Global and Semi-global Stabilizability in Certain Cascade Nonlinear Systems*

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Abstract

This paper addresses the issue of global and semi-global stabilizability of an important class of nonlinear systems, namely, a cascade of a linear, controllable system followed by an asymptotically (even exponentially) stable nonlinear system. Such structure may arise from the normal form of "minimum phase" nonlinear systems that can be rendered input-output linear by feedback. These systems are known to be stabilizable in a local sense, and in some cases global stabilizability results have also been obtained. However, it is also known that when the linear "connection" to the nonlinear system is non-minimum phase, i.e. it has zeros with positive real part, then global or semi-global stabilizability may be impossible. Indeed, it has been shown that for any given non-minimum phase linear subsystem, there exists an asymptotically stable nonlinear subsystem for which the cascade cannot be globally stabilized. We expand on the understanding of this area by establishing, for a broader class of systems, conditions under which global or semi-global stabilization is impossible for linear and nonlinear feedbacks.

Keywords: Nonlinear systems, global stabilization, cascade systems, non-minimum phase systems.

1 Introduction

The stabilizability of interconnected nonlinear systems has attracted increasing interest since the appearance in the control scene of the *normal form* and *zero dynamics* [5]. A number of recent papers have considered the problem of global stabilization of linear-nonlinear cascade systems of the form

$$
\begin{aligned}
\dot{x} &= f_0(x) + f(x,\xi,y), && x \in \mathbb{R}^l, \\
\dot{\xi} &= A\xi + Bu, && \xi \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \\
y &= C\xi, && y \in \mathbb{R}^p,
\end{aligned}
\tag{1}
$$

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where f_0 and f are smooth (i.e. C^∞) functions, $\dot{x} = f_0(x)$ is globally asymptotically stable (GAS) at $x = 0$, $f(x, 0, 0) = 0$, and the linear subsystem (A, B, C) is assumed to be stabilizable by state feedback.

Although these systems are asymptotically stabilizable by linear ξ-feedback in a local sense (e.g., [2]), the extension of such results to being global is not immediate, and further conditions may need to be applied. For example, it is known that the cascade (1) is not necessarily globally stabilizable by linear ξ-feedback only, i.e. with $u = K\xi$, due to the existence of phenomena like *peaking* [6, 15]. Another obstacle to global stabilizability has been recently identified as *unboundedness unobservability* [10], where some unmeasured states may escape in finite time while the output remains bounded.

In [7] and [14], sufficient conditions have been given to globally asymptotically stabilize these "partially linear composite systems" by a smooth feedback $u = K\xi + \nu(x, \xi)$. A key restriction on the cascades for which GAS can be achieved is that either the linear subsystem must be *weakly minimum phase* [7, 14, 13, 3, 9], or the *growth* of the connection terms is constrained [14, Proposition 5].

The conditions for GAS mentioned above are only sufficient, but as it has been pointed out [6], they are somehow close to being necessary. Furthermore, in [14] it has been proven that given a non-minimum phase linear system, one can always find a nonlinear system such that the cascade verifying all the other conditions is not globally stabilizable. Our results further expand on this idea, and, in addition to GAS, they address the problem of *semi-global stabilizability*[1].

The concept of semi-global stabilizability has been considered in many recent studies related to nonlinear stabilizability [1, 6, 2, 16, 8]. A system is semi-globally stabilizable to an equilibrium point x_e by means of a class $\mathcal{F}$ of feedback control laws, if for any a priori determined bounded set Ω of initial conditions there exist a control law in $\mathcal{F}$ that makes x_e asymptotically stable with a domain of attraction which contains Ω.

In this paper, we show that a given class of GAS nonlinear systems cascaded with *any* SISO non-minimum phase linear system is not (even semi-) globally stabilizable by linear, *time-invariant*, static ξ-feedback. Also, we give a class of SISO linear-nonlinear cascades systems for which *no control* can give global or semi-global stability. These results are illustrated by an example, where a system of the first class is semi-globally stabilized by linear *time-varying* ξ-feedback, and globally stabilized by nonlinear (ξ, x)-feedback.

Notation $\mathbb{R}$, $\mathbb{C}$, denote real and complex numbers, respectively, and $\operatorname{Re} x$ denotes the real part of the number x. $\mathbb{C}^-$ and $\mathbb{C}^+$ are the open left- and right-half planes, respectively. $\|\cdot\|$ is the Euclidean norm for matrices. The rest of the notation is fairly standard, or introduced where required.

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[1] Other authors also use the terms *potentially global* or *on compacta* stabilizability.

2 An explosive cascade

We study the following class of linear-nonlinear cascades,

$$\dot{x} = -\alpha x + \beta x^m y^{2p}, \tag{2}$$

$$\begin{aligned} \dot{\xi} &= A\xi + Bu, \\ y &= C\xi + Du, \end{aligned} \tag{3}$$

where m, p are positive integers, $m > 1$; α, β are positive real numbers; $x, u, y \in \mathbb{R}$, and $\xi \in \mathbb{R}^n$. We assume that the pair (A, B) is controllable and the linear subsystem (3) is non-minimum phase, *i.e.* its transfer function, $\mathcal{G}(s) = C(sI - A)^{-1}B + D$, has a zero at the complex frequency $s = \nu$ with positive real part, $\mathrm{Re}\,\nu > 0$. We also assume $C \neq 0$.

These cascades present two characteristics that complicate their stabilizability in a global sense: the linear "connection" (3) is non-minimum phase, and the nonlinear subsystem (2) has a strong coupling with its input. On the one hand, the non-minimum phase feature implies the existence of peaking in y with high gain ξ-feedback. On the other hand, the strong coupling implies that if y is not sufficiently small, finite time escape of x may occur. Therefore, even when the nonlinear subsystem is globally exponentially stable without input, the cascade may fail to be GAS.

