On the Stabilisation of Linear Unstable Systems with Control Constraints†

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Abstract

In this paper we consider the problem of stabilisation of linear systems with constrained control. Previous works have frequently suggested the use of low-gain feedback to avoid problems caused by saturation. This is a useful technique for open-loop stable plants, but has sensitivity problems when used for open-loop unstable plants. In this paper, we discuss this issue, together with a new control methodology. This new methodology uses two controllers, one based on a minimum-time optimal control, and another using *any* standard linear controller design. A switching strategy, with hysteresis, is suggested so that locally, the linear control behaviour is achieved, but non-locally, the maximal possible region of attraction is achieved. This control strategy, as well as the analysis of other controllers, requires computations of certain regions in the state space. In the 2^{nd} order case, methods for computing these regions are given, while for higher orders, quadratic approximations may be used. An example illustrates the above ideas.

Keywords: Control system design, Stability, Linear systems, Control nonlinearities, Saturation, Bang-bang control.

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1 Introduction

Control problems in the presence of control constraints are interesting from both practical and theoretical points of view. In most applications, constraints appear as saturation phenomena due to physical limitations of actuators and sensors. So, the existence of constraints in practical situations give rise to control problems characterised by a saturation as the dominant nonlinearity.

Thinking in terms of nonlinear control design, standard nonlinear techniques fail in providing an answer to this kind of problem. As saturation is a non-smooth and non-invertible nonlinearity, neither differential geometry nonlinear control theory (Isidori 1989) nor linearizing techniques based on the implicit function theorem (Gao, Hill and Wang 1989) are applicable, at least in their present forms.

One method of dealing with limited control action is simply to consider, at first, the unconstrained problem and then to impose a saturation nonlinearity on the unconstrained controller output. While allowing the use of standard linear design techniques, this approach may lead to important performance degradation and instability problems, as indicated in the discussion below.

As was pointed out in (Kapasouris, Athans and Stein 1990), performance degradation originates from the fact that control saturation modifies both magnitude and direction of the expected control vector. As a result, this *misprocessing* of the feedback information may deteriorate the closed-loop response, giving rise to undesirable dynamic effects such as the well-known integrator windup and local stability phenomena.

Many attempts have been made to compensate for integrator windup, for example, in (Åstrom and Rundqwist 1989) and (Campo, Morari and Nett 1989). These approaches, however, do not cover the open-loop strictly unstable plant case, where the consequences of saturation may be worse. If the plant is open-loop strictly unstable (where we use the term *strictly unstable* to denote systems with poles in the open right half plane), global stability can never be guaranteed. It is clear that with open-loop strict instability and limited control energy, there are always initial conditions which are irrecoverable (LeMay (1964), Hsu and Meyer (1968) and Ma (1991)).

A considerable bibliography classifying various approaches to the control of linear systems with constraints was recently compiled by Gilbert (1992). While many papers deal with problems of performance degradation due to saturation phenomena, very few address the problem of instability, particularly when the plant is open-loop strictly unstable. Among those, a common reference in the related literature is the work of Gutman and Hagander (1985). Their design technique *starts* with a prespecified admissible initial condition set and attempts to find a stabilizing controller based on the following idea: the unconstrained system is first stabilised by a low-gain linear state feedback control. Based on this low-gain controller, a quadratic Lyapunov function is constructed and a second feedback control is computed. Then both controllers are added and saturated. The methodology is applicable in the case of MIMO unstable plants, though, it is not systematic, as it involves some iteration based upon trial and error. The results are conservative, in that conditions for stability are sufficient but not necessary. Furthermore, there are strong reasons (Middleton 1991) for suggesting that it is unwise to use low-gain feedback for strictly unstable open-loop plants, as we will show in Section 2.

Another MIMO procedure suited to unstable plants has been suggested by Kapasouris et al. (1990), where it is introduced a new design methodology based on the implementation of a *Reference Governor*. This Reference Governor is a device for preprocessing the reference signal of the closed-loop system in such a way that no reference input could produce control signal saturation. The scheme is powerful in that the main performance difficulties originating in saturation are avoided in the compensated system. However, only BIBO stability is assured and the implementation of the logic of the Reference Governor is complicated. Moreover, it is not at all clear that a Reference Governor will help with transient stability difficulties.

In this paper, we discuss the implications – particularly in presence of control constraints – of the use of low-gain feedback controllers in open-loop unstable systems. We also suggest a new control methodology that, while preserving the linear performance close to the origin, gives the maximal possible region of attraction when the control input is saturated. The structure of the paper is the following: in Section 2, we analyse the effects of the use of low-gain strategies for these systems. Sections 3 and 4 are devoted to local stability regions, some definitions methods for computing local stability regions in certain particular cases. The new control methodology is presented in Section 5 and, finally in Section6, we supply a 2^{nd} order example to illustrate these ideas.

