Conjectures and Counterexamples on Optimal L_2 Disturbance Attenuation in Nonlinear Systems*

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Abstract

This paper considers the problem of optimal L_2 disturbance attenuation with global asymptotic stability for strict feedback nonlinear systems. It is known from previous results that this problem cannot be solved with an arbitrary level of disturbance attenuation (almost disturbance decoupling) if the disturbance input drives unstable zero dynamics of the system. In this case, the problem can only be solved to achieve a level of disturbance attenuation above a nonzero optimal bound. An explicit expression of this lowest optimal bound is known for linear systems, and an approximate bound exists for a special subclass of nonlinear systems with second order zero dynamics. A more general expression for the lowest bound remains unknown. In this paper we provide background to the problem, and discuss the feasibility of obtaining such a general expression by presenting a series of conjectures, examples and counterexamples. We first present a conjecture that might appear as a natural generalisation of the linear expression but that, as we show by means of a counterexample, is generally false. Finally, we present a second conjecture, which holds generally for the linear case, and also for a class of scalar nonlinear systems. A general proof, or a counterexample, to this conjecture are still questions open to further research.

1 Introduction

The problem of optimal L_2 disturbance attenuation with global asymptotic stability (GAS) for the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + p(x)w\ , && x \in \mathbb{R}^n, u \in \mathbb{R}, \\ y &= h(x)\ , && y \in \mathbb{R}\ , w \in \mathbb{R}, \end{aligned}$$

is that of finding a control law $u = u(x)$ such that the equilibrium at $x = 0$ of the closed loop system is globally asymptotically stable, and the system has an L_2 gain, from the exogenous disturbance input w to the regulated output y, that is less than or equal to a prescribed level of attenuation $\gamma > 0$, i.e.,

$$\int_0^t |y(\tau)|^2 d\tau \leq \gamma^2 \int_0^t |w(\tau)|^2 d\tau, \tag{1}$$

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for all $t \geq 0$ and zero initial state. When $\gamma > 0$ is arbitrary, the problem is known as that of almost disturbance decoupling with GAS.

For linear systems, the solution of the problem of disturbance decoupling with internal stability has been known for some time. In particular, in Willems (1981), it was shown that a necessary and sufficient condition for the existence of a solution is that the disturbance w does not affect the part of the system's dynamics associated with the closed right half plane zeros of the system. An equivalent statement of this condition is that the disturbance does not affect the unstable part of the system's *zero dynamics* (as defined in Isidori (1995) for nonlinear systems). More specifically, on using the *special coordinate basis* of Sannuti and Saberi (1987), the system can be represented by

$$\begin{aligned} \dot{z}_s &= A_s z_s + B_s \xi_1 + G_s w \ , \\ \dot{z}_u &= A_u z_u + B_u \xi_1 + G_u w \ , \\ \dot{\xi}_1 &= \xi_2 + P_1 w \\ \dot{\xi}_2 &= \xi_3 + P_2 w \\ &\cdots \\ \dot{\xi}_r &= u + P_r w \\ y &= \xi_1 \end{aligned} \tag{2}$$

where the eigenvalues of the matrices A_s and A_u are the system's zeros in $\mathbb{C}^-$ and $\bar{\mathbb{C}}^+$ respectively. Based on a result of Scherer (1992), Schwartz et al. (1996) presented a formula to compute the optimal value of γ, which, if the system has no zeros with zero real part (i.e., all eigenvalues of A_u lie in $\mathbb{C}^+$)[1] reduces to

$$\gamma^* \triangleq \left\{ \lambda_{\max}\{\mathcal{L}_c^{-1}\mathcal{L}_d\} \right\}^{\frac{1}{2}} , \tag{3}$$

where $\mathcal{L}_c$ and $\mathcal{L}_d$ are the control and disturbance Gramians

$$\begin{aligned} \mathcal{L}_c &\triangleq \int_0^\infty e^{-A_u t} B_u B_u^T e^{-A_u^T t} \, dt, \\ \mathcal{L}_d &\triangleq \int_0^\infty e^{-A_u t} G_u G_u^T e^{-A_u^T t} \, dt. \end{aligned}$$

Hence, we see from (3) that $\gamma^* = 0$ if and only if $\mathcal{L}_d = 0 \Leftrightarrow G_u = 0$, i.e., the disturbance does not enter the unstable zero dynamics of the system (2).[2] Otherwise, if $G_u \neq 0$, then the disturbance attenuation problem with internal stability can only be solved to a prescribed level of attenuation $\gamma > \gamma^* > 0$, and the optimal lower bound γ^* thus quantifies a fundamental obstruction to performance of the system.

