Feedback Stabilization over Signal-to-Noise Ratio Constrained Channels*

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Abstract

There has recently been significant interest in feedback stabilization problems over communication channels, including several with bit rate limited feedback. Motivated by considering one source of such bit rate limits, we study the problem of stabilization over a signal-to-noise ratio (SNR) constrained channel. We discuss both continuous and discrete time cases, and show that for either state feedback; or for output feedback delay-free, minimum phase plants, there are limitations on the ability to stabilize an unstable plant over an SNR constrained channel. These limitations in fact match precisely those that might have been inferred by considering the associated ideal Shannon capacity bit rate over the same channel.

1 Introduction

This paper discusses a feedback control system in which the measured information about the plant is fed back to the controller using a noisy channel. Such a setting arises, for example, when sensors are far from the controller and have to communicate through a (perhaps partially wireless) communication network. Feedback control over communication networks has been the general theme of a significant number of recent studies focusing on different aspects of the problem, particularly stabilization with quantization effects and limited communication data rates (e.g., Delchamps, 1990; Tatikonda et al., 1998; Brockett and Liberzon, 2000; Elia and Mitter, 2001; Nair and Evans, 2002, 2003; Ishii and Francis, 2003)

Figure 1 illustrates a basic feedback configuration of this type. Generally, if using digital communications, the link involves some pre- and post-processing of the signals that are sent through a communication channel, for example, filtering, analog-to-digital (A-D) conversion, coding, modulation, decoding, demodulation and digital-to-analog (D-A) conversion.

The case of error and delay free digital communications is a scenario of particular interest studied by Nair and Evans (2003). These authors give a necessary and sufficient condition for the asymptotic feedback stabilizability of a discrete-time LTI system,

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \qquad \forall t = 0, 1, 2, \dots \tag{1}$$

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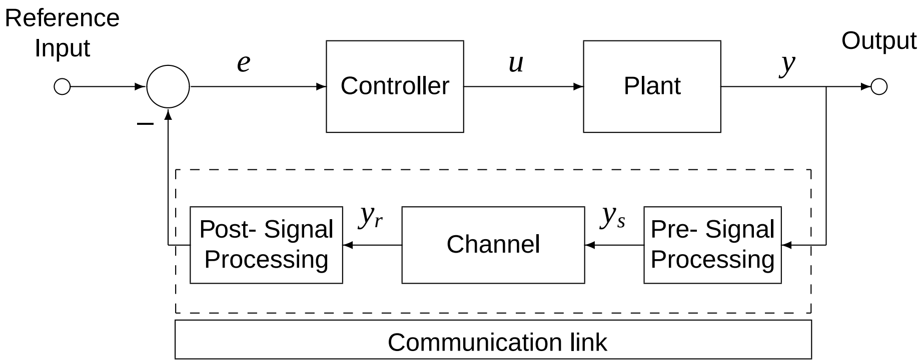

Figure 1: Control system with feedback over a communication link

through a digital channel of limited bit rate capacity. Namely, for stabilization to be feasible, it is necessary and sufficient that the data rate R (bits per interval) satisfies the condition

$$R > \sum_{|\eta_i| \geq 1} \log_2 |\eta_i| \quad \text{bits per interval,} \tag{2}$$

where η_i are the unstable eigenvalues of the matrix A. Nair and Evans obtained this result by considering the stabilization of the noiseless discrete-time system (1) by feedback through a quantized channel in which the quantizer is seen as an information encoder. They showed that the condition (2) is necessary and sufficient for the existence of a coding and control law that gives exponential convergence of the state to the origin from a random initial state.

The main motivation for the work in this paper is the observation that a bit rate limitation may be due to channel signal to noise ratio (SNR) limitations. We therefore consider SNR constrained channels and restrict all pre- and post- signal processing involved in the communication link described above to LTI filtering and D-A and A-D type operations. Thus, the communication link reduces to the noisy channel itself. Our aim in this simplified setting is to quantify the fundamental limitations arising from a simple ideal channel model that embodies two of the fundamental limiting factors in communications: noise and fixed power constraints. Other fundamental limiting factors in the problem of control over a communication link include bandwidth constraints (c.f., Dasgupta, 2003), variable time delays, and missing data and quantization effects, which are beyond the scope of this paper.

The outline of the rest of the paper is as follows. We begin in Section 2 by considering the state feedback continuous-time case. Using a minimum energy formulation, for a given channel noise intensity, we are able to exhibit the minimum signal energy required for stabilization. We follow this by an equivalent result for the output feedback case when the plant is minimum phase. In Section 3 we repeat this analysis for the discrete-time case for both state and output feedback scenarios. Finally, we conclude by discussing possible extensions of these results to performance questions, non-minimum phase plants and other channel limitations.

2 Continuous-Time Feedback Channels

A common model of a continuous-time communication channel is represented in Figure 2. Such a model is characterized by the linear input-output relation

$$r(t) = y(t) + n(t), \qquad t \in \mathbb{R}_0^+,$$

in which $n(t)$ is a continuous-time zero-mean additive white Gaussian noise (AWGN) with intensity Φ, i.e.,

$$E\{n(t)\} = 0, \quad E\{n'(t)n(\tau)\} = \Phi\delta(t-\tau), \tag{3}$$

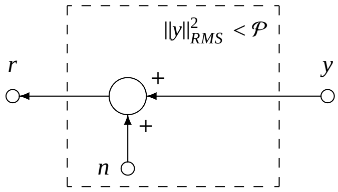

Figure 2: Continuous-time AWGN channel with an input power constraint

where $E\{\cdot\}$ represents the expectation operator, and $\delta(t)$ is the unitary impulse.

