ constant} \, . \end{aligned} \tag{45}$$

The following result shows that the solutions of (70) are bounded for ε sufficiently small and that in the limit, as $\varepsilon \to 0$, they converge arbitrarily fast to the solutions of the reduced system (44).

Lemma 3.1. Given any compact set $\mathcal{K} \subset \mathbb{R}^n$, there exists $\varepsilon^* > 0$ such that for every $\varepsilon \in (0, \varepsilon^*]$ the origin is a GAS equilibrium of system (70) with a region of attraction that includes $\mathcal{K}$. Moreover, for every initial condition $(\eta_0, z_0) \in \mathcal{K}$, the solutions of (70) coverge to those of the reduced system (44) as $\varepsilon \to 0$, uniformly on $[t_1, \infty)$, for all $t_1 > 0$.

Proof. From our assumptions, system (70) satisfies the hypothesis of Christofides & Teel (1996, Theorem 1). Therefore, there exist class-KL functions β_z, β_η and for each pair of positive numbers (δ, d), there is an $\varepsilon_d > 0$ such that if $\max\{|z_0|, |\eta_0|\} < \delta$ and $\varepsilon \in (0, \varepsilon_d)$, then for all $t \geq 0$

$$|z(t)| \leq \beta_z(|z_0|, t) + d \tag{46}$$

$$|\eta(t)| \leq \beta_\eta\left(|\eta_0|, \frac{t}{\varepsilon}\right) + d\,. \tag{47}$$

This shows the stability of (70).

Local asymptotic stability follows from . . .

To show that the solutions of (70) tend to those of (44) uniformly on $[t_1, \infty)$ as $\varepsilon \to 0$, we next show that given any arbitrary $d_1 > 0$ and $t_1 > 0$, there is some $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1]$

$$\|\eta(t)\| + \|z(t) - z_s(t)\| < c \qquad \forall t \in [t_1, \infty)\,. \tag{48}$$

First, from (46) (46), there is a finite $t_2 > 0$ and $\varepsilon_2 > 0$ such that for $\varepsilon \in (0, \varepsilon_2]$

$$|z(t)| < \frac{c}{3} \quad \text{and} \quad |\eta(t)| < \frac{d_1}{3} \qquad \forall t \in [t_2, \infty)\,. \tag{49}$$

Because the reduced system (44) is GAS, also

$$|z_s(t)| < \frac{d_1}{3} \qquad \forall t \in [t_3, \infty)\,, \tag{50}$$

for some finite time $t_3 > 0$. Then, it follows from (49) and (50) that

$$|z(t) - z_s(t)| + |\eta(t)| \leq d_1 \qquad \forall t \in [T, \infty)\,, \tag{51}$$

where $T = \max\{t_2, t_3\}$.

On the other hand, (47) implies that $\eta(t) \to 0$ as $\varepsilon \to 0$ uniformly on $(t_1, T]$, and by the continuous dependence of solutions of differential equations on the right-hand side functions, also $z(t) \to z_s(t)$ as $\varepsilon \to 0$ uniformly on $[t_1, T]$. Thus, there is an $\varepsilon_3 > 0$ such that for all $\varepsilon \in (0, \varepsilon_3]$

$$|z(t) - z_s(t)| + |\eta(t)| \leq d_1 \qquad \forall t \in [t_1, T]\,. \tag{52}$$

Hence (59) follows from (51) and (52). $\square$

Lemma 3.2. Under (ii), Assumption 2, given any compact sets $\mathcal{K}_\eta \subset \mathbb{R}^m$, $\mathcal{K}_z \subset \mathbb{R}^{n-m}$, for every initial condition $(\eta_0, z_0) \in \mathcal{K}_\eta \times \mathcal{K}_z$, there exists $\varepsilon_r > 0$ such that for all $\varepsilon \in (0, \varepsilon_r)$ the solutions of (70) are bounded and converge to the solutions of the reduced system (44) as $\varepsilon \to 0$, uniformly on $[t_1, \infty)$, for all $t_1 > 0$.