The problem of disturbance attenuation for the class of *strict feedback* nonlinear systems, given by the lower triangular structure

$$\begin{aligned} \dot{z} &= f_0(z, \xi_1) + p_0(z, \xi_1) w \\ \dot{\xi}_1 &= \xi_2 + p_1(z, \xi_1) w \\ \dot{\xi}_2 &= \xi_3 + p_2(z, \xi_1, \xi_2) w \\ &\cdots \\ \dot{\xi}_r &= u + p_r(z, \xi_1, \xi_2, \ldots, \xi_r) w \\ y &= \xi_1, \end{aligned} \tag{4}$$

has been extensively studied in the series of papers Marino et al. (1994); Isidori (1996b,a); Lin (1998). The strict feedback structure generalises that of a linear system in the special coordinate basis (2)

[1] See Schwartz et al. (1996); Isidori et al. (1999) for the general case.

[2] Note, on the other hand, that $\mathcal{L}_c$ must be full rank for the system to be stabilisable.

to nonlinear systems. As shown in these papers, disturbance attenuation to a given level γ can be achieved for the system (4) if $\exists\ \xi_1 = v^*(z)$, $v^*(0) = 0$, and a smooth proper function $V(z) > 0$ such that

$$\frac{\partial V}{\partial z} f_0(z, v^*(z)) + \frac{1}{2\gamma^2}\left[\frac{\partial V}{\partial z} p_0(z, v^*(z))\right]^2 + [v^*(z)]^2 < 0\ . \tag{5}$$

Thus, the problem reduces to a disturbance attenuation problem for the zero dynamics *with cost on the control*, i.e., the Hamilton Jacobi Issacs (HJI) Equation (5) corresponds to finding an optimal "control" $\xi_1 = \xi(z)$ achieving global asymptotic stability of the perturbed zero dynamics subsystem

$$\dot{z} = f_0(z, \xi) + p_0(z, \xi) w \tag{6}$$

minimising the L_2 gain from the disturbance w to the "control" ξ to a level γ, as specified in (1).

In Isidori et al. (1999), the authors show that if the zero dynamics can be split into *stable* and *unstable*

$$\begin{aligned} \dot{z}_s &= f_s(z_s, z_+, y) + p_s(z_s, z_+, y) w \\ \dot{z}_u &= f_u(z_u, y) + p_u(z_u, y) w\ , \end{aligned} \tag{7}$$

then the problem can be solved if it can be solved for *the unstable component* of the zero dynamics. They also define a structure for second order nonlinear zero dynamics, for which they can provide an upper bound of the optimal value γ^*.

However, an expression or procedure to compute the value of γ^* more generally for nonlinear systems remains unknown.

In this paper we concentrate on the class of nonlinear systems in which the unstable part of the zero dynamics, Equation (7), is affine on y and w, i.e., it can be written in the form

$$\dot{z}_u = f_u(z_u) + g_u(z_u) y + p_u(z_u) w.$$

We present several conjectures for the generalisation of the expression for γ^* in (3) to this class of systems. In each case we provide examples which validate or contradict the conjecture.

More specifically, the expression (3) of γ^* for linear systems is the spectral radius of a matrix involving the control and disturbance controllability Gramians. A controllability Gramian, in turn, is associated with the solution to a minimum energy optimal control problem. Hence, it might seem a natural generalisation to conjecture an equivalent expression for nonlinear systems as the ratio of two optimal value functions corresponding to minimum energy optimal control problems (§ 2). Unfortunately, this first conjecture, although valid in the linear case and some nonlinear examples, is generally false, as we show with counterexamples in § 3. Therefore, inspired on the same idea, we propose in § 4 a second conjecture for an expression of γ^*, which is not only valid for linear systems, but also generally for scalar nonlinear systems. Nevertheless, we have not as yet been able to invalidate or prove this conjecture more generally, which remains as a question open to further research.

2 A Fallacious Conjecture

We focus on the class of nonlinear systems described by

$$\dot{z} = f_0(z) + g_0(z) u + p_0(z) w \tag{8}$$

where f_0 is assumed to be *antistable*, i.e. $\dot{z} = -f_0(z)$ is assumed to be GAS, $u \in \mathbb{R}$ is the control input, and $w \in \mathbb{R}$ the external disturbance input. The system (8) corresponds to the zero dynamics of a strict feedback system as in (4). The problem of determining whether a prescribed L_2 attenuation level γ from the disturbance input w to u is achievable for the system (8) can be reduced to finding whether a well behaved (i.e. positive, sufficiently smooth) $V_\gamma(z)$ exists which satisfies the HJI Equation

$$\frac{\partial V_\gamma}{\partial z} f_0(z) - \frac{1}{2}\frac{\partial V_\gamma}{\partial z} g_0(z) g_0(z)^T \frac{\partial V_\gamma^T}{\partial z} + \frac{1}{2\gamma^2}\frac{\partial V_\gamma}{\partial z} p_0(z) p_0(z)^T \frac{\partial V_\gamma^T}{\partial z} = 0. \tag{9}$$