The input signal $y(t)$ is assumed a stationary stochastic process with *root mean square (RMS)* value

$$\|y\|_{RMS} = \left(E\{y'(t)y(t)\}\right)^{1/2}.$$

The *power* of the signal y is defined as $\|y\|^2_{RMS}$ and, for our AWGN channel model, is assumed to satisfy the constraint

$$\|y\|^2_{RMS} < \mathcal{P} \tag{4}$$

for some predetermined value $\mathcal{P} > 0$. Such a power constraint may arise either from electronic hardware limitations or regulatory constraints introduced to minimize interference to other communication system users.

As is well-known (e.g., Saberi et al., 1995, pp. 21–22), the power of the continuous-time stochastic signal y can be expressed in terms of the autocorrelation matrix $R_y(\tau) = E\{y(t)y'(t+\tau)\}$, or the power spectral density

$$S_y(\omega) = \int_{-\infty}^{\infty} R_y(\tau)e^{-j\omega\tau}d\tau,$$

as

$$\|y\|^2_{RMS} = \operatorname{trace}\left[R_y(0)\right] = \frac{1}{2\pi}\operatorname{trace}\left[\int_{-\infty}^{\infty} S_y(\omega)\,d\omega\right]. \tag{5}$$

In this section, we will use the notation $\overline{\mathbb{C}}^{+}$ and $\overline{\mathbb{C}}^{-}$ to represent respectively the closed right and left halves of the complex plane $\mathbb{C}$.

2.1 State Feedback Stabilization

We first consider the problem of finding a static state feedback gain K that stabilizes the loop of Figure 3, subject to a constraint on the power of the computed control signal y_s. In this problem we assume the system is described by the state space model[1]

$$\dot{x} = Ax + Bu, \tag{6}$$

where the pair (A, B) is stabilizable and the state x is available for feedback. The matrix state feedback gain K is assumed to asymptotically stabilize the system, and we suppose that the computed control signal y_s is fed back through a AWGN channel with a power constraint $\mathcal{P} \geq \|y_s\|^2_{RMS}$. We formalize the statement of this problem as follows.

Problem 1 (Continuous-time state feedback stabilization with a power constraint).

[1] Note that a mathematically precise treatment of the continuous-time stochastic system would require use of *Ito calculus*, etc. on the stochastic differential equation $dx = Ax\,dt + B\,du$. Under appropriate stationarity assumptions, this formulation reduces to the analysis here (Åström, 1970, §4).

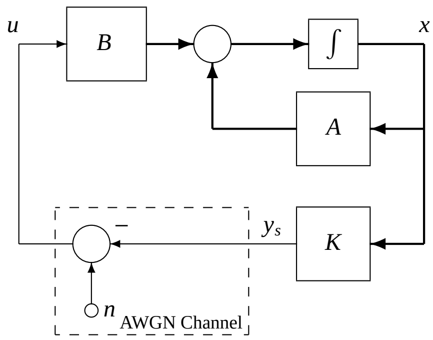

Figure 3: State feedback loop

Find a static state feedback gain K such that the closed loop system

$$\begin{aligned} \dot{x}(t) &= (A - BK)x(t) + Bn(t) \\ y_s(t) &= Kx(t) \end{aligned} \tag{7}$$

is asymptotically stable and, for a zero-mean white Gaussian noise input $n(t)$ with intensity Φ, the power of the signal $y_s(t)$ satisfies the constraint

$$\|y_s\|^2_{RMS} < \mathcal{P} \tag{8}$$

for a predetermined feasible value $\mathcal{P} > 0$. ∘

Because the closed loop system in Figure 3 is assumed asymptotically stable, the signal $y_s(t)$ resulting from the input noise $n(t)$ is a stationary stochastic process with Gaussian distribution. The power spectral density of $y_s(t)$ can then be expressed as

$$S_{y_s}(\omega) = T_K(j\omega)S_n(\omega)T'_K(-j\omega),$$

where $T_K(s)$ is the closed loop transfer function between $n(t)$ and $y_s(t)$ in Figure 3, that is,

$$T_K(s) = \frac{K(sI - A)^{-1}B}{1 + K(sI - A)^{-1}B}. \tag{9}$$

Thus, the power constraint on y_s in the system of Figure 3 may be expressed as

$$\mathcal{P} \geq \|y_s\|^2_{RMS} = \|T_K\|^2_{H_2}\Phi, \tag{10}$$

where $\|T_K\|_{H_2}$ denotes the H_2 norm of the strictly proper, stable scalar transfer function $T_K(s)$, defined as

$$\|T_K\|_{H_2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} T_K(j\omega)T_K(-j\omega)\,d\omega\right)^{1/2}.$$

The following result gives an explicit expression for the lowest value that $\|T_K\|^2_{H_2}$ can take over the class of all stabilizing gains K in the closed loop system in Figure 3.

Proposition 2.1. Consider the feedback loop of Figure 3. Let $p_k, k = 1, 2, \ldots, m$ be the eigenvalues of A in $\mathbb{C}^+$. Then,

$$\inf_{K\,:\,(A-BK)\text{ is Hurwitz}} \|T_K\|^2_{H_2} = \sum_{k=1}^{m} 2\,\mathrm{Re}\{p_k\}. \tag{11}$$

Proof. See Appendix A.1. □

From (11), we see that in order to be able to solve Problem 1, the lowest feasible value of $\mathcal{P}$ in (8) must be greater than a positive value fixed by the open loop unstable poles of the plant and the intensity of the noise; in other words, the channel SNR[2] must satisfy

$$\frac{\mathcal{P}}{\Phi} \geq \sum_{k=1}^{m} 2\,\mathrm{Re}\{p_k\}\,. \tag{12}$$