We shall derive a necessary condition for the semi-global stabilizability of system (2)-(3) that will be central to the results in §2. Define the following change of coordinates,

$$\eta(t) \triangleq \frac{e^{-\alpha' t}}{(m-1)x^{m-1}(t)}, \qquad x \neq 0,$$

with $\alpha' \triangleq \alpha(m-1)$. Then, we can write (2) as

$$\dot{\eta} = -\beta y^{2p} e^{-\alpha' t}. \tag{4}$$

The solution to (4) is given by

$$\eta(t) = \eta(0) - \beta \int_0^t e^{-\alpha' \tau} y^{2p}(\tau)\, d\tau. \tag{5}$$

Suppose that x_0 is a positive initial condition for (2). Then $\eta_0 = x_0^{(1-m)}/(m-1)$ is also positive, and so we can see from (5) that the approach of $\eta(t)$ to zero at any t would imply explosion of the nonlinear subsystem (2); i.e. the trajectory $x(t)$ starting at x_0 would escape to infinity. Therefore, a necessary condition for the stability of (2) with any positive initial condition is the following:

$$\int_0^t e^{-\alpha' \tau} y^{2p}(\tau)\, d\tau < \frac{\eta(0)}{\beta}, \qquad \text{for any } t \geq 0,$$

or equivalently, since $\eta(t)$ is a monotone function,

$$\int_0^\infty e^{-\alpha' \tau} y^{2p}(\tau)\, d\tau < \frac{\eta(0)}{\beta}, \qquad \text{for every } \eta(0) > 0 \tag{6}$$

Based on the above condition, in next section we further explore the relationship between the "degree of non-minimum phaseness" of the linear subsystem and the dynamics of the nonlinear subsystem. As we shall see, this relationship will determine the stabilizability of the cascade (2)-(3).

3 Does a non-minimum phase linear subsystem necessarily imply peaking?

This section contains our main results, which characterize stabilizability properties of the class of linear-nonlinear cascades (2)-(3) introduced before. Depending on the position of the non-minimum phase zero of the linear subsystem, we shall demonstrate

- conditions under which the cascade cannot be semi-globally stabilizable; and
- conditions under which the cascade cannot be semi-globally stabilized by using linear, time-invariant, static feedback of the linear subsystem states.

We shall also extend these issues to a more general class of cascades, where the nonlinear subsystem presents a particular type of Lyapunov function.

The main results require some properties of linear systems and matrices. The following lemma recalls an input-output relation satisfied by linear systems with non-minimum phase zeros [11].

Lemma 1
Consider a linear system

$$\begin{aligned}\dot{\xi} &= A\xi + Bu,\\ y &= C\xi + Du,\end{aligned}$$

with ξ in $\mathbb{R}^n$ and u, y in $\mathbb{R}$. If ν is a non-minimum phase zero of the transfer function $\mathcal{G}(s) = C(sI - A)^{-1}B + D$, and $y(t)$ is the bounded response to a bounded input $u(t)$ with initial condition ξ_0, then

$$\int_0^\infty e^{-\nu\tau} y(\tau)\, d\tau = C(\nu I - A)^{-1}\xi_0.$$

Proof: The proof follows trivially on noting that since $y(t)$ and $u(t)$ are bounded, $s = \nu$ is in the region of convergence of their corresponding Laplace transforms, $\mathcal{Y}(s)$ and $\mathcal{U}(s)$. Indeed, let $\mathcal{Y}(s)$ denote the Laplace transform of $y(t)$

$$\mathcal{Y}(s) = \int_0^\infty e^{-s\tau} y(\tau)\, d\tau \tag{7}$$

Evaluating $\mathcal{Y}(s)$ at $s = \nu$, we get

$$\begin{aligned}\int_0^\infty e^{-\nu\tau} y(\tau)\, d\tau &= \mathcal{Y}(\nu)\\ &= C(\nu I - A)^{-1}\xi_0 + [C(\nu I - A)^{-1}B + D]\,\mathcal{U}(\nu)\\ &= C(\nu I - A)^{-1}\xi_0\end{aligned}$$

□

We are now able to present the first of our statements, namely, conditions under which it is impossible to stabilize the cascade.

Theorem 2
The cascade system (2)-(3) is not semi-globally stabilizable if the linear subsystem (3) has a non-minimum phase zero at $s = \nu$ such that $\mathrm{Re}\,\nu > \alpha(m-1)/2p$.

Proof: From Lemma 1 we can write

$$\begin{aligned} |C(\nu I - A)^{-1}\xi_0| &= |\mathcal{Y}(\nu)| \\ &\leq \int_0^\infty e^{-\mathrm{Re}\,\nu\tau} |y(\tau)|\, d\tau. \end{aligned} \tag{8}$$

By applying Hölder's Inequality to the RHS of (8) and solving an integral, we obtain

$$\int_0^\infty e^{-\mathrm{Re}\,\nu\tau}|y(\tau)|\,d\tau \leq \left(\frac{2p-1}{2p\ \mathrm{Re}\,\nu - \alpha'}\right)^{\frac{2p-1}{2p}} \left(\int_0^\infty e^{-\alpha'\tau} y^{2p}(\tau)\,d\tau\right)^{\frac{1}{2p}}, \tag{9}$$

where, as in §1, we denoted $\alpha' = \alpha(m-1)$. Inequalities (8) and (9) give

$$\int_0^\infty e^{-\alpha'\tau} y^{2p}(\tau)\,d\tau \geq M, \tag{10}$$

where

$$M = \left(\frac{2p\ \mathrm{Re}\,\nu - \alpha'}{2p-1}\right)^{2p-1} |C(\nu I - A)^{-1}\xi_0|^{2p}.$$

Note that M is a number independent of any feedback. Moreover, M can always be made positive and arbitrarily large by appropriate choice of the initial condition ξ_0. As we have seen in §1, a necessary condition for the stability of this cascade is that (6) be true; but (10) and the previous remark on M indicate that this condition can always be violated, hence completing the proof. □

The above result specifies a condition for which the class of cascades given by (2)-(3) is not globally or semi-globally stabilizable by any control. Can we expect to be able to do anything when this condition is not satisfied? We shall see in Theorem 4 that if we restrict ourselves to linear, time invariant, static ξ-feedback, then it is again impossible to semi-globally stabilize the cascade.

Once more, a property of linear systems leads the way to the next theorem. Let (A, B) be a controllable pair, A in $\mathbb{R}^{n\times n}$, B in $\mathbb{R}^{n\times 1}$. Define $\mathbf{K}$ as the set of all row vectors K in $\mathbb{R}^{1\times n}$ such that $A_K \triangleq A + BK$ has all its eigenvalues in $\mathbb{C}^-$. Then, we have the following result.