2 Comments on Low-Gain Control

As has been know for some time (*e.g.* Stein (1989), Middleton (1991), Freudenberg and Looze (1988) and Zames and Francis (1983)), open-loop unstable plants require, in general, high-gain feedback, in order that reasonable sensitivity properties may be achieved. Despite this, one technique that has been suggested for controlling plants with constrained input, is to use low-gain control to avoid control saturation (see Gutman and Hagander (1985), Dolphus and Schmitendorf (1991) and Gilbert (1992)). In this section, we discuss a limitation of these low-gain approaches, particularly when applied to open-loop strictly unstable plants.

High sensitivity of low bandwidth control of open-loop unstable systems, in the linear controller case, has been discussed in Stein (1989), Middleton (1991) and Freudenberg and Looze (1987). It may be tempting to think that nonlinear and/or time-varying controllers may overcome this difficulty, however, we show below that at least some of the conclusions carry over to these cases as well. Consider a linear (apart from –perhaps– saturated input) plant as in Figure 1:

$$Y(s) = S_d(s)\,U(s) \tag{1}$$

where $S_d(s)$ has at least one strictly unstable pole. Let $u(t)$ be generated as:

$$u(t) = u_d(t) - u_c(t) \tag{2}$$

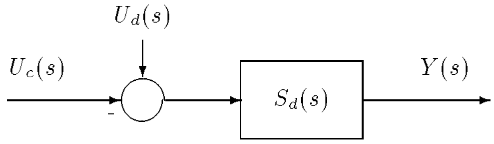

Figure 1: Strictly unstable system with control disturbance

where $u_d(t)$ denotes an exogenous input disturbance, and $u_c(t)$ is the controller output.

Then, for a unit step input disturbance, $u_d(t) = \mathbf{1}(t)$, the following holds:

$$\int_0^\infty (1 - u_c(t))e^{-pt}dt = 0 \tag{3}$$

for any p a strictly unstable pole of $S_d(s)$, and any $u_c(t)$ such that $y(t)$ is bounded. This follows as in (Middleton 1991), Lemma 1. Suppose we define the *rise time* of the controller via:

$$t_r := \arg\max_{T>0}\{u_c(t) \leq \frac{t}{T} \text{ for } t \in [0, T]\} \tag{4}$$

Then, (as in Middleton (1991), Lemma 3) we get from (3)

$$\begin{aligned}
\int_0^\infty e^{-pt}dt &= \int_0^\infty u_c(t)e^{-pt}dt && (5)\\
\frac{1}{p} &= \int_0^{t_r} u_c(t)e^{-pt}dt + \int_{t_r}^\infty u_c(t)e^{-pt}dt && (6)\\
&\leq \int_0^{t_r} \frac{t}{t_r}e^{-pt}dt + \int_{t_r}^\infty u_{max}e^{-pt}dt && (7)\\
&\leq \frac{[pt_r(u_{max} - 1) - 1]e^{-pt_r} + 1}{p^2 t_r} && (8)
\end{aligned}$$

If we define the overshoot in signal $u_c(t)$ as $u_{OS} := u_{max} - 1$, we can obtain from (8)

$$\begin{aligned}
u_{OS} &\geq \frac{(pt_r - 1)e^{pt_r} + 1}{pt_r} && (9)\\
&\geq \frac{pt_r}{2} && (10)
\end{aligned}$$

So, we have shown that, independent of the control chosen, long rise times (*i.e.* *slow* closed-loop response, as when using low-gain feedback) will lead to large overshoots in the input signal of the plant when a unit step disturbance is introduced. As is well known, these large overshoots are necessarily associated with high sensitivity characteristics, as we will show in the following example.

Example 2.1 *Let us consider the linear system displayed in Figure 2, where:*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{11}$$

$$F = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \tag{12}$$

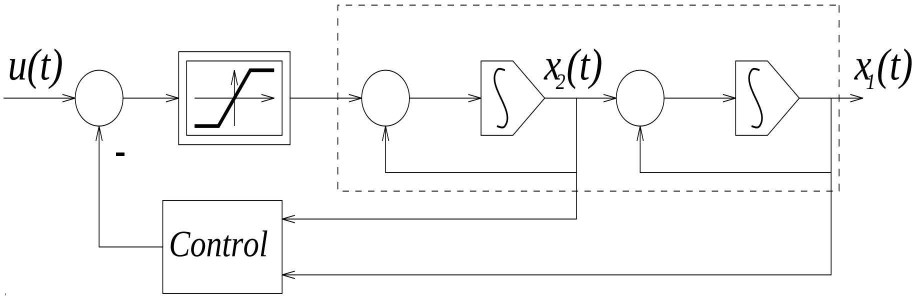

Figure 2: 2nd. order strictly unstable system.