How does this constraint relate to Nair and Evans's bound (2) on the lowest data rate required for stabilization? Suppose that the discrete time system (1) arises as the discretization with sample interval T of a continuous-time system with unstable eigenvalues $p_i \in \mathbb{C}^+$, $i = 1, 2, \ldots, m$. Then, the bound (2) establishes that the lowest data rate required for stabilization must satisfy

$$R/T > \log_2 e \sum_{\mathrm{Re}\{p_i\}\geq 0} \mathrm{Re}\{p_i\} \quad \text{bits per second.} \tag{13}$$

On the other hand, we know that the capacity C of a continuous-time AWGN channel as in Figure 2, with infinite bandwidth, power constraint $\mathcal{P} \geq \|y\|^2_{RMS}$, and noise spectral density Φ, can be made arbitrarily close to (Cover and Thomas, 1991, p. 250)

$$C = \frac{\mathcal{P}}{2\Phi}\,(\log_2 e)\,, \quad \text{bits per second.} \tag{14}$$

Note that under (12), the maximum channel capacity (14) permitted by Shannon's Theorem must satisfy

$$C \geq \log_2 e \sum_{k=1}^{m} \mathrm{Re}\{p_k\} \quad \text{bits per second.} \tag{15}$$

Therefore, assuming maximum channel capacity can be attained, Equation (15) gives the same bound (13) derived from Nair and Evans's result.

2.2 Output Feedback Stabilization

The previous section considered a simplified version of the feedback system of Figure 1 in which we were only concerned about stabilization by static state feedback over an AWGN channel. In this section, we turn to stabilization by dynamic output feedback. Under the assumption that the plant is minimum phase, we will recover in this case the same bound (12) on the required SNR for stabilization, again consistent with Nair and Evans's result.

On using the channel model of Figure 2, the feedback loop of Figure 1 reduces to the LTI loop of Figure 4, in which $P(s)$ and $C(s)$ respectively are the transfer functions of the plant and the controller, and $y(t)$ is the output of the system. We assume that the controller $C(s)$ is such that the feedback loop of Figure 4 is asymptotically stable. We also assume that the plant $P(s)$ is proper and minimum phase (it does not contain either zeros in $\mathbb{C}^+$ or time delays), although it may be unstable.

Because the closed loop system is asymptotically stable, the output $y(t)$ resulting from the input noise $n(t)$ is a stationary stochastic process with Gaussian distribution. By using the power spectral density of $y(t)$ as in the previous section,

$$\begin{aligned} \|y\|^2_{RMS} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}\,[G(j\omega)G'(-j\omega)]\,\Phi\,d\omega \\ &= \|G\|^2_{H_2}\Phi, \end{aligned} \tag{16}$$

[2] Note here that we use the term SNR as the ratio *signal power/noise intensity*. Strictly, the SNR ratio in a continuous-time channel should be defined as the ratio *signal power/noise power*, in which the noise power is $W\Phi$, where W represents the channel bandwidth (assumed infinite in this paper).

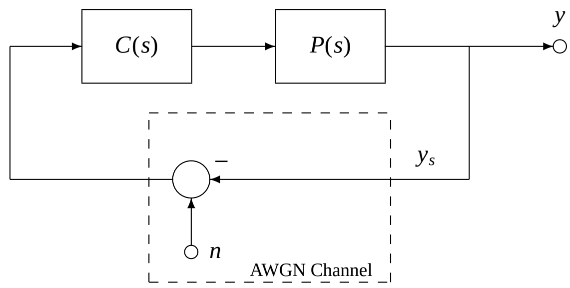

Figure 4: Simplified continuous-time feedback loop over an AWGN channel

where $G(s)$ is the closed loop transfer function between $n(t)$ and $y(t)$ in Figure 4

$$G(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}, \tag{17}$$

and $\|G\|_{H_2}$ is the H_2 norm of $G(s)$. Note that $\|G\|_{H_2}$ is finite because $G(s)$ is stable and strictly proper. Thus to find the lowest achievable value of $\|y\|_{RMS}$ we have to find the lowest achievable value of $\|G\|_{H_2}$ over the class of all stabilizing controllers.

If the plant $P(s)$ is unstable, $\|G\|_{H_2}$ has a positive lower bound that cannot be further reduced by any choice of the controller, as we show in the following proposition.

Proposition 2.2. Consider the feedback loop of Figure 4. Assume that the plant $P(s)$ is proper and minimum phase, and has m poles $p_k, k = 1, 2, \ldots, m$ in $\mathbb{C}^+$, and that $C(s)$ is such that the closed-loop is asymptotically stable. Then,

$$\|G\|_{H_2}^2 \geq \left(\sum_{k=1}^{m} 2 \operatorname{Re}\{p_k\} \right). \tag{18}$$

Proof. See Appendix A.2. □

From (14) and Proposition 2.2, we have that for the feedback loop of Figure 4, stabilization under the power constraint (20) is only possible if

$$\mathcal{P} \geq \|y\|_{RMS}^2 = \|G\|_{H_2}^2 \Phi \geq \sum_{k=1}^{m} 2 \operatorname{Re}\{p_k\} \Phi, \tag{19}$$

which is the same as (12), and hence yields, together with Shannon's Theorem, the same bound (15). Again, assuming maximum channel capacity is attained, we recover the bound (13) derived for our continuous-time setting from Nair and Evans's result.

Note that in the output feedback case, if the plant $P(s)$ in Figure 4 is *non-minimum phase*, then a *higher* lower bound on $\|y\|_{RMS}^2$, and hence also on C, should be expected. The lowest channel capacity required would then account not only for the bit rate needed for stabilization of the loop, but also for the H_2 *performance* requirement on the system output.

3 Discrete-Time Feedback Channels

Under the simplifying assumptions stated in Section 1, that all pre- and post- signal processing involved in the communication link illustrated in Figure 1 are limited to LTI filtering and sampling and hold operations, we consider now a discrete-time version for the problems discussed in the previous sections.