Proof. We divide the proof in two parts. We first show that (i) by choosing ε sufficiently small all the solutions of (70) starting in $\mathcal{K}_\eta \times \mathcal{K}_z$ converge to an arbitrarily small invariant compact set around the origin, and therefore are bounded. We then show that (ii) they in fact converge to the solutions of the reduced system (44), uniformly on $t \in (t_1, \infty)$, as $\varepsilon \to 0$.

(i) Since the reduced system (44) is globally asymptotically stable (GAS) by (ii), Assumption 2, there exists a positive definite and proper function $W_0(z)$, and a class-$\mathcal{K}_\infty$ function $\alpha_0(|z|)$ such that

$$\dot{W}_0 = \frac{\partial W_0}{\partial z}\tilde{f}_0(z) \leq -\alpha_0(|z|)\,. \tag{53}$$

The boundary-layer system (45) is also GAS, for the derivative of the Lyapunov function $W_1(\eta) = ||\eta||^2$ along the trajectories of (45) is, using (31),

$$\frac{dW_1}{d\tau} = -2\eta^T\tilde{g}(\eta, z)\eta \leq -2b||\eta||^2\,. \tag{54}$$

This part of the proof now follows Teel & Praly (1995, Lemma 2.2 and Example 2.1). Choose the numbers

$$\mu \geq \max\left\{1, \max_{\eta\in\mathcal{K}_\eta} ||\eta||^2\right\}$$

and $c \geq 1$ such that the set $\{z : W_0(z) \leq c+1\}$ contains $\mathcal{K}_z$, and introduce the composite Lyapunov function

$$W(\eta, z) = \frac{c\,W_0(z)}{c+1-W_0(z)} + \frac{\mu W_1(\eta)}{\mu+1-W_1(\eta)} \tag{55}$$

for the perturbed system (70). Using (53) and (54), the derivative of W along the trajectories of (70) satisfies

$$\dot{W} \leq -2\Phi(\eta, z) + \Psi(\eta, z)\,, \tag{56}$$

where

$$\begin{aligned}
\Phi(\eta, z) &= -\frac{c}{2[c+1-W_0(z)]^2}\alpha_0(|z|) - \frac{1}{\varepsilon}\frac{\mu}{\mu+1}b||\eta||^2\,,\\
\Psi(\eta, z) &= \frac{(c^2+\mu^2+1+c)^2}{c(c+1)}\left\|\frac{\partial W_0}{\partial z}g_0\eta\right\| + 2\frac{(c^2+\mu^2+1+\mu)^2}{\mu(\mu+1)}||\eta^T\tilde{f}(\eta, z)||\,.
\end{aligned}$$

Now, for any arbitrarily small $\rho > 0$, Lemma 2.1 in Teel & Praly (1995) guarantees the existence of $\varepsilon^*(\rho, c, \mu) > 0$ such that for all $\varepsilon \in (0, \varepsilon^*(\rho, c, \mu))$, we have that

$-\Phi(\eta,z)+\Psi(\eta,z)<0$ on the set $\mathcal{B}=\{(\eta,z):\rho\le W(\eta,z)\le 1+c^2+\mu^2\}$. Hence, from (56),

$$\dot{W}<-\Phi(\eta,z) \quad \text{on } \mathcal{B},$$

and since $\Phi(\eta,z)$ is strictly positive on $\mathcal{B}$, we conclude that there exists some finite $t_\rho>0$ at which the solutions of (70) starting in $\mathcal{B}$ are captured by the set $\mathcal{B}_\rho=\{(\eta,z):W(\eta,z)\le\rho\}$ and are thus bounded.

(ii) Consider the level set $\Gamma_\rho=\{(\eta,z):W(\eta,z)=\rho\}$ and define

$$\begin{aligned}\delta_1(\rho)&=\min_{(\eta,z)\in\Gamma_\rho}\|(\eta,z)\|\,,\\ \delta_2(\rho)&=\max_{(\eta,z)\in\Gamma_\rho}\|(\eta,z)\|\,.\end{aligned} \tag{57}$$