Because (8) is open loop unstable, we know that it is impossible to achieve disturbance attenuation to a level $\gamma \to 0$; instead, disturbance attenuation can only be achieved to a positive level γ greater than a lower optimal value $\gamma^* > 0$. We wish to obtain an expression characterising γ^* that generalises the formula (3) to this class of nonlinear systems. To do so, we start by revisiting the linear case.

Let us consider the case in which the system is linear, in which (8) reduces to

$$\dot{z} = A_0 z + B_0 u + G_0 w. \tag{10}$$

The formula (3) for γ^* in the linear case can be alternatively expressed as

$$(\gamma^*)^2 = \sup_z \left\{ \frac{V_c(z)}{V_d(z)} \right\} \tag{11}$$

where

$$V_c(z) = z^T \mathcal{L}_c^{-1} z, \quad \text{and} \quad V_d(z) = z^T \mathcal{L}_d^{-1} z \tag{12}$$

are the value functions of two *minimum energy* optimal control problems, one for the control input u and the other for the disturbance input w, consisting in

1. Find $u = u(z)$ minimising

$$J_c = \int_0^\infty |u(t)|^2 dt$$

   and such that the closed loop system

$$\dot{z} = A_0 z + B_0 u$$

   is GAS.

2. Find $w = w(z)$ minimising

$$J_d = \int_0^\infty |w(t)|^2 dt$$

   and such that the closed loop system

$$\dot{z} = A_0 z + G_0 w$$

   is GAS.

The matrices $\mathcal{L}_c^{-1}$ and $\mathcal{L}_d^{-1}$ in the minimum energy value functions $V_c(z)$ and $V_d(z)$ are the unique positive definite solutions to the Riccati Equations

$$\mathcal{L}_c^{-1} A_0 + A_0^T \mathcal{L}_c^{-1} - \mathcal{L}_c^{-1} B_0 B_0^T \mathcal{L}_c^{-1} = 0 \tag{13}$$

$$\mathcal{L}_d^{-1} A_0 + A_0^T \mathcal{L}_d^{-1} - \mathcal{L}_d^{-1} G_0 G_0^T \mathcal{L}_d^{-1} = 0. \tag{14}$$

Hence, it would seem reasonable to conjecture the general expression of γ^* for the nonlinear system (8) using the same formula (11), where $V_c(z)$ and $V_d(z)$ are the (sufficiently smooth) positive definite solutions to Hamilton Jacobi Bellman (HJB) equations generalising the Riccati Equations (13) and (14) to nonlinear systems.

Conjecture 1 *The optimal level of L_2 disturbance attenuation from w to u for the system* (8) *is given by*

$$(\gamma^*)^2 = \sup_z \left\{ \frac{V_c(z)}{V_d(z)} \right\} \tag{15}$$

where $V_c(z)$ and $V_d(z)$ are the (sufficiently smooth) positive definite solutions the HJB Equations

$$\frac{\partial V_c}{\partial z} f_0(z) - \frac{1}{2} \frac{\partial V_c}{\partial z} g_0(z) g_0(z)^T \frac{\partial V_c^T}{\partial z} = 0, \tag{16}$$

$$\frac{\partial V_d}{\partial z} f_0(z) - \frac{1}{2} \frac{\partial V_d}{\partial z} p_0(z) p_0(z)^T \frac{\partial V_d^T}{\partial z} = 0, \tag{17}$$

which correspond to solution of the the minimum energy optimal control problems

1. *Find $u = u(z)$ minimising*

   $$J_c = \int_0^\infty |u(t)|^2 dt$$

   and such that the closed loop system

   $$\dot{z} = f_0(z)z + g_0(z)u$$

   is GAS.

2. *Find $w = w(z)$ minimising*

   $$J_d = \int_0^\infty |w(t)|^2 dt$$

   and such that the closed loop system

   $$\dot{z} = f_0(z) + p_0(z)w$$

   is GAS.