A common model of a discrete-time communication channel is defined by the linear input-output relation

$$r(t) = y(t) + n(t), \qquad t = 0, 1, 2, \ldots, \tag{20}$$

in which $n(t)$ represents a zero-mean, discrete-time white Gaussian noise, and the input signal $y(t)$ is required to satisfy a power constraint. This channel model, illustrated in Figure 5, is usually referred to as the *discrete-time AWGN channel*, and is widely used in Communications (e.g., Proakis and Salehi 1994, §10; Cover and Thomas 1991, §10; Forney and Ungerboeck 1998). The discrete-time AWGN channel model is also useful to represent, to some extent, the effects of roundoff and quantization in A-D and D-A converters (Gray, 1990).

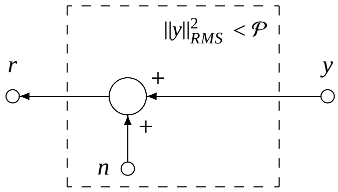

Figure 5: Discrete-time AWGN channel with power constraint

The zero-mean, discrete-time white Gaussian noise $n(t)$ in (20) is assumed to have intensity Φ, i.e.,

$$E\{n(t)\} = 0, \quad E\{n'(t)n(\tau)\} = \Phi\delta(t-\tau), \qquad \text{where} \quad \delta(t-\tau) = \begin{cases} 1 & \text{if } t = \tau \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

The input signal $y(t)$ is assumed to be a discrete-time stationary stochastic process with autocorrelation matrix

$$R_y(\tau) = E\{y(t)y'(t+\tau)\},$$

and power spectral density

$$S_y(\omega) = \sum_{-\infty}^{\infty} R_y(\tau)e^{-j\omega\tau}, \quad -\pi \le \omega \le \pi.$$

For a discrete-time stochastic signal $y(t)$ we have

$$\|y\|_{RMS} = E\left\{y(t)'y(t)\right\}^{1/2}.$$

Its *power* $\|y\|^2_{RMS}$ is given by

$$\|y\|^2_{RMS} = \operatorname{trace}\left[R_y(0)\right] = \frac{1}{2\pi}\operatorname{trace}\left[\int_{-\pi}^{\pi} S_y(\omega)\,d\omega\right]. \tag{22}$$

The input power constraint in the channel model of Figure 5 is enforced by requiring that $\|y\|^2_{RMS}$ be bounded by some predetermined positive value $\mathcal{P}$, $\|y\|^2_{RMS} < \mathcal{P}$.

3.1 State Feedback Stabilization

Consider a discrete-time version of the state feedback stabilization problem discussed in Section 2.1, illustrated in Figure 6. The computed state feedback signal y_s is fed back over a discrete-time AWGN channel with an input power constraint

$$\|y_s\|^2_{RMS} < \mathcal{P}, \tag{23}$$

Under these conditions, we pose the following stabilization problem.

Problem 2 (State feedback stabilization with a power constraint).

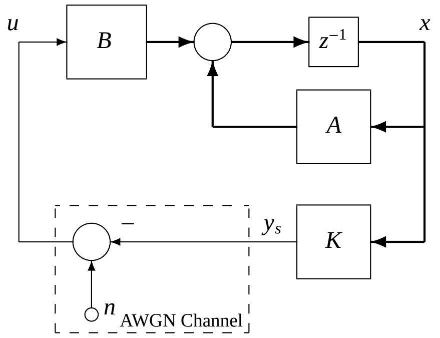

Figure 6: Discrete-time state feedback loop

Find a static state feedback gain K such that the closed loop system

$$\begin{aligned} x(t+1) &= (A - BK)x(t) + Bn(t) \\ y_s(t) &= Kx(t) \end{aligned} \tag{24}$$

is asymptotically stable and, for a zero-mean white Gaussian noise input $n(t)$ with intensity Φ, the power of the signal $y_s(t)$ satisfies the constraint

$$\|y_s\|_{RMS}^2 < \mathcal{P} \tag{25}$$

for a predetermined feasible value $\mathcal{P} > 0$. ∘

Note that the power constraint (23) is a constraint on the H_2 norm of the transfer function between n and y_s in the loop of Figure 6. Indeed, it is well-known (e.g. Saberi et al., 1995, p. 23) that the power of the signal $y_s(t)$ resulting from the input noise $n(t)$ is given by

$$\|y_s\|_{RMS}^2 = \|T_K\|_{H_2}^2 \Phi, \tag{26}$$

where $T_K(z)$ is the transfer function

$$T_K(z) = K(zI - A + BK)^{-1}B, \tag{27}$$

and $\|T_K\|_{H_2}$ now represents the H_2 norm of a proper, stable scalar discrete transfer function, defined as

$$\|T_K\|_{H_2} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} T_K(e^{j\theta}) T_K(e^{-j\theta})\, d\theta \right)^{\frac{1}{2}} \tag{28}$$

$$= \left(\sum_{t=0}^{\infty} |\mathcal{T}_K(t)|^2 \right)^{\frac{1}{2}}, \tag{29}$$

where $\mathcal{T}_K(t)$ is the discrete-time impulse response of the transfer function $T_K(z)$.

The following proposition states necessary and sufficient conditions for Problem 2 to be solvable in terms of the feasible SNR[3] in the noisy feedback channel with power constraint $\mathcal{P}$ and noise power Φ.

Proposition 3.1. There exists a state feedback gain K solving Problem 2 if and only if the power constraint (25) satisfies

$$\frac{\mathcal{P}}{\Phi} > \left(\prod_{|\eta_i| \geq 1} |\eta_i|^2 - 1 \right), \tag{30}$$

where $\{\eta_i : |\eta_i| \geq 1\}$ are the unstable eigenvalues of A in (24).