Note that $\delta_1(\rho)$ is the radius of the largest ball included in $\mathcal{B}_\rho$, while $\delta_2(\rho)$ is the radius of the smallest ball including $\mathcal{B}_\rho$. Since from part (i) $\mathcal{B}_\rho$ is invariant for $\varepsilon\in(0,\varepsilon^*(\rho,c,\mu))$, we have that

$$\begin{aligned}&\|(\eta(t_\theta),z(t_\theta))\|<\delta_1(\rho) \quad \text{for some } t_\theta>0 \quad\Rightarrow\\ &\qquad (\eta(t_\theta),z(t_\theta))\in\mathcal{B}_\rho \quad\Rightarrow\\ &\qquad\qquad \|(\eta(t),z(t))\|\le\delta_2(\rho) \quad \text{for all } t\in[t_\theta,\infty)\,.\end{aligned} \tag{58}$$

To show that the solutions of (70) tend to those of (44) uniformly in $[t_1,\infty)$ as $\varepsilon\to0$, we will next show that given any $\theta>0$ and $t_1>0$, there exists some $\varepsilon_\theta>0$ such that for all $\varepsilon\in(0,\varepsilon_\theta)$

$$\|\eta(t)\|+\|z(t)-z_s(t)\|<\theta \qquad \forall t\in[t_1,\infty)\,. \tag{59}$$

To this end, given $\theta>0$, let $\rho_\theta>0$ be such that

$$\frac{\delta_1(\rho_\theta)}{2\sqrt{2}}+2\delta_2(\rho_\theta)\le\theta\,, \tag{60}$$

and chose $\varepsilon^*(\rho_\theta,c,\mu)$, as in part (i), to make $\mathcal{B}_{\rho_\theta}$ invariant.

Since (44) is GAS, there exists some finite $t_\theta>t_1$ such that

$$\|z_s(t)\|<\frac{\delta_1(\rho_\theta)}{2\sqrt{2}} \qquad \forall t\in[t_\theta,\infty)\,. \tag{61}$$

Using Tikhonov's theorem (e.g., Hoppensteadt 1966, Lemma 4), one can find some $\varepsilon_\theta>0$, $\varepsilon_\theta\le\varepsilon^*(\rho_\theta,c,\mu)$, such that for all $\varepsilon\in(0,\varepsilon_\theta)$,

$$\|\eta(t)\|+\|z(t)-z_s(t)\|<\frac{\delta_1(\rho_\theta)}{2\sqrt{2}} \qquad \forall t\in[t_1,t_\theta]\,. \tag{62}$$

Then at $t=t_\theta$ we have that

$$\|\eta(t_\theta)\|+\|z(t_\theta)\|\le\|\eta(t_\theta)\|+\|z(t_\theta)-z_s(t_\theta)\|+\|z_s(t_\theta)\|<\frac{\delta_1(\rho_\theta)}{\sqrt{2}}\,,$$

which implies that $\|(\eta(t_\theta), z(t_\theta))\| < \delta_1(\rho_\theta)$. It follows from (58) that

$$\|\eta(t)\| + \|z(t)\| \leq 2\delta_2(\rho) \quad \forall t \in [t_\theta, \infty) \,. \tag{63}$$

Hence, from (60), (61) and (63) we finally obtain

$$\|\eta(t)\| + \|z(t) - z_s(t)\| < \frac{\delta_1(\rho_\theta)}{2\sqrt{2}} + 2\delta_2(\rho_\theta) \leq \theta \quad \forall t \in [t_\theta, \infty) \,,$$

which combined with (62) gives (59). Since θ and t_1 were arbitrary, the result follows. □

3.3 The Nonlinear Analog of the Complementary Sensitivity Invariant

We are now ready to state and prove our main result.

Theorem 3.3 (Limiting Optimal Cost). Under Assumption 2, given any compact sets $\mathcal{K}_y \subset \mathbb{R}^m$, $\mathcal{K}_z \subset \mathbb{R}^{n-m}$ such that $\mathcal{K}_y \times \mathcal{K}_z \subset \mathcal{D}$, for every initial condition $(y_0, z_0) \in \mathcal{K}_y \times \mathcal{K}_z$, the limiting optimal cost of the nonlinear cheap control problem (30), (32) is

$$\lim_{\varepsilon \to 0} V(y_0, z_0, \varepsilon) = V_0(z_0) \,,$$

where $V_0(z)$ is the optimal value function of the minimum energy problem for the zero dynamics (35) controlled by the output y.