For linear systems the conjecture is obviously correct, as is easy to see if we let $V_c(z) = z^T \mathcal{L}_c^{-1} z$ and $V_d(z) = z^T \mathcal{L}_d^{-1} z$, and define

$$V_\gamma(z) = z^T \left(\mathcal{L}_c^{-1} - \tfrac{1}{\gamma^2} \mathcal{L}_d^{-1} \right) z. \tag{18}$$

The function $V_\gamma(z)$ of (18) is a solution of the HJI equation (9), and it is positive definite if $\forall z \neq 0$

$$z^T \left(\mathcal{L}_c^{-1} - \tfrac{1}{\gamma^2} \mathcal{L}_d^{-1} \right) z > 0 \quad \Leftrightarrow \quad \gamma^2 > \sup_{z \neq 0} \frac{z^T \mathcal{L}_c^{-1} z}{z^T \mathcal{L}_d^{-1} z} = \sup_{x = \mathcal{L}_d^{-1/2} z \neq 0} \frac{x^T \mathcal{L}_d^{1/2} \mathcal{L}_c^{-1} \mathcal{L}_d^{1/2} x}{x^T x}$$

$$= \lambda_{\max} \left\{ \mathcal{L}_c^{-1} \mathcal{L}_d \right\} = (\gamma^*)^2,$$

namely, the formula (3) for γ^* of Isidori et al. (1999).

It could be argued that Conjecture 1 has some appeal, in that the formula (15) for the optimal value of disturbance rejection can be interpreted as a ratio between the minimum control energies required by the control and the disturbance inputs to independently stabilise the system.

Unfortunately, as we will show in the next section, Conjecture 1 is in general false. Nevertheless, we note that for some classes of nonlinear systems the conjecture does hold, as is the case of the following example.

2.1 A first order example validating Conjecture 1

Consider the system

$$\dot{z} = z + u + \frac{1}{1+z^2}w. \tag{19}$$

In this case, the HJI equation (9) becomes

$$\begin{aligned}
&\frac{\partial V_\gamma}{\partial z}z - \frac{1}{2}\left(\frac{\partial V_\gamma}{\partial z}\right)^2 + \frac{1}{2\gamma^2}\left(\frac{\partial V_\gamma}{\partial z}\right)^2\left(\frac{1}{1+z^2}\right)^2 = 0 \\
\Leftrightarrow \quad & 2z - \frac{\partial V_\gamma}{\partial z}\left(1 - \frac{1}{\gamma^2}\frac{1}{(1+z^2)^2}\right) = 0 \\
\Leftrightarrow \quad & \frac{\partial V_\gamma}{\partial z} = \frac{2\gamma^2 z\left(1+z^2\right)^2}{\gamma^2\left(1+z^2\right)^2 - 1}.
\end{aligned} \tag{20}$$

Provided $\partial V_\gamma / \partial z$ is positive and well defined for all z, the RHS of Equation (20) can be integrated to give a suitable value function V_γ. It turns out that this is so if $\gamma > 1 = \gamma^*$, and we obtain

$$V_\gamma(z) = z^2 + \frac{1}{2\gamma}\log\left(\frac{\gamma z^2 + \gamma - 1}{\gamma z^2 + \gamma + 1}\right).$$

On the other hand, we can now solve for the control minimum energy problem (16), which fairly trivially gives

$$V_c(z) = z^2, \tag{21}$$

and also for the disturbance minimum energy problem (17), which gives

$$\begin{aligned}
&\frac{\partial V_d}{\partial z}z - \frac{1}{2}\left(\frac{\partial V_d}{\partial z}\right)^2\left(\frac{1}{1+z^2}\right)^2 = 0 \\
\Leftrightarrow \quad & z - \frac{1}{2}\frac{\partial V_d}{\partial z}\left(\frac{1}{1+z^2}\right)^2 = 0 \\
\Leftrightarrow \quad & \frac{\partial V_d}{\partial z} = 2z\left(1+z^2\right)^2 \\
\Leftrightarrow \quad & V_d(z) = z^2 + z^4 + \frac{1}{3}z^6.
\end{aligned} \tag{22}$$

From (21) and (22) we see that the supremum of $V_c(z)/V_d(z)$ occurs at $z = 0$ and matches $\gamma^* = 1$, validating the conjectured expression (15).