[3] In this case, our treatment of SNR is consistent with *signal power/noise power*, since in discrete-time the noise power is precisely $\|n\|_{RMS} = \Phi$. Compare Footnote 2 on Page 5.

Proof. See Appendix A.3. □

Thus, we see that for the feedback stabilization problem to have a solution, the lowest feasible input power constraint (25) for the AWGN feedback channel must be greater than a positive value fixed by the open loop unstable poles of the plant and the intensity of the noise.

By using the constraint (30) on Shannon's bound on the capacity of a discrete-time AWNG channel (e.g., Cover and Thomas, 1991, § 10)

$$C = \frac{1}{2}\log_2\left(1 + \frac{\mathcal{P}}{\Phi}\right) \quad \text{bits per interval,} \tag{31}$$

we recover again Nair and Evans's bound (2) on the lowest data rate necessary for stabilization,

$$C > \sum_{|\eta_i|\geq 1} \log_2 |\eta_i| \qquad \text{bits per interval.} \tag{32}$$

3.2 Output Feedback Stabilization

We consider the discrete-time output feedback loop pictured in Figure 7, in which we have used the AWGN channel model of Figure 5 to represent the noisy feedback channel. We intend to find the lowest value of $\|y\|^2_{RMS}$ over the set of all stabilizing controllers $C(z)$ for a zero-mean, white Gaussian noise $n(k)$ with intensity Φ.

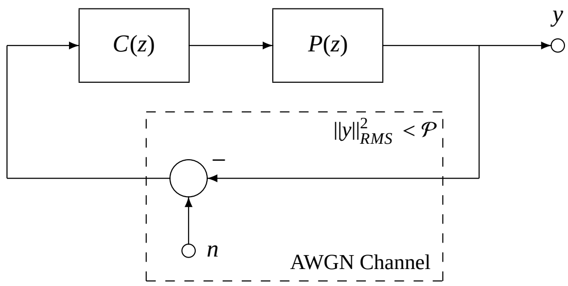

Figure 7: Simplified discrete-time feedback loop over an AWGN channel

Note that

$$Y(z) = G(z)N(z) = \frac{C(z)P(z)}{1 + C(z)P(z)}N(z) \tag{33}$$

Since $n(t)$ is white with power spectral density $S_n(\omega) = \Phi$, the power spectral density of the output $y(t)$ is given by

$$S_y(\omega) = |G(e^{j\omega})|^2\Phi, \tag{34}$$

and therefore the output power is

$$\begin{aligned} \|y\|^2_{RMS} &= \left(\int_{-\pi}^{\pi} |G(e^{j\omega})|^2 d\omega\right)\Phi \\ &= \|G\|^2_{H_2}\Phi. \end{aligned} \tag{35}$$

We are therefore trying to solve an H_2 optimization problem.

Proposition 3.2. Consider the feedback loop of Figure 7. Assume that the plant $P(z)$ is minimum phase, strictly proper with relative degree 1, and has m poles p_k, $k = 1, 2, \ldots, m$, in $\mathbb{D}^{\complement}$, and that $C(z)$ is such that the closed loop is asymptotically stable. Then,

$$\|G\|^2_{H_2} \geq \left[\prod_{i=1}^{m} |p_i|^2\right] - 1 \tag{36}$$

Proof. See Appendix A.4. □

4 Conclusions

In this paper we have been motivated by control over bit rate limited channels to consider stabilization over SNR limited channels. We have considered the simple case where there is essentially no encoding or decoding present, and looked at the limits of achievable stabilization of linear controls. For both state feedback and for delay free minimum phase plants, we obtain results equivalent to those that would be obtained if delay and error free, Shannon channel capacity digital communication could be performed on the same channel. The results include both continuous-time and discrete-time channels.

Extensions of this work would include looking at output feedback non-minimum phase plants where, at least in the present framework, a deterioration in the achievable H_2 performance would suggest that stabilization will be more difficult. In addition, channel bandwidth constraints, and more general control performance questions than simple stabilization could be considered within the framework suggested in this paper. More complex questions including the potential use of time varying or nonlinear elements in the coding and decoding are also of interest.

A Appendix

A.1 Continuous-Time State Feedback

Proof of Proposition 2.1. Consider the following minimum energy stabilization problem.

Problem 3 (Continuous-time minimum energy stabilization). Find $y_s = -Kx$ minimizing the cost function

$$J(x(0)) = \int_0^\infty |y_s(t)|^2 dt \tag{37}$$

and such that the system

$$\dot{x} = Ax + By_s, \quad x(0) = B, \tag{38}$$

is asymptotically stable. ◦

The value function (37) can be computed explicitly as

$$\begin{aligned}
J(x(0)) &= \int_0^\infty \left|Ke^{(A-BK)t}x(0)\right|^2 dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|K(j\omega I - A + BK)^{-1}B\right|^2 d\omega, && \text{by Parseval's Theorem,} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|K(j\omega I - A)^{-1}B\left[I + K(j\omega I - A)^{-1}B\right]\right|^2 d\omega, && \text{using the Matrix Inversion Lemma,} \\
&= \|T_K\|_{H_2}^2,
\end{aligned} \tag{39}$$

where $T_K(s)$ is the transfer function defined in (9). Hence, the right hand side of (11) is precisely the optimal value of (37) with $x(0) = B$.