Proof. By construction, our candidate function $\hat{V}$ from (39) satisfies the HJB equation (33) up to $O(\varepsilon)$, and thus tends to an exact solution as $\varepsilon \to 0$. Since, as shown in Lemma 3.2, the $O(\varepsilon)$ truncation of the control (34) constructed from $\hat{V}$ (semiglobally) asymptotically stabilizes the closed-loop system (42) as $\varepsilon \to 0$, we conclude that $\hat{V}$ is indeed an approximation of the optimal value function V. □

Thus, in the cheap control problem for NMP nonlinear systems, the output y cannot be steered arbitrarily rapidly to zero, because it must first attend to the task of stabilizing the zero dynamics. The fast part εV_1 of the value function $V(y, z, \varepsilon)$ is negligible, so that the lowest achievable cost is given by $V_0(z)$, where z is the initial state of the zero-dynamics subsystem.

The linear property that the optimal poles are placed as the mirror image of the NMP zeros has its nonlinear analog in the following "mirroring property".

Corollary 3.4 (Mirroring Property). The optimal closed-loop zero-dynamics subsystem is the z-equation of the reduced system (44), and along the trajectories of the open- and closed-loop zero-dynamics subsystems the value function $V_0(z)$ satisfies

$$\dot{V}_0^{\mathrm{OL}}(z) = -\dot{V}_0^{\mathrm{CL}}(z) \,, \tag{64}$$

where

$$\dot{V}_0^{\mathrm{OL}}(z) \triangleq \frac{\partial V_0}{\partial z}(z) f_0(z) \qquad \text{and} \qquad \dot{V}_0^{\mathrm{CL}}(z) \triangleq \frac{\partial V_0}{\partial z}(z) \tilde{f}_0(z) \,.$$

Proof. Follows immediately from (37) and (44). □

We observe that (64) is a consequence of the Hamiltonian decomposition into a stable and an unstable manifold (Anderson & Kokotović 1987). For antistable systems, the unstable manifold corresponds to the solution $V_0(z) \equiv 0$ of (37).

As in the linear case, we can formulate the cheap control problem to steer the output $y(t)$ to the setpoint $r = 1$. Including a feedforward control term $\bar{u} = -g^{-1}(r, \bar{z})f(r, \bar{z})$, where $\bar{z}$ is determined from $f_0(\bar{z})+g_0(\bar{z})r = 0$, we can introduce the new variables $e = y-r$, $\tilde{z} = z - \bar{z}$ and $\tilde{u} = u - \bar{u}$. The limiting cost $V_0(\tilde{z})$ of this setpoint problem would then yield the nonlinear analog of the complementary sensitivity invariant.

3.4 Example

Example 3.1. We illustrate our results with the second-order system

$$\dot{y} = z^4 + u\,, \qquad \dot{z} = \frac{z^3}{2} + y\,, \tag{65}$$

whose nonlinear zero dynamics $\dot{z} = z^3/2$ are antistable. We compute the solution of the cheap-control HJB equation

$$\frac{\partial V}{\partial y}z^4 + \frac{\partial V}{\partial z}\left[\frac{z^3}{2} + y\right] + \frac{1}{2}y^2 - \frac{1}{2\varepsilon^2}\left(\frac{\partial V}{\partial y}\right)^2 = 0 \tag{66}$$

as an expansion of the form $\hat{V} = V_0(z) + \varepsilon V_1(y, z) + O(\varepsilon^2)$. From its substitution in (75) we obtain

$$\frac{\partial V_0}{\partial z}\left[\frac{z^3}{2} + y\right] + \frac{1}{2}y^2 - \frac{1}{2}\left[\frac{\partial V_1}{\partial y}\right]^2 + O(\varepsilon) = 0\,. \tag{67}$$