2.2 A second order example validating Conjecture 1

Consider the system

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{bmatrix} z_1^3 + z_2 \\ z_2^3 - z_1 + 2z_1^2 z_2\end{bmatrix} + \begin{bmatrix}z_1\\ z_2\end{bmatrix}u + \begin{bmatrix}z_1\\ z_2\end{bmatrix}\sqrt{\frac{1+z_1^2+z_2^2}{2+z_1^2+z_2^2}}w. \tag{23}$$

It is not difficult to verify that the function

$$V_\gamma(z) = \frac{\gamma^2}{\gamma^2-1}\left(z_1^2 + z_2^2 - \frac{1}{\gamma^2-1}\log\left[\frac{\gamma^2 + (\gamma^2-1)(1+z_1^2+z_2^2)}{2\gamma^2-1}\right]\right)$$

satisfies the corresponding HJI Equation (9), and is well defined and positive definite for all $\gamma > 1 = \gamma^*$. On the other hand, from the minimum energy HJB Equations (16) and (17) for u and w, we obtain

$$V_c(z) = z_1^2 + z_2^2, \quad \text{and} \quad V_d(z) = z_1^2 + z_2^2 + \log(1 + z_1^2 + z_2^2).$$

Hence

$$\sup_z \frac{V_c(z)}{V_d(z)} = 1,$$

validating Conjecture 1.

3 Counterexamples to Conjecture 1

Conjecture 1 is however generally false. In this section we present two counterexamples that invalidate (15). The first example illustrates a system for which V_γ is well defined for all $\gamma > \gamma^*$, but for which the minimum energy function V_d is not well defined, and hence (15) cannot be computed. The second example presents a system for which all V_γ, V_c and V_d are well defined, but the value of γ^* is different to the conjectured value (15).

3.1 Example: γ^* exists, but V_d is not well behaved.

In this example, we illustrate that if $p_0(z)$ in (8) tends to zero as $z \to 0$, then a well behaved V_d may not exist, even though a γ^* does. Consider the system

$$\dot{z} = z + u + \frac{z}{1+z^2}w. \tag{24}$$

In this case, the HJI Equation (9) becomes

$$\frac{\partial V_\gamma}{\partial z}z - \frac{1}{2}\left(\frac{\partial V_\gamma}{\partial z}\right)^2 + \frac{1}{2\gamma^2}\left(\frac{\partial V_\gamma}{\partial z}\right)^2\left(\frac{z}{1+z^2}\right)^2 = 0 \tag{25}$$

$$\Leftrightarrow \quad 2z - \frac{\partial V_\gamma}{\partial z}\left(1 - \frac{1}{\gamma^2}\frac{z^2}{\left(1+z^2\right)^2}\right) = 0$$

$$\Leftrightarrow \quad \frac{\partial V_\gamma}{\partial z} = \frac{2z\gamma^2\left(1+z^2\right)^2}{\gamma^2\left(1+z^2\right)^2 - z^2}. \tag{26}$$

Now, provided $\partial V_\gamma/\partial z$ is positive, and well defined for all $z > 0$ (and conversely negative, and well defined for all $z < 0$) the RHS of Equation (26) can be integrated to give a suitable value function V_γ. It turns out that provided $\gamma > \frac{1}{2}$ then the RHS of Equation (26) is positive and well defined for all $z > 0$. Similar arguments can be made for negative z.

On the other hand, if we try to solve for V_d we get

$$\frac{\partial V_d}{\partial z}z - \frac{1}{2}\left(\frac{\partial V_d}{\partial z}\right)^2\left(\frac{z}{1+z^2}\right)^2 = 0$$

$$\Leftrightarrow \quad z - \frac{1}{2}\frac{\partial V_d}{\partial z}\left(\frac{z}{1+z^2}\right)^2 = 0$$

$$\Leftrightarrow \quad \frac{\partial V_d}{\partial z} = \frac{2\left(1+z^2\right)^2}{z}. \tag{27}$$

We see from (27) that there is no $\mathcal{C}^1$ solution for V_d valid in a neighbourhood of $z = 0$.

3.2 Example: $V_\gamma(z)$, V_c and V_d exist, Conjecture 1 invalid

In this example we illustrate a case where Conjecture 1 is invalid, even though both V_c and V_d exist. Consider the system

$$\dot{z} = z + u + \left(\frac{1+z^2}{1+\frac{1}{3}z^4}\right)w \tag{28}$$

In this case, the HJI Equation (9) becomes

$$\frac{\partial V_\gamma}{\partial z}z - \frac{1}{2}\left(\frac{\partial V_\gamma}{\partial z}\right)^2 + \frac{1}{2\gamma^2}\left(\frac{\partial V_\gamma}{\partial z}\right)^2\left(\frac{1+z^2}{1+\frac{1}{3}z^4}\right)^2 = 0 \tag{29}$$

$$\Leftrightarrow \qquad 2z - \frac{\partial V_\gamma}{\partial z}\left(1 - \frac{1}{\gamma^2}\left(\frac{1+z^2}{1+\frac{1}{3}z^4}\right)^2\right) = 0$$

$$\Leftrightarrow \qquad \frac{\partial V_\gamma}{\partial z} = \frac{2z\gamma^2\left(1+\frac{1}{3}z^4\right)^2}{\gamma^2\left(1+\frac{1}{3}z^4\right)^2 - \left(1+z^2\right)^2}\,. \tag{30}$$

Provided $\partial V_\gamma/\partial z$ is positive and well defined for all $z > 0$, the RHS of Equation (30) can be integrated to give a suitable value function V_γ. It turns out that provided $\gamma > \frac{3}{2}$ then the RHS of Equation (30) is positive and well defined for all $z > 0$. Similar arguments can be made for negative z.