Lemma 3
If ν is a complex number with positive real part, then

$$\sup_{K\in\mathbf{K}} \|(\nu I - A_K)^{-1}\| < \infty. \tag{11}$$

Proof: The proof, though perhaps intuitively reasonable, is a little long and involved, and so it is deferred to the Appendix. □

Remark 1 The above result relies on the fact that the linear system associated with (A, B) is single-input. Indeed, Lemma 3 is not true for multi-input systems. This can be seen by means of the following simple counterexample: $A = 0^{2\times 2}, B = I^{2\times 2}$, choose

$$K = \begin{bmatrix} -\rho_1 & 0 \\ -\rho_2 & -\rho_3 \end{bmatrix}, \tag{12}$$

with ρ_i, $i = 1, 2, 3$, positive numbers. Then write

$$\begin{aligned} (\nu I - A_K)^{-1} &= (\nu I - K)^{-1} \\ &= \begin{bmatrix} (\nu+\rho_1)^{-1} & 0 \\ \rho_2(\nu+\rho_1)^{-1}(\nu+\rho_3)^{-1} & (\nu+\rho_3)^{-1} \end{bmatrix}. \end{aligned} \tag{13}$$

Clearly from (13), $\|(\nu I - A_K)^{-1}\|$ grows without bound with ρ_2, even though stability of A_K is independent of ρ_2.

Theorem 4
If the linear subsystem (3) is non-minimum phase, then the cascade system (2)-(3) is not semi-globally stabilizable by linear, time-invariant, static ξ-feedback, that is, $u = K\xi$, *with* K *in* $\mathbb{R}^{1\times n}$.

Proof: Suppose we stabilized the linear subsystem (3) with a state feedback $u = K\xi$, so that all the eigenvalues of the closed loop evolution matrix, $\lambda_i\{A_K\}$, with $i = 1, 2, \ldots, n$, are in $\mathbb{C}^-$. Let ν be a non-minimum phase zero of the linear subsystem, $\mathcal{Y}(s)$ the Laplace transform of the closed-loop output, and α' as defined in §1. Then, we can write

$$\begin{aligned} \mathcal{Y}(\alpha') &= (C + DK)(\alpha' I - A_K)^{-1}\xi_0 \\ &= (C + DK)\left(\nu I - A + (\alpha' - \nu)I - BK\right)^{-1}\xi_0 \end{aligned}$$

By applying the Matrix Inversion Lemma to the last equation, we get

$$\mathcal{Y}(\alpha') = (C + DK)(\nu I - A)^{-1}\left(I - H\left(I + (\nu I - A)^{-1}H\right)^{-1}(\nu I - A)^{-1}\right)\xi_0,$$

where $H = (\alpha' - \nu)I - BK$. Using the fact that $\mathcal{G}(\nu) = D + C(\nu I - A)^{-1}B = 0$, and after some manipulation, we obtain

$$\mathcal{Y}(\alpha') = C(\nu I - A)^{-1}\left(I - (\alpha' - \nu)(\alpha' I - A_K)^{-1}\right)\xi_0. \tag{14}$$

Now suppose that we choose the initial condition ξ_0 proportional to the complex conjugate transpose of the row vector $C(\nu I - A)^{-1}$, i.e.

$$\xi_0 = \gamma\,(\bar{\nu} I - A^T)^{-1}C^T, \tag{15}$$

with γ a real number. Notice that the vector $(\bar{\nu} I - A^T)^{-1}C^T$ is non-null, since $(\nu I - A)^{-1}$ is full-rank and $C \neq 0$. Then, substituting (15) in (14) and taking absolute values on both sides gives

$$\begin{aligned} |\mathcal{Y}(\alpha')| &= |\gamma|\,|C(\nu I - A)^{-1}\left(I - (\alpha' - \nu)(\alpha' I - A_K)^{-1}\right)(\bar{\nu} I - A^T)^{-1}C^T| \\ &\geq |\gamma|\,\|C(\nu I - A)^{-1}\|^2\,\underline{\sigma}, \end{aligned} \tag{16}$$

where $\underline{\sigma}$ denotes the smallest singular value of the matrix $\Gamma \triangleq I - (\alpha' - \nu)(\alpha' I - A_K)^{-1}$, for short. Note that

$$\begin{aligned}
\underline{\sigma} &= \|\Gamma\|^{-1} \\
&= \|(\nu I - A_K)^{-1}(\alpha' I - A_K)\|^{-1} \\
&= \|I + (\alpha' - \nu)(\nu I - A_K)^{-1}\|^{-1} \\
&\geq \frac{1}{1 + |\alpha' - \nu| \, \|(\nu I - A_K)^{-1}\|}
\end{aligned} \tag{17}$$

Lemma 3 assures that $\|(\nu I - A_K)^{-1}\|$ is uniformly bounded on $\mathbf{K}$; denote this bound by μ. Then, we get from (16) and (17),

$$|\mathcal{Y}(\alpha')| \geq \frac{|\gamma| \, \|C(\nu I - A)^{-1}\|^2}{1 + |\alpha' - \nu| \, \mu}. \tag{18}$$

On the other hand, we can bound $|\mathcal{Y}(\alpha')|$ and use Hölder's Inequality to get

$$\begin{aligned}
|\mathcal{Y}(\alpha')| &\leq \int_0^\infty e^{-\alpha' \tau} |y(\tau)| \, d\tau \\
&\leq \left(\frac{1}{\alpha'}\right)^{\frac{2p-1}{2p}} \left(\int_0^\infty e^{-\alpha' \tau} y^{2p}(\tau) \, d\tau\right)^{\frac{1}{2p}}.
\end{aligned} \tag{19}$$

From (18) and (19) together, it follows

$$\int_0^\infty e^{-\alpha' \tau} y^{2p}(\tau) \, d\tau \geq M,$$

where

$$M = \frac{(\alpha')^{2p-1} \, |\gamma|^{2p} \, \|C(\nu I - A)^{-1}\|^{4p}}{(1 + |\alpha' - \nu| \, \mu)^{2p}}.$$

Here, as in the proof of Theorem 2, M is a number independent of feedback K. Moreover, M can be set positive and arbitrarily large by selection of the number γ. Therefore, the necessary condition for stability (6) can always be violated, and so the proof is completed. □

The issues addressed by Theorem 2 and Theorem 4 can be extended to cascaded linear-nonlinear systems with the nonlinear subsystem having a particular type of Lyapunov function.

