Inherent Design Limitations for Linear Sampled-data Feedback Systems

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Abstract

There is a well-developed theory describing inherent design limitations for linear time invariant feedback systems consisting of an analog plant and analog controller. This theory describes limitations on achievable performance present when the plant has nonminimum phase zeros, unstable poles, and/or time delays. The parallel theory for linear time invariant discrete time systems is less interesting because it describes system behavior only at sampling instants. In this paper, we develop a theory of design limitations for sampled-data feedback systems wherein we consider the response of the *analog* system output. To do this, we use the fact that the steady state response of a hybrid feedback system to a sinusoidal input consists of a fundamental component at the frequency of the input together with infinitely many harmonics at frequencies spaced integer multiples of the sampling frequency away from the fundamental. This fact allows us to define fundamental sensitivity and complementary sensitivity functions that relate the fundamental component of the response to the input signal. These sensitivity and complementary sensitivity functions must satisfy integral relations analogous to the Bode and Poisson integrals for purely analog systems. The relations show, for example, that design limitations due to nonminimum phase zeros of the analog plant constrain the response of the sampled-data feedback system regardless of whether the discretized system is minimum phase and independently of the choice of hold function.

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1 Introduction and Motivation

Recent years have seen renewed interest in hybrid, or sampled-data, feedback control systems. Two research directions in particular have generated much activity. First, various optimal control problems have been stated and solved for hybrid systems (*cf.* the reference list in [37]). Second, several researchers have explored the potential ability of nonstandard hold functions, periodic digital controllers, and multirate sampling to circumvent design limitations inherent to linear time invariant systems (e.g., [23, 14, 24, 8, 46, 20]).

Despite these advances in synthesis, there is as yet no well-developed theory of inherent design limitations for hybrid feedback systems. For analog feedback systems, on the other hand, many results on design limitations are available. Bode first stated the sensitivity integral theorem in [3]. The importance of this integral for feedback control was emphasized by Horowitz [22]. Later extensions were obtained by several researchers; of particular relevance to the present paper are the results of [18, 30]. Briefly, the theory describes how plant properties such as nonminimum phase zeros, unstable poles, and time delays limit the achievable performance of a feedback system consisting of a linear time invariant plant and continuous-time, or analog, controller. These limitations manifest themselves as tradeoffs between desirable system properties in different frequency ranges, and are expressed mathematically using Bode and Poisson integrals. A parallel theory of inherent design limitations for purely discrete time feedback systems is also available ([30, 29, 31, 38]). Unfortunately, this theory only describes system behavior at sampling instants, and thus does not provide a satisfactory framework for analyzing the intersample behavior of the analog signals.

An apparent obstacle to the development of a theory of design limitations for hybrid systems is that they are periodically time-varying due to the action of the sampler. As explained in textbooks on sampled-data control, this fact implies that one cannot use transfer functions to describe system input-output properties[1]. However, it is possible to calculate the Laplace transform of the response of a hybrid system to a particular input. Hence one may evaluate the steady state response of a stable hybrid system to a sinusoidal input of given frequency. For analog systems, the response to such an input is a sinusoid of the same frequency as the input, but with amplitude and phase modified according to the transfer function of the system evaluated at the input frequency. The response of a stable hybrid system to an input signal, on the other hand, consists of a sum of infinitely many sinusoids spaced at integer multiples of the sampling frequency away from the frequency of the input. We shall refer to that component with the same frequency as the input as the *fundamental*, and the other components as *harmonics*. Each of these components is governed by a *response function* with many properties similar to those of a transfer function. In particular, the response functions have sufficient structure to allow complex analysis to be applied to derive a set of formulas analogous to the Bode and Poisson integrals. As in the continuous-time case, these integrals describe tradeoffs between system properties in different frequency ranges. Frequency response properties of hybrid systems have been discussed in several recent papers, among them [1, 13, 19, 26, 40, 41, 44, 45]. The frequency response of periodic analog feedback systems is treated in [43].

We now outline the remainder of the paper. Section 2 contains definitions and preliminary results. We review the properties of nonminimum phase zeros and nonpathological sampling for generalized hold functions that were developed in [28]. We also quote the infinite summation formula showing that the response of the discretized plant at a given frequency depends upon that of the analog plant at infinitely many frequencies. This fact is crucial to understanding, not only frequency response, but also sensitivity and robustness properties of hybrid systems. In Section 3 we define the fundamental sensitivity, fundamental complementary sensitivity, and the harmonic response