The control minimum energy problem (16) gives $V_c(z) = z^2$. Also, we solve (17) for V_d and get

$$\frac{\partial V_d}{\partial z}z - \frac{1}{2}\left(\frac{\partial V_d}{\partial z}\right)^2\left(\frac{1+z^2}{1+\frac{1}{3}z^4}\right)^2 = 0$$

$$\Leftrightarrow \qquad z - \frac{1}{2}\frac{\partial V_d}{\partial z}\left(\frac{1+z^2}{1+\frac{1}{3}z^4}\right)^2 = 0$$

$$\Leftrightarrow \qquad \frac{\partial V_d}{\partial z} = \frac{2z\left(1+\frac{1}{3}z^4\right)^2}{\left(1+z^2\right)^2}$$

$$\Leftrightarrow \qquad V_d(z) = \frac{1}{27}z^6 - \frac{1}{9}z^4 + z^2 + \frac{16z^2}{9\left(1+z^2\right)} - \frac{16}{9}\ln\left(1+z^2\right). \tag{31}$$

From (31) we see that

$$\frac{V_c(z)}{V_d(z)} = \frac{z^2}{\frac{1}{27}z^6 - \frac{1}{9}z^4 + z^2 + \frac{16z^2}{9(1+z^2)} - \frac{16}{9}\ln\left(1+z^2\right)} \tag{32}$$

which is plotted on Figure 1. The turning point can be found numerically to be at $z = 1.3192$ and the value of the maximum is (again numerically):

$$\sup_z\left(\frac{V_c(z)}{V_d(z)}\right) = \left.\frac{V_c(z)}{V_d(z)}\right|_{z=1.3192} = 1.8595 \tag{33}$$

However, $(\gamma^*)^2 = 2.25$ which does not match (33).

4 A Second Conjecture

For the sorts of examples considered, Conjecture 1 seems to work if $\sup_z\left(V_c(z)/V_d(z)\right)$ is achieved at either $z = 0$ or $z \to \infty$. In the example of § 3.2, the supremum is achieved elsewhere, and the formula (11) does not hold. We propose below Conjecture 2 as an alternative way to extend the linear formula (3). This second conjecture gives the correct result for the counterexamples to Conjecture 1, as well as for the general linear case and the scalar nonlinear case ($z \in \mathbb{R}$). In addition, we present in this section two second order nonlinear examples, where the disturbance enters "matched" and "unmatched", which also validate Conjecture 2. To present, we have not been able yet to find a proof or a counterexample to this conjecture.