We recognize the first two terms of (77) as the Hamiltonian corresponding to finding an optimal y that asymptotically stabilizes the zero-dynamics subsystem $\dot{z} = z^3/2 + y$ with minimum energy (L_2 norm). The HJB equation associated to this Hamiltonian is

$$\frac{\partial V_0}{\partial z}\frac{z^3}{2} - \frac{1}{2}\left[\frac{\partial V_0}{\partial z}\right]^2 = 0\,, \quad V_0(0) = 0\,,$$

and has the positive definite solution

$$V_0 = \frac{z^4}{4}\,, \tag{68}$$

which yields the "optimal output" $y = -z^3$. The substitution of V_0 in (77) gives $\partial V_1/\partial y = y + z^3$, and hence the "truncation" of the cheap control

$$\hat{u}(y, z, \varepsilon) = -\frac{1}{\varepsilon}\frac{\partial V_1}{\partial y} = -\frac{1}{\varepsilon}\left(y + z^3\right)\,. \tag{69}$$

Making the change of coordinates $y \mapsto \eta = y + z^3$ in the closed-loop system (74), (72), results in the singular perturbed system

$$\begin{aligned} \varepsilon\dot{\eta} &= -\eta + \varepsilon\left[z^4 - \frac{3}{2}z^5 + 3z^2\eta\right]\,, \\ \dot{z} &= -\frac{z^3}{2} + \eta\,. \end{aligned} \tag{70}$$

The derivative of the Lyapunov function $W(\eta, z) = \eta^2/2 + V_0(z)$ along (70)

$$\dot{W} = \begin{bmatrix} \eta \\ z \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -1 + \varepsilon z^2 & \frac{z^2}{2} + \varepsilon\left(\frac{1}{2} - \frac{3}{4}z^4\right) \\ \frac{z^2}{2} + \varepsilon\left(\frac{1}{2} - \frac{3}{4}z^4\right) & -\frac{z^4}{2} \end{bmatrix} \begin{bmatrix} \eta \\ z \end{bmatrix}$$

We conclude that the optimal limiting cost for the cheap control problem is

$$\lim_{\varepsilon \to 0} V(y, z, \varepsilon) = \frac{z^4}{4} .$$

The "minimum-energy" output (cf. (38))

$$y = \alpha_0(z) = -z^3 . \tag{71}$$

Using V_0 from (78) in (77) we obtain $\frac{\partial V_1}{\partial y} = y + z^3$, yielding the optimal cheap control

$$\hat{u}(y, z, \varepsilon) = -\frac{1}{\varepsilon}\frac{\partial^{\mathrm{T}} V_1}{\partial y} = -\frac{1}{\varepsilon}\left(y + z^3\right) . \tag{72}$$

Making the change of coordinates $y \to \eta = y - \alpha_0(z) = y + z^3$ in the closed-loop system (74), (72), results in the singular perturbed system

$$\begin{aligned} \varepsilon\dot{\eta} &= -\eta + \varepsilon\left[z - \frac{3}{2}z^5 + 3z^2\eta\right] , \\ \dot{z} &= -\frac{z^3}{2} + \eta . \end{aligned} \tag{73}$$

The derivative of the Lyapunov function $W(\eta, z) = \eta^2/2 + V_0(z)$ along (70)

$$\dot{W} = \begin{bmatrix} \eta \\ z \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -1 + \varepsilon z^2 & \frac{z^2}{2} + \varepsilon\left(\frac{1}{2} - \frac{3}{4}z^4\right) \\ \frac{z^2}{2} + \varepsilon\left(\frac{1}{2} - \frac{3}{4}z^4\right) & -\frac{z^4}{2} \end{bmatrix} \begin{bmatrix} \eta \\ z \end{bmatrix}$$

We conclude that the optimal limiting cost for the cheap control problem is

$$\lim_{\varepsilon \to 0} V(y, z, \varepsilon) = \frac{z^4}{4} .$$

The linearizarion around zero of system (74)

$$\begin{aligned} \dot{y} &= z + u, \\ \dot{z} &= y, \end{aligned}$$

has a zero on the imaginary axis and hence can be asymptotically stabilized by cheap control with a cost that tends to zero as we increase the gain. This is easily seen by computing the leading term of the solution to the ARE (4)

$$P = \begin{bmatrix} \varepsilon\sqrt{1 + 4\varepsilon^2} & 2\varepsilon^2 \\ 2\varepsilon^2 & \varepsilon\sqrt{1 + 4\varepsilon^2} \end{bmatrix} ,$$

which vanishes as $\varepsilon \to 0$. Hence, an analysis based on the linear approximation of the system will not indicate any limitation in achieving an arbitrarily small cost.