[1] An interesting notion of transfer function defined using lifting techniques is developed in [44, 45]

functions. These functions have many properties in common with transfer functions, and govern the steady state response of a hybrid system to sinusoidal disturbance or noise inputs. As with their analog counterparts, the fundamental sensitivity and complementary sensitivity functions (denoted $S_{fun}(s)$ and $T_{fun}(s)$, respectively) may be used to quantify the costs and benefits of feedback. We also discuss the use of high gain feedback in a sampled-data system. In Section 4 we show that $S_{fun}(s)$ and $T_{fun}(s)$ may be used to describe differential sensitivity and robust stability properties of a hybrid system. In particular, consider the problem of maintaining robust stability against multiplicative uncertainty which is unstructured and *linear time-invariant*. We show that for robust stability against such uncertainty it is *necessary* that $T_{fun}(s)$ be sufficiently small. Section 5 is devoted to a catalogue of interpolation constraints for $S_{fun}(s)$ and $T_{fun}(s)$. As with their analog counterparts, the values of these functions at points in the ORHP is constrained by poles, zeros, and time delays in the plant and controller. Unlike the analog case, the constraints imposed by the compensator manifest themselves differently than do those imposed by the plant, and this fact leads to interesting design interpretations. In Section 6, these interpolation constraints are used to derive generalizations of the Bode and Poisson integrals to $S_{fun}(s)$ and $T_{fun}(s)$. Design implications of these integrals are discussed in detail. Of particular interest is the fact that nonminimum phase zeros of the analog plant impose inherent tradeoffs upon $S_{fun}(j\omega)$. Nonminimum phase zeros of the discretized plant, on the other hand, do not. A summary discussion of the costs and benefits of sampled-data feedback is given in Section 7.

2 Preliminaries

2.1 Notation

We shall consider the single-input, single-output, hybrid feedback system depicted in Figure 2.1.1, where $P(s)$ and $F(s)$ are the transfer functions of the analog plant and anti-aliasing filter, $C_d(z)$ is the transfer function of the digital controller, $r(t)$, $d(t)$ and $n(t)$ are the command, disturbance and noise inputs, $y(t)$ is the system output, $u(t)$ and $\{u_k\}$ are the analog and discrete control inputs, and $v(t)$ and $\{v_k\}$ are the analog and sampled outputs of the anti-aliasing filter. Denote the sampling period by T and the associated sampling and Nyquist frequencies by $\omega_s \triangleq 2\pi/T$ and $\omega_N \triangleq \pi/T$, respectively. We shall use the term *baseband* to denote the frequency range $\Omega_N = (-\omega_N, \omega_N]$.

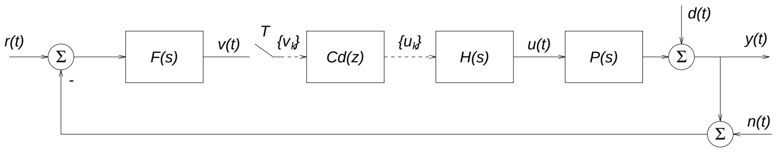

Figure 2.1.1: Hybrid Feedback System

The sampling operation mapping continuous time signal into a discrete sequence is denoted $\mathcal{S}_T$, and defined by $\mathcal{S}_T\{v(t)\} = \{v_k\}$, where $v_k \triangleq v(kT)$. We shall denote the Laplace transform of an analog signal by an upper case letter, *e.g.* $U(s) = \mathcal{L}\{u(t)\} = \int_0^\infty e^{-st}\,u(t)\,dt$, and the Z-transform of a discrete sequence by an upper case letter with a "$\sim$", *e.g.* , $\tilde{U}(z) = \mathcal{Z}\{u(t)\} = \sum_{k=0}^\infty u_k\,z^{-k}$. We denote the open and closed right halves of the s-plane by ORHP and CRHP, respectively. We denote the open and closed unit disks by $D \triangleq \{z : |z| < 1\}$ and $\bar{D} \triangleq \{z : |z| \le 1\}$, respectively, and we denote their complements by D^c and $\bar{D}^c$. A rational function of s (respectively z) is *stable* if it has no poles in the CRHP (respectively, in D^c). A rational function of s (respectively z) is *minimum phase* if it has no zeros in the ORHP (resp., in $\bar{D}^c$). Otherwise it is nonminimum phase, and the corresponding zeros are termed *nonmimimum phase (NMP) zeros.* Given a stable, proper transfer function $G(s)$, we define $\|G\|_\infty \triangleq \sup_\omega |G(j\omega)|$.

We shall assume throughout the paper that the following conditions are satisfied:

Assumption 2.1.1 *The plant, prefilter, and compensator are each free of unstable hidden modes and satisfy the following additional hypotheses*

$$P(s) = P_0(s)\, e^{-s\tau}, \quad \text{where } P_0(s) \text{ is proper and rational and } \tau \geq 0 \tag{2.1.1}$$

$$F(s) \text{ is strictly proper, rational and has no poles or zeros in the CRHP} \tag{2.1.2}$$

$$C_d(z) \text{ is proper and rational, with relative degree } N_C \geq 0 \tag{2.1.3}$$

□

The assumption that the prefilter $F(s)$ is strictly proper implies that the sampling operation is well defined (*cf.* [37]). The assumptions that $F(s)$ has no poles or zeros in the CRHP may be removed, and are invoked only to facilitate discussion. In practice antialiasing filters will satisfy these assumptions[2].