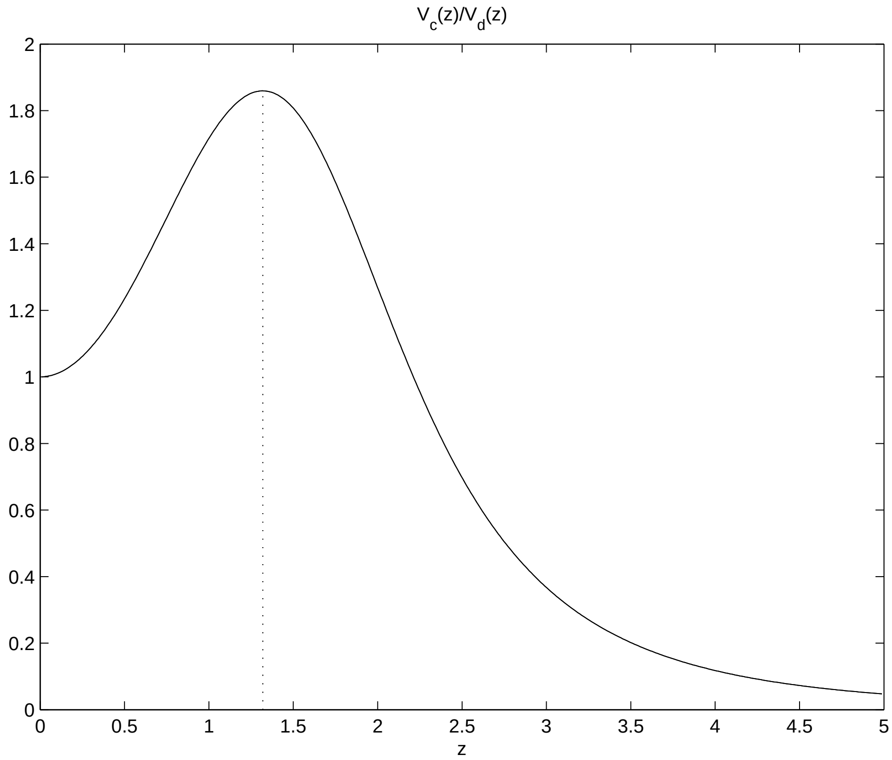

Figure 1: Plot of V_c/V_d

Conjecture 2 *The optimal level of L_2 disturbance attenuation from w to u for the system* (8) *is given by*

$$(\gamma^*)^2 = \sup_z \left(\frac{\frac{\partial V_c(z)}{\partial z} z}{\frac{\partial V_d(z)}{\partial z} z} \right) \tag{34}$$

where $V_c(z)$ and $V_d(z)$ are the (sufficiently smooth) positive definite solutions the HJB Equations (16) *and* (17)*, which correspond to solution of the the minimum energy optimal control problems defined in Conjecture 1 for the control and disturbance inputs.*

In the linear case, $V_c(z)$ and $V_d(z)$ are both quadratic in z and the RHS of (34) is the same as that of (11), thus Conjecture 2 also holds for linear systems.

4.1 Scalar systems

For general scalar systems, i.e., where in (8) $z \in \mathbb{R}$, Conjecture 2 is valid. Indeed, Equation (34) then reduces to

$$(\gamma^*)^2 = \sup_z \left(\frac{\frac{\partial V_c(z)}{\partial z}}{\frac{\partial V_d(z)}{\partial z}} \right), \tag{35}$$

and the corresponding HJI and HJB Equations (9), (16) and (17) respectively reduce to

$$f_0(z) = \frac{1}{2} \left(g_0^2(z) - \frac{1}{\gamma^2} p_0^2(z) \right) \frac{\partial V_\gamma(z)}{\partial z}, \tag{36}$$

$$f_0(z) = \frac{1}{2} \left(g_0^2(z) \right) \frac{\partial V_c(z)}{\partial z}, \tag{37}$$

$$f_0(z) = \frac{1}{2} \left(p_0^2(z) \right) \frac{\partial V_d(z)}{\partial z}. \tag{38}$$

From (36) it follows that

$$\frac{\partial V_\gamma(z)}{\partial z} = \frac{f_0(z)}{\frac{1}{2}\left(g_0^2(z) - \frac{1}{\gamma^2}p_0^2(z)\right)} \tag{39}$$

Since by assumption, $\dot{z} = -f_0(z)$ is GAS, it follows that (39) can be integrated to give a positive definite function V_γ if and only if

$$\gamma > \gamma^* \triangleq \sqrt{\sup_z \left(\frac{p_0^2(z)}{g_0^2(z)}\right)}. \tag{40}$$

However, note also from (37) and (38) that

$$\begin{aligned} \frac{\partial V_c(z)}{\partial z} &= \frac{f_0(z)}{\frac{1}{2}\left(g_0^2(z)\right)}, && (41)\\ \frac{\partial V_d(z)}{\partial z} &= \frac{f_0(z)}{\frac{1}{2}\left(p_0^2(z)\right)}, && (42) \end{aligned}$$

from which we conclude that Conjecture (34) is true in the scalar case.

4.2 Additive input disturbance

If the disturbance enters additively with the control input, then the value of γ^* is trivially found provided that there is a smooth positive definite solution to the minimum energy control problem associated with (16). In this case, both Conjectures 1 and 2 will correctly predit the value of γ^*.

More specifically, let

$$\dot{z} = f_0(z) + g_0(z)(u + \alpha w)$$

for some $\alpha \in \mathbb{R}$. Suppose that there exists a sufficiently smooth positive definite solution V_c to the HJB Equation (16). It can then be verified that $V_d = \frac{1}{\alpha^2}V_c$ is a well defined solution to the HJB Equation (17), and

$$V_\gamma(z) = \frac{\gamma^2}{\gamma^2 - \alpha^2}V_c(z)$$

will be a well defined solution to the HJI Equation (9) for all $\gamma > |\alpha| = \gamma^*$.