We consider two possible state-feedback gains F to stabilise the system:

$$F_{LG} = \begin{bmatrix} 4 & 4 \end{bmatrix} \tag{13}$$

a low-gain F, *that will give closed-loop eigenvalues $\{-1, -1\}$; and*

$$F_{HG} = \begin{bmatrix} 121 & 22 \end{bmatrix} \tag{14}$$

a high-gain F, *that will give closed-loop eigenvalues $\{-10, -10\}$. Figure 3 shows the control signal $u_c(t)$ response of the system for each F and unit step disturbance $u_d(t) = \mathbf{1}(t)$. Denoting $S_d(s)$ as the local sensitivity function, we can get an estimate of $\|S_d(s)\|_{\mathcal{L}_1}$ from the unit step responses. We can write:*

$$\|S_d(s)\|_{\mathcal{L}_1} = \int_0^\infty |s_d(t)|dt \tag{15}$$

$s_d(t)$ being the impulse response, i.e. *the derivative of the step response $h_d(t)$. Then, we have:*

$$\begin{aligned} \|S_d(s)\|_{\mathcal{L}_1} &= \int_0^\infty |\dot{h}_d(t)|dt & (16) \\ &= \int_0^{t_1} \dot{h}_d(t)dt + \int_{t_1}^{t_2} -\dot{h}_d(t)dt + \cdots & (17) \\ &= h_d(t_1) - h_d(0) + h_d(t_1) - h_d(t_2) + \cdots & (18) \end{aligned}$$

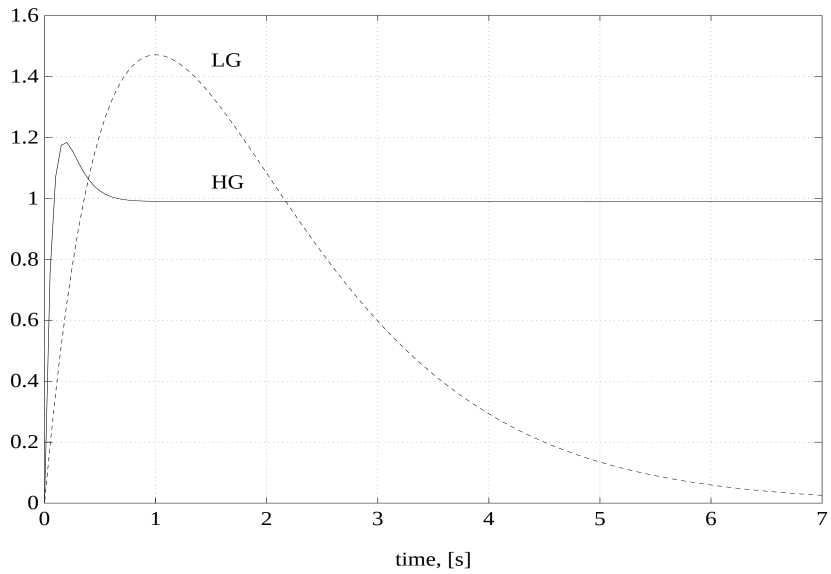

Figure 3: Step disturbance response of $u_c(t)$. LG: using F_{LG}, HG: using F_{HG}

Where intervals $[0, t_1], [t_1, t_2], \cdots$ are those where $\dot{h}_d(t)$ is sign definite. From step responses of Figure 3 we have:

$$\|S_d^{LG}(s)\|_{\mathcal{L}_1} \approx 1.47 + 1.47 - 0.06 = 2.88 \tag{19}$$

$$\|S_d^{HG}(s)\|_{\mathcal{L}_1} \approx 1.18 + 1.18 - 0.99 = 1.38 \tag{20}$$

Considering that the minimum possible of the norm of the sensitivity function is 1 for any controller, we can see that while F_{HG} gives a 38% deterioration from the minimum, F_{LG} gives a 188%. So, it is clear that sensitivity properties of the low-gain feedback are quite poor.

We have shown in this section, that low-gain feedback in open-loop strictly unstable plants leads to a closed-loop system with high sensitivity to input disturbances. When in addition the input is constrained, a high sensitivity means that disturbances more easily produce instability, as the overshoots generated by the poor disturbance response, could bring the system out of the local stability region. For these reasons, we do not recommend low-gain feedback strategies in open-loop strictly unstable plants with input constraints.