Remark A.1. Note that Problem 3 does not have a well-defined solution if the matrix A in (38) has eigenvalues on the $j\omega$-axis. However, the result can then still be proved by using perturbation arguments in the same way suggested in Remark A.2 for the discrete-time case. Thus, for the rest of the proof, we can assume without loss of generality that A does not have eigenvalues on the $j\omega$-axis. ◦

Without loss of generality assume that the system (38) is in the modal canonical form, that is,

$$\dot{x} = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix} x + \begin{bmatrix} B_s \\ B_u \end{bmatrix} y_s,$$

where A_s is Hurwitz and A_u is *anti-Hurwitz* (that is, $-A_u$ is Hurwitz). Because the system is assumed stabilizable, the pair (A_u, B_u) is controllable and the minimum energy problem above has a well defined solution given by

$$K^* = B^T P, \tag{40}$$

where P is the unique symmetric positive semi-definite solution of the Riccati equation

$$A^T P + PA - PBB^T P = 0. \tag{41}$$

It is not difficult to verify that P has the form

$$P = \begin{bmatrix} 0 & 0 \\ 0 & P_u \end{bmatrix}, \tag{42}$$

where P_u is the unique symmetric and positive definite solution of the reduced order Riccati equation

$$P_u A_u + A_u^T P_u - P_u B_u B_u^T P_u = 0. \tag{43}$$

Thus, the optimal value (37) for $x(0) = B$ is then

$$\begin{aligned} J^*(B) &= B^T PB = B_u^T P_u B_u \\ &= \operatorname{trace} P_u^{1/2} B_u B_u^T P_u^{1/2} \\ &= \operatorname{trace}(P_u^{-1/2} A_u^T P_u^{1/2} + P_u^{1/2} A_u P_u^{-1/2}) \\ &= 2 \operatorname{trace} A_u = 2 \sum_{k=1}^{m} \operatorname{Re}\{p_k\}, \end{aligned}$$

which, on using (39), proves (11). □

A.2 Continuous-Time Output Feedback

Proof of Proposition 2.2. To compute the lowest value of $\|G\|_{H_2}$ over the class of all stabilizing controllers, we apply a technique used in Chen et al. (2000). This is based on considering the spaces

$$\begin{aligned} L_2 &= \left\{ G(s) : \int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega < \infty \right\}, \\ H_2 &= L_2 \cap \left\{ G(s) : \text{analytic in } \bar{\mathbb{C}}^+ \right\} \end{aligned}$$

and the orthogonal complement of H_2

$$H_2^\perp = L_2 \cap \left\{ G(s) : \text{analytic in } \bar{\mathbb{C}}^- \right\}$$

We start by deriving an expression for $G(s)$ based on a parameterization of all stabilizing controllers. Represent $P(s)$ by a coprime factorization

$$P(s) = \frac{N(s)}{B_p(s)},$$

where $N(s) \in RH_\infty$ (the space of proper, stable, rational functions), and

$$B_p(s) = \prod_{k=1}^{m} \frac{s - p_k}{s + \bar{p}_k}$$

is the Blaschke product of all poles of $P(s)$ in $\mathbb{C}^+$. By using the well-known Youla controller parameterization (e.g. Doyle et al., 1992, §5.4), we can represent any stabilizing controller for the feedback loop in Figure 4 by[4]

$$C = \frac{X + B_p Q}{Y - NQ}, \tag{44}$$

where Q, X and Y are in RH_∞, with X and Y satisfying the Bezout identity

$$NX + B_p Y = 1. \tag{45}$$

By replacing (44) in (45), we find that G can be expressed as

$$G = \left(1 - B_p(Y - NQ)\right) = N(X + B_p Q), \tag{46}$$

where the last equality follows from the fact that $X + B_p Q$ and $Y - NQ$ are also coprime and satisfy the Bezout identity

$$N(X + B_p Q) + B_p(Y - NQ) = 1.$$

Thus, from (46), the problem of finding the lowest value of $\|G\|_{H_2}$ over the class of stabilizing controllers reduces to that of finding

$$\inf_{Q\in RH_\infty} \left\|1 - B_p Y + B_p N Q\right\|_{H_2}. \tag{47}$$

Now,

$$\begin{aligned}
\inf_{Q\in RH_\infty} \left\|1 - B_p Y + B_p N Q\right\|_{L_2} &= \inf_{Q\in RH_\infty} \left\|B_p^{-1} - Y + NQ\right\|_{L_2}^2, && \text{since } B_p \text{ is all pass}\\
&= \inf_{Q\in RH_\infty} \left\|\left(1 - B_p^{-1}\right) + (1 - Y + NQ)\right\|_{L_2}^2, \\
&= \left\|1 - B_p^{-1}\right\|_{L_2}^2 + \inf_{Q\in RH_\infty} \|1 - Y + NQ\|_{L_2}^2, && (48)
\end{aligned}$$

where the last line follows since $(1 - B_p^{-1})$ is both strictly proper and anti-stable, and therefore is in $H_2^\perp$, and conversely, $(1 - Y + NQ)$ is strictly proper and stable and therefore in H_2.

Assuming the plant is minimum phase, then we may take Q arbitrarily close to $N^{-1}(1 - Y)$. Indeed, because $(1-Y)$ is stable, given any $\varepsilon > 0$, there always exists some $Q_\varepsilon \in RH_\infty$ such that $\|1-Y+NQ\|_{L_2} < \varepsilon$, which shows in (48) that $\inf_{Q\in RH_\infty} \|1 - Y + NQ\|_{L_2}^2 = 0$.

On the other hand, note that

$$\begin{aligned}
\left\|1 - B_p^{-1}\right\|_{L_2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(1 - B_p^{-1}(j\omega)\right)\left(1 - B_p^{-1}(-j\omega)\right) d\omega, \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(1 - B_p^{-1}(j\omega)\right)\left(1 - B_p(j\omega)\right) d\omega, \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(1 - B_p^{-1}(j\omega)\right) + \left(1 - B_p(j\omega)\right) d\omega, \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \left(1 - B_p(j\omega)\right) d\omega, && \text{by conjugate symmetry.}
\end{aligned}$$

[4] Dependency on s is suppressed to simplify notation when convenient.