Theorem 5
Consider the system

$$\begin{aligned}
\dot{x} &= f(x, y), \\
\dot{\xi} &= A\xi + Bu, \\
y &= C\xi + Du,
\end{aligned} \tag{20}$$

where the linear subsystem is non-minimum phase, i.e. has a zero $\nu \in \mathbb{C}^+$, and the nonlinear subsystem $\dot{x} = f(x, 0)$ has a Lyapunov function $V(x) \geq 0$, $V(x) = 0 \Leftrightarrow x = 0$, and $V(x) \to \infty \Rightarrow \|x\| \to \infty$, such that

$$\dot{V} \geq -\alpha V + \beta V^n y^{2p}, \tag{21}$$

with α, β positive real numbers, and n, p positive integers, $n > 1$. Then:

(i) The cascade (20) is not semi-globally stabilizable by linear, time-invariant, static ξ-feedback.

(ii) If Re $\nu > \alpha(m-1)/2p$, the cascade (20) is not semi-globally stabilizable at all.

Proof: Note that if V and W satisfy

$$\begin{aligned} \dot{V} &\geq -\alpha V + \beta V^n y^{2p} \\ \dot{W} &= -\alpha W + \beta W^n y^{2p}, \end{aligned}$$

and $V(t=0) = W(t=0)$, then $V \geq W$ for all $t > 0$, see for example [4]. The proof then follows by the same arguments used in the proofs of Theorem 2 and Theorem 4 applied to the cascade

$$\begin{aligned} \dot{W} &= -\alpha W + \beta W^n y^{2p} \\ \dot{\xi} &= A\xi + Bu \\ y &= C\xi + Du. \end{aligned}$$

$\square$

To illustrate these results, next section presents examples of a systems in the class referred to in Theorem 4, where we could expect to stabilize the cascade, although not using a feedback $u = K\xi$. In fact, we achieve semi-global stability using linear static *time-varying* ξ-feedback, and global stability using nonlinear (ξ, x)-feedback.

4 Examples: a dynamic "gambit".

As we have seen in the results above, the question of the linear subsystem being non-minimum phase is critical to the stabilizability of the cascade, particularly if we are restricted to a control law $u = K\xi$. Nevertheless, the situation is different if we use a linear *time-varying* feedback $u = K(t)\xi$, or a nonlinear (ξ, x)-feedback. Indeed, as we shall see in the following examples, if the non-minimum phase zeros of the linear subsytem are "not too fast" (see Theorem 2), global and semi-global stability can be obtained using a switching control strategy.

This control strategy consists of two stages. First, annihilate the output of the linear subsystem to let the nonlinear state go to zero, while allowing the linear states to grow exponentially. Second, after a suitable period of time switch back to stabilize the linear states. Since the linear states grow at a slower rate than the decay of the nonlinear state, it is always possible to find a switching time without finite time escape of x. This stratagem is somehow reminiscent of a chess gambit, where a piece (stability of the linear states) has to be initially sacrificed to ultimately win the match (global stability of the cascade).

The first example has a linear subsystem with relative degree zero. In this case the output can be annihilated instantaneously in the first stage of the stabilization procedure. The second example illustrates how this technique can be generalized when the relative degree of the linear subsystem is greater than zero. Computations are more involved in the later case, and so we only address semi-global stabilizability.

4.1 Example 1: Linear subsystem with relative degree zero

We now introduce the example. Consider the cascade

$$\begin{aligned} \dot{x} &= -3x + x^2y^2 \\ \dot{\xi} &= u \\ y &= \xi - u, \end{aligned} \tag{22}$$

where x, y, u are in $\mathbb{R}$. The linear subsystem has a non-minimum phase zero $\nu = 1$, and so $\nu < \alpha'/2p = 3/2$. Hence, by Theorem 4, we know that linear static ξ-feedback cannot — in a global sense — stabilize the cascade.

Remark 2 Note that $x = 0$ is an invariant manifold of system (22) independently of the control law. Hence, if at any time, T say, $x(T) > 0$ ($x(T) < 0$), then for $t > T$, $x(t) \geq 0$ ($x(t) \leq 0$). This will allow us to deal with each case separately.

We first show how the cascade (22) is semi-globally stabilized using linear time-varying ξ-feedback, and later on, how global stability is achieved by nonlinear (ξ, x)-feedback.

4.1.1 Semi-global stability

Let the initial conditions be x_0, ξ_0. We propose the following control strategy,

$$u = \begin{cases} -\xi & \text{if } x_0 \leq 0, \text{ or if } x_0 > 0 \text{ and } t \geq T, \\ \xi & \text{if } x_0 > 0 \text{ and } t < T, \end{cases} \tag{23}$$

where

$$T = \log^+\left(\frac{4\xi_0^2 x_0}{5}\right) + \epsilon, \tag{24}$$

with $\log^+(\cdot) \triangleq \max\{\log(\cdot), 0\}$, and ϵ a small positive number. This control law is fixed if $x_0 \leq 0$, and has at most one switching at $t = T$, if $x_0 > 0$.

Case 1 If $x_0 \leq 0$, then the control law $u = -\xi$ gives u, ξ, x bounded for $t \geq 0$, and moreover, $u(t), \xi(t), x(t) \to 0$ as $t \to \infty$.

Proof: The differential equations become

$$\begin{aligned} \dot{x} &= -3x + x^2y^2 \\ \dot{\xi} &= -\xi \\ y &= 2\xi. \end{aligned} \tag{25}$$

Recall that, by Remark 2, $x_0 \leq 0$ implies that $x(t) \leq 0$ for $t > 0$. Let $V = -x + \xi^2/2$ be a Lyapunov function candidate for (25) with $x \leq 0$. Then

$$\dot{V} = 3x - x^2y^2 - \xi^2,$$

and so $\dot{V} \leq 0$ for $x \leq 0$ and any ξ, and $\dot{V} = 0$ if and only if $x = 0$ and $\xi = 0$. Hence, the origin is an asymptotically stable equilibrium for system (25) with initial conditions $x_0 < 0$, ξ_0. □

Case 2 If $x_0 > 0$, then the switching control law given in (23) gives u, ξ, x bounded for $t \geq 0$, and moreover, $u(t), \xi(t), x(t) \to 0$ as $t \to \infty$.