3 Local Stability Regions

Consider a linear time-invariant system with input constraints as follows:

$$\dot{x} = Ax + Bu \qquad \|u\| \leq 1 \tag{21}$$

Where (A, B) is controllable and A is completely strictly unstable, *i.e.* $\sigma(A) \subset \mathbf{R}_\circ^+$, where $\sigma(\cdot)$ denotes the spectrum. Suppose that we calculate a linear feedback control F to stabilise the system neglecting input constraints.

$$u = Fx, \qquad F : \sigma(A + BF) \subset \mathbf{R}_\circ^- \tag{22}$$

It is easy to show that, with restricted control energy, only local stability can be guaranteed. To see this, consider a positive definite matrix P such that $A^T P + PA = Q$ for some positive definite Q. Then, we can write

$$\dot{V}(x) = x^T (A^T P + PA)x + 2x^T PBu \tag{23}$$

Consider any *saturation* control on the control vector $u = Fx$ such that $\|u_{sat}\| \leq 1$ in all cases. Then, when the input is saturated, we can rewrite (23) as follows:

$$\dot{V}(x) = x^T (A^T P + PA)x + 2x^T PBu_{sat} \tag{24}$$

Then we have

$$\begin{aligned} \dot{V}(x) &\geq x^T Qx - 2\,\|x^T\, P\, B\| && (25)\\ &\geq x^T Qx - 2\,\gamma\, V^{\frac{1}{2}}(x)\,\|P^{\frac{1}{2}}\, B\| && (26) \end{aligned}$$

as we can write $V(x) = \|P^{\frac{1}{2}}\, x\|_2^2$, with $\gamma > 0$ a constant depending on the norm used. From (26) is clear that there exist positive α and β such that

$$\dot{V}(x) \geq \alpha\, V(x) - \beta\, V^{\frac{1}{2}}(x) \tag{27}$$

So, if an initial condition x_0 is big enough, such that $V(x_0) > \frac{\beta^2}{\alpha^2}$, then $\dot{V}(x_0) > 0$ and the system becomes unstable. Thus, it is clear that, for these systems, stability is only local. In the same context, we introduce the following definitions:

A state $x \in \mathbf{R}^n$ is *recoverable* if it is an initial condition from which the system can be driven to 0 by a suitable control, $u(\cdot)$, which satisfies the control constraint.

The *stability region* associated with a given controller is the set of all the initial states of the system for which that controller gives asymptotic stability. In case of a linear feedback control law $u = Fx$, we denote the associated stability region as $\mathcal{A}_F$.

The *maximum stability region* is the biggest stability region obtainable for the system. We denote this region as $\mathcal{A}_\infty$. It is clearly the set of all the recoverable states.

Some properties of the stability regions are; for completely strictly open loop unstable plants:

- They are open bounded sets.

- They are positively invariant, *i.e.* every trajectory $x(t)$ that starts inside the set belongs to the set $\forall t > 0$.
- They are symmetric with respect to the origin (for symmetric constraints on u).

We define the *region of linear operation* associated with a given controller as the set of all the states that can be driven to 0 by the controller, without violating the control constraints. In the state feedback case, we denote this region as $\mathcal{S}_F$.

It is not difficult to see that $\mathcal{A}_\infty \supset \mathcal{A}_F \supset \mathcal{S}_F$ for all stabilising linear feedback gains F.

In order to analyse stability properties and the degree of conservativeness of a given control design for the system (21), we will be interested in the following questions:

- What is the actual stability region $\mathcal{A}_F$ associated with that controller?
- What is maximal stability region $\mathcal{A}_\infty$ for the system considered ?
- Which control laws will give $\mathcal{A} = \mathcal{A}_\infty$ or $\mathcal{A}$ close to $\mathcal{A}_\infty$?

In Section 4 we will show a procedure to compute these regions for a given system of dimension 2 and we will refer some works that could be useful in estimating them for dimension $n \geq 2$.

4 Computing Stability Regions

The knowledge of the regions of stability associated with these systems is important in developing less conservative control methodologies. A criterion of evaluating different control designs may be that of comparing the regions of admissible states which they produce.

An idea that has been used extensively in optimal approaches is that of *reversing time*, see for example Hsu and Meyer (1968) and LeMay (1964). This device consists in running the *problem* backward, analysing the *reversed* system associated with (21):

$$\dot{x} = -Ax - Bu \qquad \|u\| \leq 1 \tag{28}$$

This stratagem has been commonly used to generate switching surfaces in times when analog computers were more popular than digital computers. However, the same idea may be very useful, in its present digital implementation, to study the stability of completely strictly open loop unstable systems with control constraints.