We now use contour integration around the clockwise-oriented contour in $\bar{\mathbb{C}}^{+}$, which consists of the imaginary axis, closed with a semi-circular region of arbitrarily large radius R in the right half plane,

$$\begin{aligned}\left\|1 - B_p^{-1}\right\|_{L_2}^2 &= \underbrace{\left[\frac{1}{j\pi}\oint_{\bar{\mathbb{C}}^+}\left(1 - B_p(s)\right)ds\right]}_{=0,\text{ since }(1-B_p)\text{ is analytic in }\bar{\mathbb{C}}^+} - \lim_{R\to\infty}\left[\frac{1}{\pi}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}\left(1 - B_p\left(Re^{j\theta}\right)\right)Re^{j\theta}d\theta\right],\\ &= -\lim_{R\to\infty}\left[\frac{1}{\pi}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}\left(\frac{c_1}{Re^{j\theta}} + \frac{c_2}{(Re^{j\theta})^2} + \cdots\right)Re^{j\theta}d\theta\right], \qquad \text{since } (1 - B_p) \text{ is strictly proper,}\\ &= c_1 \triangleq \lim_{s\to\infty}\left(s\left(1 - B_p(s)\right)\right)\\ &= 2\sum_{k=1}^{m}\mathrm{Re}\left\{p_k\right\},\end{aligned}$$

which completes the proof.

□

A.3 Discrete-Time State Feedback

Proof of Proposition 3.1. Because of the relation (26), the lowest value of $\mathcal{P}$ for which Problem 2 admits a solution is that in which K, among the set of all stabilizing state feedback gains, minimizes $\|G\|_{H_2}$. The value of $\|T_K\|_{H_2}$ may be expressed in terms of K, on using (29), by

$$\|T_K\|_{H_2}^2 = \sum_{t=0}^{\infty}\left|K(A - BK)^t B\right|^2. \tag{49}$$

An explicit expression of the lowest value of $\|T_K\|_{H_2}$ over all stabilizing Ks can be obtained by recognizing the RHS of (49) as the optimal cost of the following auxiliary *minimum energy stabilization problem* from a particular initial condition.

Problem 4 (Minimum energy stabilization). Find y_s minimizing the cost

$$J(x(0)) = \sum_{t=0}^{\infty} y_s(t)' y_s(t) \tag{50}$$

and such that the system

$$x(t+1) = Ax(t) + By_s(t), \qquad \text{with } x(0) = B. \tag{51}$$

is asymptotically stable. ○

The optimal y_s in Problem 4 is well-known to be (e.g. Anderson and Moore, 1971, §14)

$$y_s{}^* = -K^* x, \quad \text{where } K^* = (1 + B'SB)^{-1}B'SA, \tag{52}$$

and S is the unique symmetric positive semidefinite solution of the discrete algebraic Riccati equation

$$A'SA - S = A'SB(1 + B'SB)^{-1}B'SA. \tag{53}$$

The evaluation of (50) at a state feedback $y_s = -Kx$ together with the expression of the closed-loop response $x(t) = (A - BK)^t B$, which follows from (51), yields

$$\sum_{t=0}^{\infty} y_s{}'(t) y_s(t) = \sum_{t=0}^{\infty} |K(A - BK)^t B|^2 = J(B),$$

which with (49) shows that $\|T_K\|_{H_2}^2 = J(B)$.

The optimal cost for Problem 4 is achieved by $y_s{}^*$ in (52) and may be computed explicitly by

$$J(B) = B'SB. \tag{54}$$

We next show that

$$B'SB = \prod_{|\eta_i|\geq 1} |\eta_i|^2 - 1. \tag{55}$$

Remark A.2. Note that if A in (51) has eigenvalues on the unit circle, Problem 4 as stated is not well defined, in the sense that $y_s{}^*$ in (52) will not be stabilizing. Without loss of generality we will consider then that A has no eigenvalues on the unit circle. Otherwise, our argument could be carried out with identical conclusions by considering the minimum energy problem as a limiting case of an ε-parameterized family of optimal control problems with cost

$$J_\varepsilon(x(0)) = \sum_{t=0}^{\infty} \varepsilon^2 x(t)'x(t) + y_s(t)'y_s(t), \quad \varepsilon > 0.$$

For any $\varepsilon > 0$ the corresponding state-feedback stabilization problem has a well defined solution. The minimum energy optimal cost would then be taken as $\lim_{\varepsilon\to 0} J_\varepsilon(x(0))$. ◦

Without loss of generality assume that A and B in (7) are in the modal canonical form, that is,

$$A = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}, \quad B = \begin{bmatrix} B_s \\ B_u \end{bmatrix},$$

where A_s is Hurwitz and A_u is *anti-Hurwitz* (that is, A_u^{-1} is Hurwitz). If the pair (A, B) is assumed stabilizable, the pair (A_u, B_u) is controllable and the minimum energy problem above has a well defined solution given by (52), where S can be shown to have the form

$$S = \begin{bmatrix} 0 & 0 \\ 0 & S_u \end{bmatrix}, \tag{56}$$

where S_u is the unique symmetric and positive definite solution of the reduced order discrete algebraic Riccati equation

$$A_u'S_uA_u - S_u = A_u'S_uB_u(1 + B_u'S_uB_u)^{-1}B_u'S_uA_u. \tag{57}$$

Then the minimum energy state feedback gain (52) is given as $K^* = \begin{bmatrix} 0 & (1 + B_u'S_uB_u)^{-1}B_u'S_uA_u \end{bmatrix}$ and yields the closed loop spectrum

$$\begin{aligned}
\sigma\{A - BK^*\} &= \sigma\left\{\begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix} - \begin{bmatrix} 0 & B_s(1 + B_u'S_uB_u)^{-1}B_u'S_uA_u \\ 0 & B_u(1 + B_u'S_uB_u)^{-1}B_u'S_uA_u \end{bmatrix}\right\} \\
&= \sigma\{A_s\} \cup \sigma\left\{A_u - B_u(1 + B_u'S_uB_u)^{-1}B_u'S_uA_u\right\} \\
&= \sigma\{A_s\} \cup \sigma\left\{S_u^{-1}A_u'S_u\right\}, \qquad \text{on using (57),} \\
&= \sigma\{A_s\} \cup \sigma\left\{A_u^{-1}\right\}.
\end{aligned} \tag{58}$$