Proof: First, for $t < T$, the control is set to $u = \xi$, so the system has the form

$$\begin{aligned} \dot{x} &= -3x, \\ \dot{\xi} &= \xi, \end{aligned}$$

then $x(t) = e^{-3t}x_0$, and $\xi(t) = e^t\xi_0$. This obviously makes the linear subsystem unstable, but note that the condition $\nu < \alpha'/2p$ assures us that $x(t)$ goes to zero faster than $|\xi(t)|$ increases.

Then, at $t = T$ — and for all $t > T$ — set the control to $u = -\xi$, to render the linear subsystem stable. The linear state will evolve as $\xi(t) = e^{-(t-2T)}\xi_0$, and the output, $y = 2\xi$, will display a big peak before starting to decrease. Let us see that, by the choice of T, this peak does not provoke finite-time escape of x, and furthermore, gives exponential decay of all variables.

Recalling the change of coordinates introduced in §1, we have the associated variable $\eta(t) = e^{-3t}/x(t)$, and for $t \geq T$ we can write

$$\begin{aligned} \eta(t) &= x_0^{-1} - \frac{4}{5}\xi_0^2 e^{4T}(e^{-5T} - e^{-5t}) \\ &\geq x_0^{-1}\left(1 - \frac{4}{5}\xi_0^2 x_0 e^{-T}\right) \\ &\geq x_0^{-1}(1 - e^{-\epsilon}), \end{aligned} \tag{26}$$

where the last step uses the choice of T in (24). Returning to the original variables, we have obtained

$$x(t) \leq \frac{e^{-3t}x_0}{1 - e^{-\epsilon}}, \qquad t \geq T \tag{27}$$

showing the boundedness of the nonlinear state, x, for every time $t \geq T$. Since T is finite for finite initial conditions, the linear state, ξ, and the control, u, are also exponentially bounded and the result follows. □

This strategy can be applied to every given set of initial conditions, thus resulting in semi-global stability. Note that the semi-global property provided by this time-varying controller is *not uniform in time*, though.

From (24), we can also draw the boundary of the basin of attraction corresponding to a given T. In particular, for $\xi_0^2 x_0 \leq 5/4$, the cascade is stabilized without switching by using the fixed control law $u = -\xi$. Figure 1 shows, in the state-space, different boundaries for three values of T.

Figure 2 displays simulation responses for $\xi_0 = -1$, $x_0 = 2$, and different switching times. For these initial conditions, (24) gives $T = 0.47 + \epsilon$. It can be seen from the plots that the critical case with $\epsilon = 0$ still stabilizes the system, although with a big peak in x.

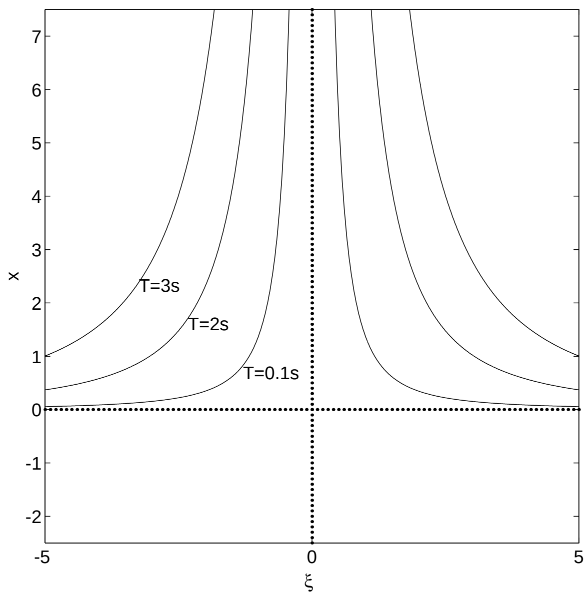

Figure 1: Basins of attraction for different T's.

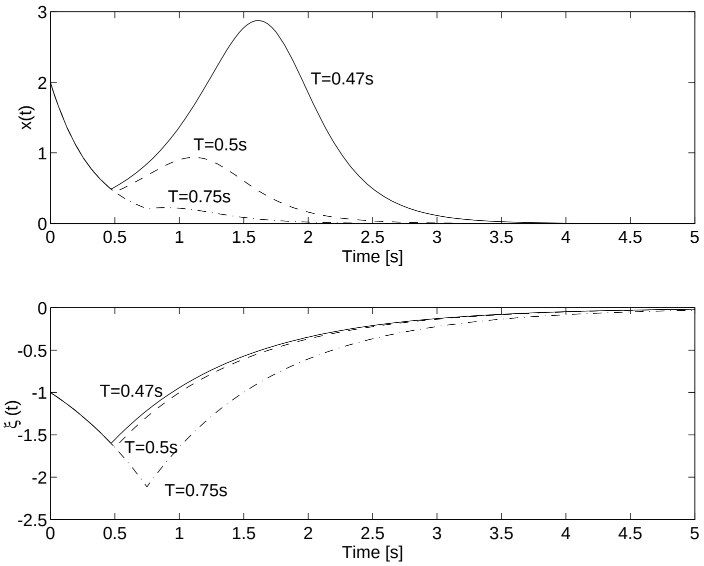

Figure 2: Time responses for semi-global stability.

4.1.2 Global stability

Alternatively, a nonlinear switching strategy can be devised if we also feed back the nonlinear state x. In this case, we can use the control law

$$u = \begin{cases} \xi & \text{whenever } x\xi^2 \geq 5e^{-\epsilon}/4, \\ -\xi & \text{whenever } x\xi^2 < 5e^{-\epsilon}/4, \end{cases}$$

with ϵ a small positive number. Hence, at most one switching occurs, and the system is *globally* stabilized (in this case the stability is uniform in time). This is easily seen from the following arguments.

Case 3 If, at any time $t = T$, we have $x(T) < 0$, then the control law $u(t) = -\xi(t)$ gives u, ξ, x bounded for $t \geq T$, and moreover, $u(t), \xi(t), x(t) \to 0$ as $t \to \infty$.

Proof: The proof follows the same steps than the proof of Case 1, with $x(T)$ instead of x_0. □

Case 4 If, at any time $t = T$, we have $x(T) > 0$ and $x(T)\,\xi^2(T) \leq 5e^{-\epsilon}/4$, then the control law $u = -\xi$ causes u, ξ, x to be bounded for $t \geq T$, and moreover, $u(t), \xi(t), x(t) \to 0$ as $t \to \infty$ without further switching.