As we know from Section 3, stability regions for these systems are open bounded sets. This means that the boundary, completely formed by solution trajectories of the system (Hahn 1967), does not belong to the set.

As the origin is a local attractive point of the forward time system, and outside the stability region the system is unstable, it is clear that the boundary must be repulsive. If we analyse the behaviour of the associated reversed system (28), it will not be difficult to see that the same boundary will be attractive, and so, all the solution trajectories from any initial condition will be bounded. This property makes also attractive the idea of using the reversing time trick in studying stability limits of these systems.

Simulation results show that for single-input systems of order $n = 2$, the stability boundary is, in general for linear controlled cases, a limit cycle, which in the limit case – $\mathcal{A}_\infty$ – may degenerate into an heteroclinic orbit, with two singular points on it.

In general, the problem of computing and characterising these regions is not trivial, particularly when considering systems of order higher than 2. However, in many cases the regions can be approximated. In particular, a quadratic estimation of the region $\mathcal{A}_F$ certainly exists, as if we reverse the Lyapunov argument employed in Section 3, we get from:

$$V(x) = x^T P x \tag{29}$$

that, when the control saturates, gives

$$\begin{aligned} \dot{V}(x) &= x^T(A^T P + PA)x + 2\, x^T P\, B\, u_{sat} && (30) \\ &= x^T Q^{\frac{1}{2}} Q^{\frac{1}{2}} x + 2\, x^T Q^{\frac{1}{2}} Q^{-\frac{1}{2}} P\, B\, u_{sat} && (31) \end{aligned}$$

Where $Q^{\frac{1}{2}}$ denotes the symmetric positive square root of Q. Then, completing squares, we obtain

$$\begin{aligned} \dot{V}(x) &= x^T Q^{\frac{1}{2}} Q^{\frac{1}{2}} x + 2\, x^T Q^{\frac{1}{2}} Q^{-\frac{1}{2}} P\, B\, u_{sat} + u_{sat}^T B^T P Q^{-\frac{1}{2}} Q^{-\frac{1}{2}} P\, B\, u_{sat} && (32) \\ &\quad - u_{sat}^T B^T P Q^{-\frac{1}{2}} Q^{-\frac{1}{2}} P\, B\, u_{sat} && (33) \\ &= (Q^{\frac{1}{2}} x + Q^{-\frac{1}{2}} P\, B\, u_{sat})^T (Q^{\frac{1}{2}} x + Q^{-\frac{1}{2}} P\, B\, u_{sat}) - (P\, B\, u_{sat})^T Q^{-1} (P\, B\, u_{sat}) && (34) \end{aligned}$$

So, if

$$\|Q^{\frac{1}{2}} x + Q^{-\frac{1}{2}} P\, B\, u_{sat}\| < \|Q^{-\frac{1}{2}} P\, B\, u_{sat}\| \tag{35}$$

i.e.

$$V(x) < 4\, \delta \|Q^{-\frac{1}{2}} P\, B\, u_{sat}\|^2, \qquad \delta > 0 \tag{36}$$

then $\dot{V}(x) < 0$ and we have local asymptotic stability conditions for the system in saturating operation.

The problem of selecting appropriated matrices P and Q, can be managed by applying a remarkable technique suggested in Sabin and Summers (1990). In that paper, symbolic algebra programming and IMSL routines are combined to obtain optimal quadratic approximates of this kind of sets. While the paper is restricted to single-input systems, the methodology is applicable to higher-dimensional systems ($n \geq 3$). In addition, projections of the estimates onto any space of dimension $\leq n - 1$ can be readily found.

The region of linear operation $\mathcal{S}_F$ is a positive-invariant subset of the region where $\|u\| = \|Fx\| < 1$. Specifically, in 2^{nd} order systems it is clear that there is a *limit* pair of symmetric initial conditions $\pm\bar{x}_0$, such that

$$F\bar{x}_0 = 1 \quad \text{and} \quad F(A\bar{x}_0 + BF\bar{x}_0) = 0 \tag{37}$$

The 2nd condition in (37) means that the trajectory starting at $\bar{x}_0$ will be tangent to $\|Fx\| = 1$. This is the extreme initial condition that will give rise to a trajectory entirely contained in the region $\|Fx\| < 1$. Then, it is not difficult to approximate $\mathcal{S}_F$ with the knowledge of F and $\bar{x}_0$, this one easily computes based on (37). For higher orders, the same quadratic techniques as for $\mathcal{A}_F$ may be applied.