On the other hand,

$$\begin{aligned}
A - B(1 + B'SB)^{-1}B'SA &= [I - B(1 + B'SB)^{-1}B'S]A \\
&= (I + BB'S)^{-1}A,
\end{aligned}$$

by the Matrix Inversion Lemma, which together with (58) implies that

$$\begin{aligned}
\det\{A_s\}\det\left\{A_u^{-1}\right\} &= \det\left\{(I + BB'S)^{-1}A\right\} \\
&= (1 + B'SB)^{-1}\det\{A_s\}\det\{A_u\} \\
\Leftrightarrow \qquad (1 + B'SB) &= \det\{A_u\}^2,
\end{aligned}$$

from which we conclude (55), completing the proof. □

A.4 Discrete-Time Output Feedback

Proof of Proposition 3.2. Here we follow similar lines to Toker et al. (2002). We consider

$$\begin{aligned} L_2(\mathbb{D}) &= \left\{ f(z) : \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{j\theta})|^2 d\theta < \infty \right\} \\ H_2(\mathbb{D}) &= L_2(\mathbb{D}) \cap \left\{ f(z) : \text{analytic in } \mathbb{D}^{\complement} \right\} \end{aligned}$$

and the orthogonal complement of $H_2(\mathbb{D})$

$$H_2^{\perp}(\mathbb{D}) = L_2(\mathbb{D}) \cap \{ f(z) : \text{strictly proper and analytic in } \mathbb{D} \}$$

Let

$$P(z) = \frac{N(z)}{B_p(z)},$$

where N is a stable and proper rational function, and

$$B_p(z) = \prod_{i=1}^{m} \frac{z - \eta_i}{1 - z\overline{\eta_i}}$$

is here the discrete-time Blaschke product of all poles of P outside the unit disk $\mathbb{D}$. The class of all discrete-time stabilizing controllers for P in the feedback loop of Figure 7 is parameterized by

$$C(z) = \frac{X(z) + B_p(z)Q(z)}{Y(z) - Q(z)N(z)},$$

where Q, X and Y are stable and proper rational functions which satisfy the Bezout identity

$$X(z)N(z) + B_p(z)Y(z) = 1 \tag{59}$$

Then we have that

$$\begin{aligned} G(z) &= X(z)N(z) + B_p(z)Q(z)N(z) \\ &= 1 - B_p(z)Y(z) + B_p(z)Q(z)N(z). \end{aligned}$$

Thus,

$$\begin{aligned} \|G\|^2_{H_2(\mathbb{D})} &= \inf_{Q(z)} \left\| 1 - B_p(z)\left(Y(s) - Q(z)N(z)\right) \right\|^2_{H_2(\mathbb{D})} \\ &= \inf_{Q(z)} \left\| \left(B_p(z)\right)^{-1} - \left(B_p(\infty)\right)^{-1} + \left(B_p(\infty)\right)^{-1} - Y(z) + Q(z)N(z) \right\|^2_{L_2(\mathbb{D})} \\ &= \left\| \left(B_p(z)\right)^{-1} - \left(B_p(\infty)\right)^{-1} \right\|^2_{L_2(\mathbb{D})} + \inf_{Q(z)} \left\| \left(B_p(\infty)\right)^{-1} - Y(z) + Q(z)N(z) \right\|^2_{L_2(\mathbb{D})}. \end{aligned} \tag{60}$$

First note that

$$\begin{aligned}
\left\|\left(B_p(z)\right)^{-1} - \left(B_p(\infty)\right)^{-1}\right\|^2_{L_2(\mathbb{D})} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[B_p^{-1}(e^{j\theta}) - B_p^{-1}(\infty)\right]\left[B_p^{-1}(e^{-j\theta}) - B_p^{-1}(\infty)\right]d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1 + B_p^{-2}(\infty) - B_p^{-1}(\infty)\left[B_p(e^{j\theta}) + B_p^{-1}(e^{j\theta})\right]\right]d\theta \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1 + B_p^{-2}(\infty)}{2} - B_p^{-1}(\infty)B_p(e^{j\theta})\right]d\theta \\
&= \frac{1}{j\pi}\oint_{\circlearrowleft\partial\mathbb{D}}\left[\frac{1 + B_p^{-2}(\infty)}{2} - B_p^{-1}(\infty)B_p(z)\right]\frac{d\theta}{z} \\
&= 2\left[\frac{1 + B_p^{-2}(\infty)}{2} - B_p^{-1}(\infty)B_p^{-1}(0)\right], \qquad \text{since } B_p^{-1}(z) \text{ is analytic in } \mathbb{D}, \\
&= \prod_{i=1}^{m}|\eta_i|^2 - 1.
\end{aligned}$$

On the other hand, note in (60) that

$$\inf_{Q(z)}\left\|\left(B_p(\infty)\right)^{-1} - Y(z) + Q(z)N(z)\right\|^2_{H_2(\mathbb{D})} = 0$$

since we can take

$$Q(z) = \frac{Y(z) - B_p^{-1}(\infty)}{N(z)},$$

which is stable and proper because $N(z)$ is minimum phase and has relative degree 1, and $Y(\infty) = B_p^{-1}(\infty)$ from the Bezout identity (59).

□

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