Proof: For this case, the system's equations are given by (25) again. Let $V = x\xi^2$; then it is easy to check that

$$\begin{aligned} \dot{V} &= -5V(1 - \frac{4}{5}V) \\ &\leq -5V(1 - e^{-\epsilon}), \end{aligned}$$

and so V goes to zero exponentially,

$$V(t) \leq V(T)e^{-\delta(t-T)}, \qquad \delta = 5(1 - e^{-\epsilon}), \tag{28}$$

i.e. there is no further switching for $t > T$.

The exponential decrease of V determines that also x goes exponentially to zero. Indeed, the differential equation for x in (25) can be written as

$$\dot{x} = -3x + 4xV.$$

Its solution for $t \geq T$ is non-negative, by Remark 2, and more specifically,

$$\begin{aligned} x(t) &= e^{-3(t-T)}x(T) + \int_T^t e^{-3(t-\tau)}x(\tau)V(\tau)d\tau \\ &\leq e^{-3(t-T)}x(T) + \int_T^t e^{-3(t-\tau)}x(\tau)V(T)e^{-\delta(\tau-T)}d\tau. \end{aligned} \tag{29}$$

Multiplying both sides of (29) by e^{3t}, and denoting $z(t) = e^{3t}x(t)$, we get

$$z(t) \leq z(T) + e^{\delta T}V(T)\int_T^t z(\tau)e^{-\delta\tau}d\tau. \tag{30}$$

Applying the Bellman-Gronwall Lemma to (30), it follows that

$$\begin{aligned} z(t) &\leq z(T)e^{\left(e^{\delta T}V(T)\int_T^t e^{-\delta\tau}d\tau\right)} \\ &\leq z(T)e^{\frac{V(T)}{\delta}}. \end{aligned} \tag{31}$$

Finally, (31) gives

$$x(t) \leq e^{-3(t-T)}x(T)e^{\frac{x(T)\xi^2(T)}{5(1-e^{-\epsilon})}}.$$

On the other hand, note that ξ and u are already exponentially decreasing to zero. In effect, with this control law we have that $\xi(t) = \xi(T)\,e^{-(t-T)} = -u(t)$. □

Case 5 If, at any time $t = T$, we have $x(T)\,\xi^2(T) > 5e^{-\epsilon}/4$, then the control law $u = \xi$ causes $x\xi^2$ to decrease exponentially to zero.

Proof: For this case, the system's equations become

$$\begin{aligned} \dot{x} &= -3x \\ \dot{\xi} &= \xi. \end{aligned}$$

Then,

$$\begin{aligned} \frac{d(x\xi^2)}{dt} &= -3x\xi^2 + 2x\xi^2 \\ &= -x\xi^2. \end{aligned}$$

□

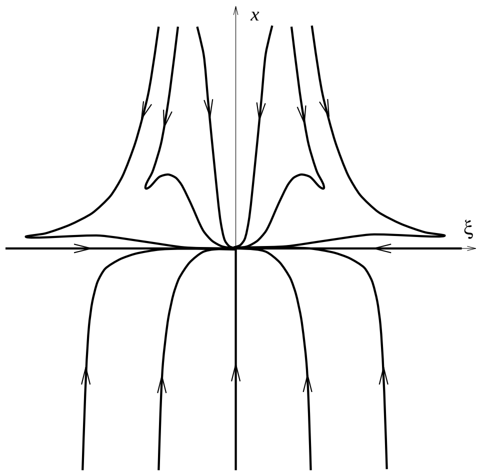

Figure 3: Phase portrait of the system globally stabilized with nonlinear (ξ, x)-feedback.

From Cases 3, 4 and 5, we can see that any pair of initial conditions will produce at most one switching of the controller, rendering the system asymptotically stable to zero. Note that in Case 5, the linear subsystem will be unstable, ξ growing exponentially, but only for a finite interval of time, i.e. until $x(T)\,\xi^2(T) \leq 5e^{-\epsilon}/4$ for some finite T. Figure 3 shows the phase portrait of the closed loop system.

4.2 Example 2: Linear subsystem with relative degree one

This control strategy for semi-global stabilizability can be generalized to a cascade where the linear subsystem has relative degree greater than zero. In this case, though, we cannot force the linear subsystem output to zero immediately. Instead, we force it to approach zero exponentially rapidly, and this is sufficient to avoid finite-time escape of $x(t)$ in a first stage. The determination of a suitable switching time T for ultimate exponential boundedness of all variables follows the same lines given above, but the computation is much more involved.

Consider now the cascade

$$\begin{aligned}\dot{x} &= -3x + x^2y^2 \\ \dot{y} &= u - z \\ \dot{z} &= u.\end{aligned}$$

The conditions are similar to those in Example 1, i.e. same nonlinear subsystem and a linear nonminimum phase zero at $\nu = 1$. Let the initial conditions be x0, $x_0 > 0$, y_0 and z_0, respectively, and set the control to $u = -Ky + z$, $K \geq 0$. This place the closed loop eigenvalues of the linear subsystem at $-K$ and 1, and so the corresponding responses are given by

$$\begin{aligned}y(t) &= e^{-y_0^2 x_0 t} y_0 \\ z(t) &= (z_0 - \frac{y_0^3 x_0}{y_0^2 x_0 + 1})e^t + \frac{y_0^3 x_0}{y_0^2 x_0 + 1} e^{-y_0^2 x_0 t}\end{aligned}$$

Introducing again the change of coordinates $\eta(t) = e^{-3t}/x(t)$, we have that

$$\begin{aligned}\eta(t) &= x_0^{-1} - \int_0^t e^{-3\tau} y(\tau)^2 \, d\tau \\ &= x_0^{-1} - y_0^2 \int_0^t e^{-(2K+3)\tau} \, d\tau \\ &\geq x_0^{-1}\left(1 - \frac{y_0^2 x_0}{2K+3}\right). \end{aligned} \tag{32}$$

The selection $K = y_0^2 x_0$ determines from (32) the boundedness of $x(t)$ for $t < T$, namely,

$$x(t) \leq e^{-3t} x_0 \frac{2y_0^2 x_0 + 3}{y_0^2 x_0 + 3}.$$

Now, at $t = T$ and for all $t > T$ switch the control to $u = y - 3z$, placing both eigenvalues of the linear subsystem at $s = -1$. Notice that before going to zero, $y(t)$ will peak, and so T has to be chosen large enough to guarantee that this peak does not break the stability of $x(t)$. The computation is much more intricate in this example than in the previous one.