5 A Control Methodology

In this Section we suggest a means of control, which tries to give both the maximal possible region of stability, together with prespecified linear behaviour when the control becomes small. The design algorithm is summarised as follows:

1. Design a linear feedback controller $u = Fx$. We designate this as the *linear control law.*
2. Compute:
   (a) $\mathcal{A}_F$ the region of stability associated with that controller.
   (b) $\mathcal{S}_F$ the region where control never saturates.
3. Compute the maximal region of stability $\mathcal{A}_\infty$ and the associated optimal – minimum time – control switching surfaces and law. We designate this as the *nonlinear control law.*

Then, the proposed control methodology switches *with memory* between the nonlinear control law and the linear control law as follows:

- If previously using the nonlinear control law, keep using it until $x \in \mathcal{S}_F$.
- If previously using the linear control law, keep using it until $x \notin \mathcal{A}_F$.

The proposed scheme is showed in Figure 4.

In practice, as we pointed out in Section 4, regions $\mathcal{A}_F$ and $\mathcal{S}_F$ can be replaced by quadratic approximations $\Omega_\mathcal{A}$ and $\Omega_\mathcal{S}$, such that they verify $\Omega_\mathcal{S} \subset \mathcal{A}_F$ and $\Omega_\mathcal{S} \subset \Omega_\mathcal{A} \subset \mathcal{A}_\infty$. In this way, it is much easier to check the state for switching conditions. Indeed, when the dimensionality is greater than 2, it appears to be very difficult to compute, and represent, $\mathcal{S}_F$ and $\mathcal{A}_F$ exactly.

Note that the minimum-time optimal control law gives the maximal possible region of attraction. For the second order case, the required switching curves, for this optimal controller, are easy to compute and represent. In higher dimensional cases, this need not be so, and some approximation may be required. To our knowledge, there remain several open research problems in this area.

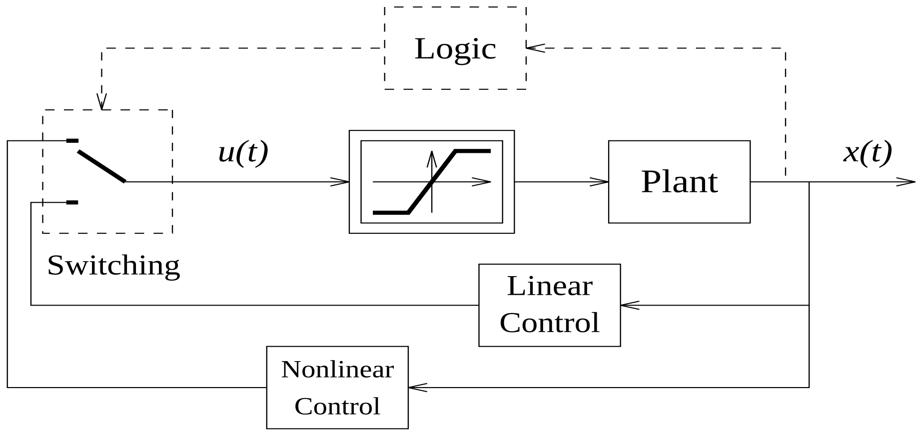

Figure 4: Proposed Control Scheme

6 Example

Consider again the system of Figure 2 introduced in (11) and (12). We compute two linear quadratic feedback gains F with the state weight matrix

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{38}$$

and two different control weights $R_1 = 0.01$ and $R_2 = 100$. Solving the corresponding algebraic Riccati equations, we obtain

$$F_1 = [-26.5948, -13.4173] \tag{39}$$
$$F_2 = [-4.0174, -4.0075] \tag{40}$$

With these two linear feedback gains, we plot the stability regions for this system following the reversing-time procedures described in Section 4. They are displayed in Figure 5 (a) and (b).

Note that the region of linear operation $\mathcal{S}_F$, is very restricted, even in the case of F_2, where it is bigger, but at the cost of a higher sensitivity. So, we see that control methodologies that keep the states of the system in these regions, may be remarkably conservative.

The minimum-time optimal control for this system gives the switching curves

$$\gamma^- : x_1 + x_2 + (1 - x_2)\log(1 - x_2) = 0 \qquad x_2 < 1 \tag{41}$$
$$\gamma^+ : x_1 + x_2 - (1 + x_2)\log(1 + x_2) = 0 \qquad x_2 > -1 \tag{42}$$

and the optimal – bang-bang – law is

$$\text{if} \quad \gamma^- \geq 0 \quad \text{or} \quad \gamma^+ > 0 \quad \text{then} \quad u = -1 \tag{43}$$

$$\text{else} \quad u = 1 \tag{44}$$

Quadratic approximations of the regions $\mathcal{A}_{F_1}$ and $\mathcal{A}_{F_2}$ are:

$$\Omega_{\mathcal{A}_1} : x^T P x - 0.045 \leq 0 \tag{45}$$
$$\Omega_{\mathcal{A}_2} : x^T P x - 0.075 \leq 0 \tag{46}$$

where

$$P = \begin{bmatrix} 0.50 & 0.25 \\ 0.25 & 0.25 \end{bmatrix} \tag{47}$$

As $\Omega_{\mathcal{S}}$ regions, we used

$$\Omega_{\mathcal{S}_1} : \quad \|F_1 x\| \leq 0 \qquad \|x\| \leq 0.5 \tag{48}$$
$$\Omega_{\mathcal{S}_2} : \quad x^T P x - 0.02 \leq 0 \tag{49}$$

These regions, together with the switching curves for the minimum-time controller, are displayed in Figure 5 (c) and (d). Time responses to the initial condition $x_0 = [0, -0.6]^T$ are displayed in Figure 6. We note that the predominant behaviour is due to the minimum-time optimal control law.

In Figure 7, we show the response of the system, with zero initial conditions, to a white noise input disturbance of 50% peak, uniform distribution and $10/3s^{-1}$ sampling rate. We can see clearly that the system with the *lower* gain F_2 presents a remarkable higher sensitivity to this disturbance than with the *higher* gain F_1. As we stated in Section 2, this feature can be critical, as a transient overshoot originating in some disturbance could bring the system to unrecoverable instability.

7 Conclusions

In this paper, we have seen that low-gain control gives undesirable sensitivity problems for open-loop unstable plants. Therefore, for open-loop unstable plants with control constraints, we have argued that many existing approaches, based on low-gain control, are inappropriate. We have proposed a new methodology for controlling such plants based on switching between a linear controller, and the optimal minimum-time controller. This methodology requires computation/approximation of certain regions of attraction in the state space. For a second order, single input system, these regions can be readily computed. For higher order and/or multiple inputs, these computations are much more difficult, and approximations for the optimal minimum-time control, in such cases, remain an open research problem.