For a concrete example take the system given by

$$\begin{aligned} f_0(z_1, z_2) &= \begin{bmatrix} z_1 \\ 2z_1 + 2z_2^3 \end{bmatrix} \\ p_0^T &= g_0^T = \begin{bmatrix} 1 & 1 \end{bmatrix}. \end{aligned} \tag{43}$$

Then the positive definite function

$$V_\gamma(z_1, z_2) = \left(\frac{\gamma^2}{\gamma^2 - 1}\right)\left(z_1^2 + z_2^4\right) \tag{44}$$

satisfies the corresponding HJI Equation (9). To test this, we evaluate

$$\frac{\partial V_\gamma}{\partial z} = \left(\frac{\gamma^2}{\gamma^2 - 1}\right)\begin{bmatrix} 2z_1 & 4z_2^3 \end{bmatrix} \tag{45}$$

and so

$$\begin{aligned} \frac{\partial V_\gamma}{\partial z}f_0 &= \left(\frac{\gamma^2}{\gamma^2 - 1}\right)2\left(z_1 + 2z_2^3\right)^2 \\ \frac{\partial V_\gamma}{\partial z}g_0 &= \frac{\partial V_\gamma}{\partial z}p_0 = \left(\frac{\gamma^2}{\gamma^2 - 1}\right)2\left(z_1 + 2z_2^3\right). \end{aligned} \tag{46}$$

From (46), the HJI Equation (9) becomes

$$\begin{aligned} & \frac{\partial V_\gamma}{\partial z} f_0(z) - \frac{1}{2}\frac{\partial V_\gamma}{\partial z} g_0(z) g_0(z)^T \frac{\partial V_\gamma^T}{\partial z} + \frac{1}{2\gamma^2}\frac{\partial V_\gamma}{\partial z} p_0(z) p_0(z)^T \frac{\partial V_\gamma^T}{\partial z} \qquad (47) \\ = & \left(\frac{\gamma^2}{\gamma^2-1}\right) 2\left(z_1 + 2z_2^3\right)^2 - \frac{1}{2}\left(1 - \frac{1}{\gamma^2}\right)\left(\left(\frac{\gamma^2}{\gamma^2-1}\right) 2\left(z_1+2z_2^3\right)\right)^2 \\ = & \left(\frac{\gamma^2}{\gamma^2-1}\right) 2\left(z_1 + 2z_2^3\right)^2 - 2\left(\frac{\gamma^2}{\gamma^2-1}\right)\left(z_1+2z_2^3\right)^2 \\ = & \; 0 \end{aligned}$$

as required. The minimum energy problems have identical solutions, $V_c(z) = V_d(z) = \left(z_1^2 + z_2^4\right)$, and hence both Conjectures 1 and 2 correctly predict $\gamma^* = 1$.

4.3 A non-matched disturbance example

Finally, we present a second order example in which the disturbance enters "unmatched", namely, affecting the state equation on different vector directions to those on which the control acts. This particular example validates both Conjectures 1 and 2. Consider the system (8) with

$$f_0(z_1, z_2) = \begin{bmatrix} z_1^3 + z_1 z_2^2 \\ z_1^2 z_2 + z_2^3 \end{bmatrix}, \quad g_0(z_1, z_2) = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad p_0(z_1, z_2) = \begin{bmatrix} z_1 + z_2^2 \\ z_2 - z_1 z_2 \end{bmatrix}.$$

On defining the function

$$\phi(z) = z_1^2 + z_2^2,$$

it can be verified that

$$V_c(z) = \phi(z), \quad V_d(z) = \phi(z), \quad \text{and} \quad V_\gamma(z) = \frac{\gamma^2}{\gamma^2 - 1}\phi(z)$$

are respectively solutions to the HJB Equations (16) and (17), and the HJI Equation (9). Because $\phi(z)$ is positive definite, for $V_\gamma(z)$ to be positive definite we require that $\gamma > 1 = \gamma^*$. Thus γ^* is as predicted by both Conjectures 1 and 2.

5 Conclusions

We have considered the problem of optimal L_2 disturbance attenuation for non minimum phase strict feedback nonlinear systems in which the disturbance affects the system's unstable zero dynamics. In this case it is impossible to achieve almost disturbance decoupling and the optimal value of disturbance attenuation γ^* must be positive.

In attempting to generalise a formula for γ^* reported in Isidori et al. (1999) for linear systems, we have proposed two conjectures for nonlinear systems. The first conjecture appears as, arguably, a natural extension of the formula for the linear case. However, as we show by counterexample, our first conjecture is in general invalid. Our second conjecture, a modification of the first one, holds generally for linear systems, as well as for *scalar* nonlinear systems. We have presented several examples illustrating that this second conjecture also holds for some second order nonlinear systems with "matching" and "unmatching" disturbance. To present, we have been unable as yet to find a proof or a counterexample to our second conjecture, which remains the subject of further study.

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