The evolution of $y(t)$ for $t \geq T$ is given by

$$y(t) = (\delta + \gamma t)e^{-t}, \tag{33}$$

where

$$\begin{aligned}\delta &= 4Te^{2T} z_0 + \left(e^{-y_0^2 x_0 T}\left(2T\frac{y_0^2 x_0 - 1}{y_0^2 x_0 + 1} + 1\right) - 4Te^T \frac{y_0^2 x_0}{y_0^2 x_0 + 1}\right) e^T y_0, \\ \gamma &= -4e^{2T} z_0 + \left(2e^{-y_0^2 x_0 T}\frac{y_0^2 x_0 - 1}{y_0^2 x_0 + 1} - 4e^T\right) e^T y_0.\end{aligned}$$

Now we can write

$$\begin{aligned}\eta(t) &= \eta(T) - \int_T^{\infty} |\delta + \gamma\tau|\, e^{-5\tau}\, d\tau \\ &\geq \eta(T) - \sup_{t\geq T}\left((|\delta| + |\gamma t|)e^{-2t}\right)^2 \int_T^{\infty} e^{-\tau} d\tau. \end{aligned} \tag{34}$$

Assuming in addition $K > 1$ and $T > 1$, then it can be showed that

$$\sup_{t\geq T}\left((|\delta| + |\gamma t|)e^{-2t}\right)^2 \leq 64T^2(|z_0| + |y_0|)^2. \tag{35}$$

Solving the integral on the RHS of (34), and using (35), we arrive to

$$\eta(t) \geq \eta(T) - 64T^2(|z_0| + |y_0|)^2 e^{-T}. \tag{36}$$

Let T be found as the solution of

$$T^2 e^{-T} \leq \frac{y_0^2 x_0 + 3}{32 x_0 (2y_0^2 x_0 + 3)(|z_0| + |y_0|)^2}$$

Recalling from (32) that $\eta(t) \geq x_0^{-1}(y_0^2 x_0 + 3)/(2y_0^2 x_0 + 3)$ for all t, in particular for $t = T$, then we can see from (36) that this choice of T implies that

$$\eta(t) \geq \frac{y_0^2 x_0 + 3}{2x_0(2y_0^2 x_0 + 3)}, \tag{37}$$

and consequently, from (37), we finally get

$$x(t) \leq 2e^{-3t} x_0 \frac{(2y_0^2 x_0 + 3)}{y_0^2 x_0 + 3}, \qquad t \geq T,$$

and therefore, the cascade is stabilized. As in Example 1, T can be recomputed for every set of initial conditions, rendering semi-global stability, which results non-uniform in time, as in Example 1. Figure 4 shows the evolution of the states for $y_0 = 0.5$, $z_0 = -1$ and $x_0 = 1$, which give $T = 3.05$s.

We conjecture that a nonlinear control law can also be generated using these ideas to give global stability when the linear subsystem has relative degree greater than zero. Yet, as for the case of semi-global stability, we should also expect much more involved computations.

5 Conclusions

We have considered the problem of global/semi-global stabilization of a class of nonlinear systems in the form of a cascade of a linear, controllable system, and a nonlinear, exponentially stable system. Key features of such systems are that:

- The linear "connection" to the nonlinear system is non-minimum phase.
- The nonlinear system has a strong coupling with its input.

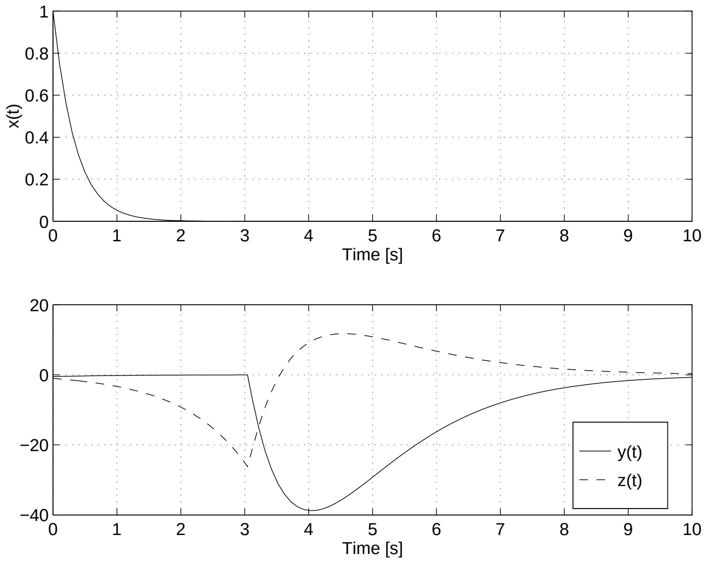

Figure 4: Time responses for the example with relative degree 1, T=3.05s.

The non-minimum phase property implies that, with high gain ξ-feedback, peaking occurs. The strong connection implies that if there is sufficient "energy" in the input to the nonlinear system, then a finite time escape may occur.

By consideration of the peaking phenomena, we have demonstrated families of these cascades for which global/semi-global stabilization by linear time-invariant feedback of ξ is not possible.

We have also shown that for systems with "slow" zero dynamics (x stability) relative to the linear non-minimum phase zero location, even semi-global stabilization is impossible.

By example, we have illustrated nonlinear and linear time-varying feedbacks that can give global/semi-global stabilization for the remaining case, i.e. "fast" zero dynamics. In this case, the approach is somewhat counterintuitive, namely:

(i) Firstly, force the input to the nonlinear system to zero, using unstable feedback on the nonlinear system. In this case, the nonlinear state x converges to zero rapidly, while the linear state ξ diverges.

(ii) After a suitable period of time, switch to a linear stabilizing feedback, to cause the linear state to return to zero.

This procedure has an interpretation in terms of invertibility of the linear subsystem. Referring to the internal model paradigm [12], an ideal controller would be constructed using the inverse of the plant. When the plant (our linear subsystem) is non-minimum phase, we cannot invert it, as it would render an unstable controller. However, as our control strategy suggests, depending on the relative speed of the non-minimum zero we could invert it "for a while". This period of "disconnection" allows the stabilization of the nonlinear subsystem in a way that, ultimately, the whole cascade remains stable when switching to a stabilizing controller for the linear part.