Example 3.2. To solve the cheap control problem for the system

$$\dot{y} = z^4 + u\,, \qquad \dot{z} = \frac{z^3}{2} + y\,, \tag{74}$$

we need to find a positive semi-definite function V satisfying the HJB equation

$$\frac{\partial V}{\partial y} z^4 + \frac{\partial V}{\partial z}\left[\frac{z^3}{2} + y\right] + \frac{1}{2}y^2 - \frac{1}{2\varepsilon^2}\left(\frac{\partial V}{\partial y}\right)^2 = 0\,, \tag{75}$$

and such that the control

$$u = -\frac{1}{\varepsilon^2}\frac{\partial V}{\partial y} \tag{76}$$

asymptotically stabilizes (74) when $\varepsilon \to 0$.

To resolve the singularity of (75) at $\varepsilon = 0$, we seek a solution in the form $\hat{V}(y, z, \varepsilon) = V_0(z) + \varepsilon V_1(y, z) + O(\varepsilon^2)$, by determining appropriate functions V_0 and V_1. The substitution of $\hat{V}$ into (75) yields

$$\frac{\partial V_0}{\partial z}\left[\frac{z^3}{2} + y\right] + \frac{1}{2}y^2 - \frac{1}{2}\left[\frac{\partial V_1}{\partial y}\right]^2 + O(\varepsilon) = 0\,. \tag{77}$$

A careful inspection of (77) gives conditions for a suitable V_0: the first two terms of (77) are the Hamiltonian corresponding to the optimization problem: Find the minimum energy (L_2-norm) output y that asymptotically stabilizes the zero-dynamics subsystem $\dot{z} = z^3/2 + y$. The HJB equation for this problem

$$\frac{\partial V_0}{\partial z}\frac{z^3}{2} - \frac{1}{2}\left[\frac{\partial V_0}{\partial z}\right]^2 = 0\,, \quad V_0(0) = 0\,,$$

has the positive definite solution

$$V_0 = \frac{z^4}{4}\,, \tag{78}$$

The linearizarion around zero of system (74)

$$\begin{aligned} \dot{y} &= z + u, \\ \dot{z} &= y, \end{aligned}$$

has a zero on the imaginary axis and hence can be asymptotically stabilized by cheap control with a cost that tends to zero as we increase the gain. This is easily seen by computing the leading term of the solution to the ARE (4)

$$P = \begin{bmatrix} \varepsilon\sqrt{1+4\varepsilon^2} & 2\varepsilon^2 \\ 2\varepsilon^2 & \varepsilon\sqrt{1+4\varepsilon^2} \end{bmatrix},$$

which vanishes as $\varepsilon \to 0$. Hence, an analysis based on the linear approximation of the system will not indicate any limitation in achieving an arbitrarily small cost.

4 Conclusions

We have linked the two sets of results that quantify feedback limitations in linear control systems. The T-invariant is the minimum cost achieved by the cheap control problem of transferring the system output to a unit setpoint.

After establishing this link for linear systems, we have shown that the cheap control problem for nonlinear systems becomes, in the limiting case, the minimum energy problem for the zero dynamics driven by the output of the system. Thus the lowest achievable cost is the amount of control energy needed to stabilize the unstable zero dynamics. For nonlinear systems this a fundamental limitation of feedback analogous to that given by the T-invariant.

Our analysis has also provided a definite answer to the heretofore unanswered question whether the feedback limitation for linear systems, expressed by the T-invariant, can be reduced (or even eliminated) with an unconstrained nonlinear controller. The answer to this question is negative, because the analysis of the cheap control problem proves that the lowest cost achieved with a linear controller cannot be further reduced by any nonlinear controller. This limitation of feedback control is caused by the zero dynamics in both linear and nonlinear systems.

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