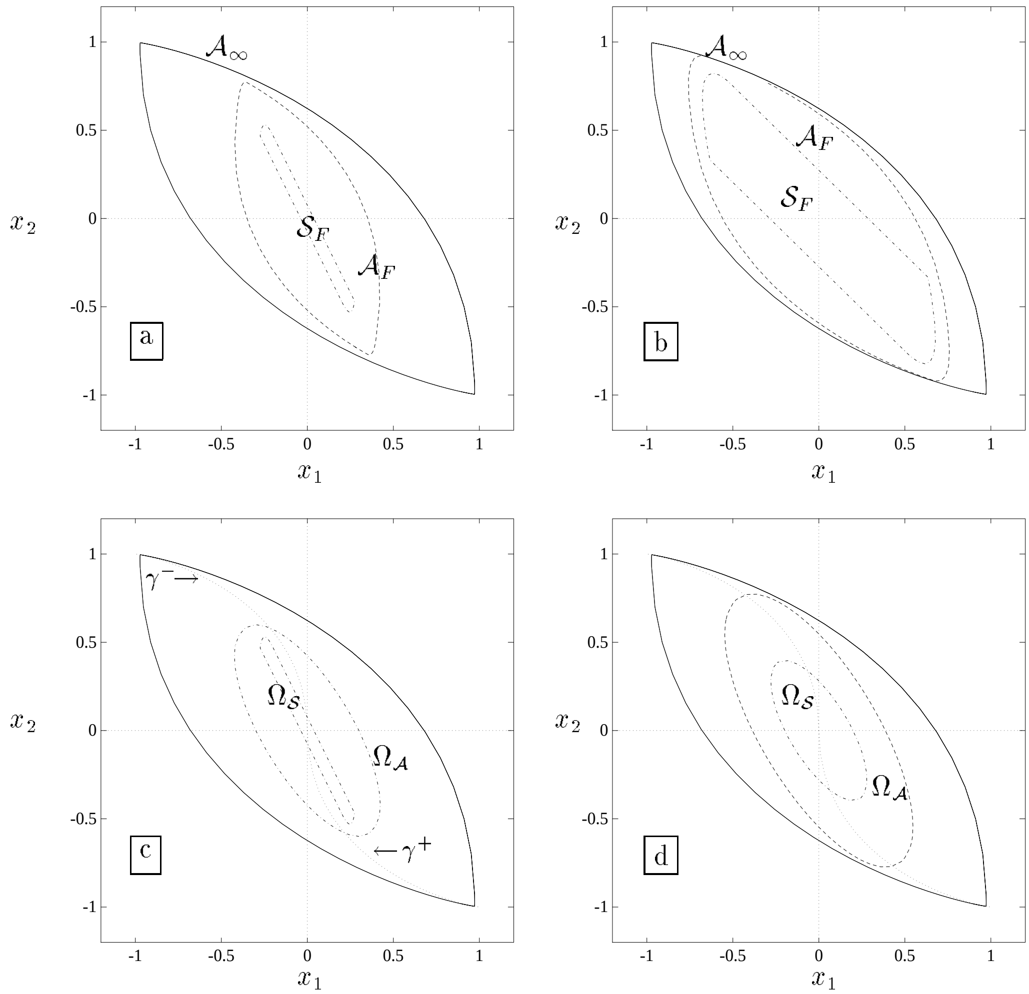

Figure 5: Example - (a) and (b): Regions of stability. (c) and (d): Approximations and switching curves. F_1: (a) and (c). F_2: (b) and (d)

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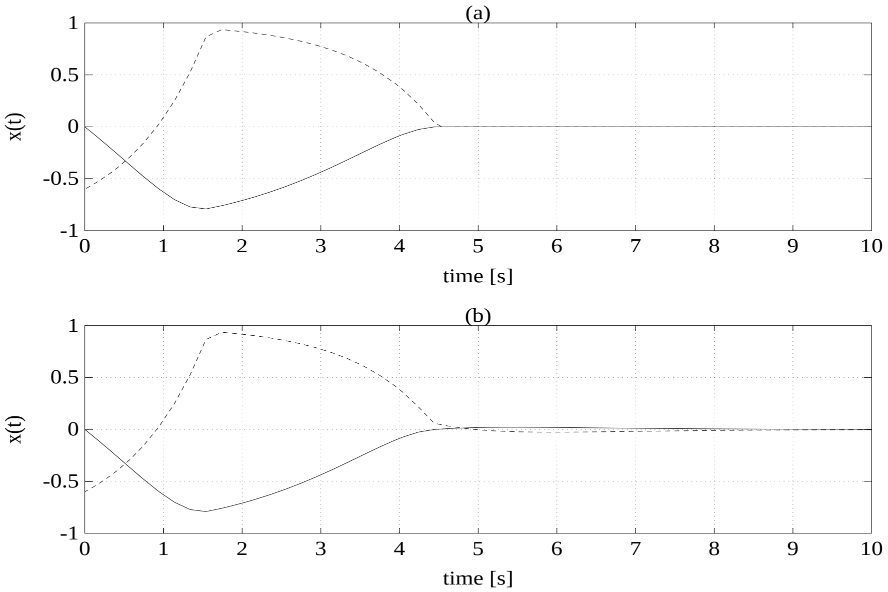

Figure 6: System response to $x_0 = [0, -0.6]^T$ with feedback gains F_1 (a) and F_2 (b).

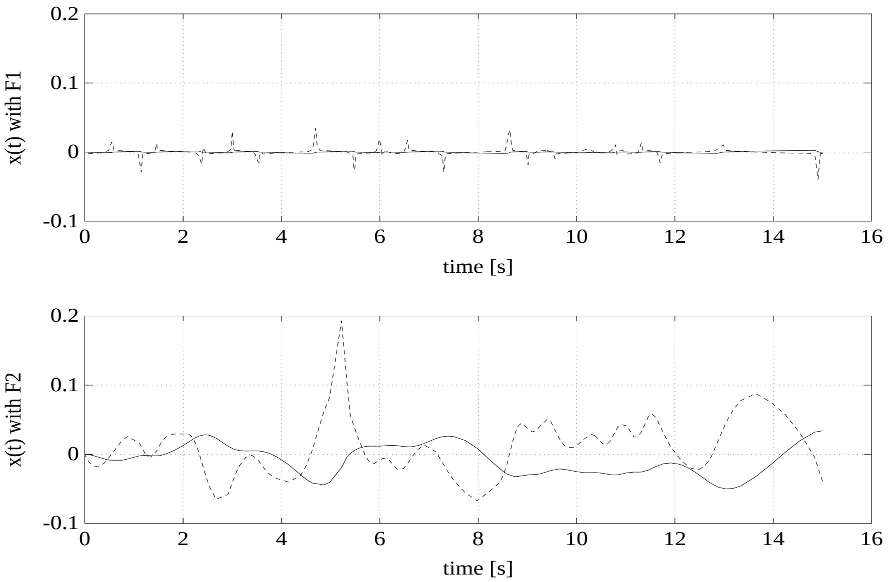

Figure 7: System sensitivity: response to a white noise disturbance with F_1 (upper plot) and F_2