Appendix

Proof of Lemma 3 Since (A,B) is a controllable pair, without lose of generality assume (A,B) in the controllable canonical form, and K a stabilizing feedback gain,

$$A = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$K = \begin{bmatrix} -k_{n-1} & -k_{n-2} & \cdots & -k_1 & -k_0 \end{bmatrix}.$$

We want to prove that

$$\sup_{K\in\mathbf{K}} \|(\nu I - A_K)^{-1}\| < \infty,$$

that is,

$$\sup_{K\in\mathbf{K}} \left\| \begin{bmatrix} \nu + a_{n-1} + k_{n-1} & a_{n-2} + k_{n-2} & \cdots & a_0 + k_0 \\ -1 & \nu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \nu \end{bmatrix}^{-1} \right\| < \infty.$$

Note that we can write

$$(\nu I - A_K)^{-1} = \begin{bmatrix} \frac{\nu^{n-1}}{h(\nu)} & -\frac{p_{n-2}(\nu)}{h(\nu)} & -\nu\,\frac{p_{n-3}(\nu)}{h(\nu)} & -\nu^2\,\frac{p_{n-4}(\nu)}{h(\nu)} & \cdots & -\nu^{n-2}\,\frac{p_0(\nu)}{h(\nu)} \\ \frac{\nu^{n-2}}{h(\nu)} & \frac{q_{n-2}(\nu)}{\nu\ h(\nu)} & -\frac{p_{n-3}(\nu)}{h(\nu)} & -\nu\,\frac{p_{n-4}(\nu)}{h(\nu)} & \cdots & -\nu^{n-3}\,\frac{p_0(\nu)}{h(\nu)} \\ \frac{\nu^{n-3}}{h(\nu)} & \frac{q_{n-2}(\nu)}{\nu^2\, h(\nu)} & \frac{q_{n-3}(\nu)}{\nu\ h(\nu)} & -\frac{p_{n-4}(\nu)}{h(\nu)} & \cdots & -\nu^{n-4}\,\frac{p_0(\nu)}{h(\nu)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\nu}{h(\nu)} & \frac{q_{n-2}(\nu)}{\nu^{n-2}\, h(\nu)} & \frac{q_{n-3}(\nu)}{\nu^{n-3}\, h(\nu)} & \frac{q_{n-4}(\nu)}{\nu^{n-4}\, h(\nu)} & \cdots & -\frac{p_0(\nu)}{h(\nu)} \\ \frac{1}{h(\nu)} & \frac{q_{n-2}(\nu)}{\nu^{n-1}\, h(\nu)} & \frac{q_{n-3}(\nu)}{\nu^{n-2}\, h(\nu)} & \frac{q_{n-4}(\nu)}{\nu^{n-3}\, h(\nu)} & \cdots & \frac{q_0(\nu)}{\nu\ h(\nu)} \end{bmatrix},$$

where $h(\nu)$ is the characteristic polynomial of A_K, and q_i and p_i are parts of h as given by

$$\begin{aligned} h(\nu) = {} & \underbrace{\nu^n + (a_{n-1}+k_{n-1})\,\nu^{n-1} + \cdots + (a_{i+1}+k_{i+1})\,\nu^{i+1}}_{q_i(\nu)} \\ & + \underbrace{(a_i+k_i)\,\nu^i + \cdots + a_0 + k_0}_{p_i(\nu)} \end{aligned} \tag{38}$$

All the elements of the matrix $(\nu I - A_K)^{-1}$ have $h(\nu)$ as common denominator, and parts of $h(\nu)$ as numerators. Moreover, $h(\nu)$, $q_i(\nu)$ and $p_i(\nu)$ are linear in k_j, $i,j = 1,2\cdots,n$. We shall see that each of these elements is bounded for all $\nu \in \mathbb{C}^+$ and any stabilizing K.

If $\{\lambda_i, i = 1, 2, \cdots, n\}$ are the eigenvalues of A_K, we have that $h(\nu) = \prod_{i=1}^{n}(\nu - \lambda_i)$. Note that, since A_K is stable, every λ_i belongs to $\mathbb{C}^-$. Therefore, the coefficients of $h(\nu)$ in (38) can be written as

$$(a_i + k_i) = \sum_{\ell_1 \neq \ell_2 \neq \cdots \neq \ell_{n-i}} \lambda_{\ell_1}\lambda_{\ell_2}\cdots\lambda_{\ell_{n-i}}. \tag{39}$$

Then, for each quotient of the form $(a_i + k_i)/h(\nu)$, we have the following,

$$\begin{aligned} \left|\frac{(a_i + k_i)}{h(\nu)}\right| &= \left|\frac{\sum \lambda_{\ell_1}\lambda_{\ell_2}\cdots\lambda_{\ell_{n-i}}}{\prod_{i=1}^{n}(\nu - \lambda_i)}\right| \\ &= \left|\sum \frac{\lambda_{\ell_1}\lambda_{\ell_2}\cdots\lambda_{\ell_{n-i}}}{(\nu - \lambda_1)(\nu - \lambda_2)\cdots(\nu - \lambda_n)}\right| \\ &\leq \frac{1}{\operatorname{Re}\nu^i} \sum \left|\frac{\lambda_{\ell_1}}{\nu - \lambda_{\ell_1}}\right| \cdots \left|\frac{\lambda_{\ell_{n-i}}}{\nu - \lambda_{\ell_{n-i}}}\right| \end{aligned} \tag{40}$$

where, for simplicity, we dropped the indices in the summation sign (39). Now, each of the factors inside the summation is of the form

$$f(\lambda) = \left|\frac{\lambda}{\nu - \lambda}\right| \leq \frac{|\nu|}{\operatorname{Re}\nu} \qquad \text{with } \lambda \text{ in } \mathbb{C}^-. \tag{41}$$

From (40) and (41), we obtain

$$\left|\frac{(a_i + k_i)}{h(\nu)}\right| \leq \frac{|\nu|^{n-i}\, n!}{\operatorname{Re}\nu^n\, i!(n-i)!}, \tag{42}$$

and so, each element of the matrix $(\nu I - A_K)^{-1}$ is bounded for every

$$K = [k_{n-1}, k_{n-2}, \ldots, k_0]$$

such that $h(\cdot)$ is Hurwitz, completing the proof. □

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