The next result is essential to understanding frequency response properties of hybrid feedback systems. Unfortunately, despite the fact that the result appears in many textbooks (e.g., [2, 16, 25, 32]), it is difficult to find a proof that is rigorous and self-contained, and which clearly delineates the classes of signals to which it is applicable. Indeed, this fact has stimulated discussion in the past (*cf.* [35, 4, 33, 34, 9]). The following statement is adapted from [9, 10].

Lemma 2.1.1 *Consider $G(s)$, the Laplace transform of the function $g(t)$ defined for $t \geq 0$. Let $x + jy$, $-\infty < y < \infty$, be a vertical line in the domain of convergence of $G(s)$. Assume that the following conditions are satisfied:*

(i) $\lim_{t\to 0^+} g(t) = 0$

(ii) $g(t)$ is continuous in the neighborhood of $t = kT$, $k = 1, 2, \ldots$

(iii) the infinite sum

$$\sum_{k=-\infty}^{\infty} G(s + jk\omega_s) \tag{2.1.4}$$

is uniformly convergent for $s = x + jy$, $y \in (-\omega_n, \omega_n)$.

Define

$$\tilde{G}(z) \triangleq \mathcal{Z}\{\mathcal{S}_T\{\mathcal{L}^{-1}\{G(s)\}\}\} \ . \tag{2.1.5}$$

Then

$$\tilde{G}(e^{sT}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G(s + jk\omega_s) \ . \tag{2.1.6}$$

□

[2] The requirement of an antialiasing filter may be removed by assuming instead that (a) $P_o(s)$ in (2.1.1) is strictly proper and (b) the exogeneous signals have Laplace transforms satisfying Assumption 2.1.2 with the upper bounds replaced by $M_r/|s|^2$, etc.

We shall need to apply Lemma 2.1.1 to the response of a strictly proper antialiasing filter to an exogenous input signal. To insure validity of the formula (2.1.6), we shall assume throughout the paper that the exogenous input signals in Figure 2.1.1 satisfy the following conditions:

Assumption 2.1.2 *There exist constants M_R, M_D, and M_N such that the Laplace transforms of the signals $r(t)$, $d(t)$, and $n(t)$ satisfy $|R(s)| \leq M_r/|s|$, $|D(s)| \leq M_d/|s|$, and $|N(s)| \leq M_n/|s|$, $\forall$ s in the region of convergence of each transform.* □

It is straightforward to verify that this assumption is satisfied by signals that are steps, ramps, sinusoids, exponentials, or that lie in $\mathcal{L}^2$. Signals that contain impulses are excluded.

2.2 The Hold Device

The hold device is a generalized sampled-data hold function (GSHF), defined by

$$u(t) = h(t - kT)u_k, \qquad kT \leq t < (k+1)T \ . \tag{2.2.1}$$

We shall assume throughout the paper that the hold function satisfies the following conditions.

Assumption 2.2.1 *Assume that*

(a) $|h(t)|$ *is bounded on* $[0, T]$.

(b) $h(t)$ *has finitely many points of discontinuity* $\tau_i \in (0, T), i = 1, \cdots, n$.

(c) *Define* $\tau_0 = 0$, $\tau_{n+1} = T$ *and* $h'(t) = \frac{dh(t)}{dt}$. *Then* $\exists M_i < \infty$ *such that*

$$\sup_{t \in (\tau_i, \tau_{i+1})} |h'(t)| \leq M_i, i = 0, 1, \cdots, n.$$

□

As discussed in [28], to this hold device there is an associated frequency response function

$$H(s) \triangleq \int_0^T e^{-sT}\, h(t)\, dt. \tag{2.2.2}$$

These functions may have infintely many nonminimum phase zeros, with associated transmission blocking properties [28].

In important special cases it is possible to derive closed form expressions for the response function.

Lemma 2.2.1 ***Hold Response Functions [28]***
Let (2.2.1) be given by

(a) *the response of a finite-dimensional linear time invariant (FDLTI) system*

$$h(t) = Ke^{L(T-t)}M, \qquad 0 \leq t < T \ . \tag{2.2.3}$$

Then

$$H(s) = K(sI + L)^{-1}(e^{LT} - e^{-sT}I)M \ . \tag{2.2.4}$$

(b) *a piecewise constant hold function*

$$h(t) = \begin{cases} \alpha_0 & , \quad t \in [0, \frac{T}{N}) \\ \alpha_1 & , \quad t \in [\frac{T}{N}, \frac{2T}{N}) \\ \vdots & \\ \alpha_{N-1} & , \quad t \in [(\frac{N-1}{N})T, T) \ . \end{cases} \tag{2.2.5}$$

Then

$$H(s) = \left(\frac{1 - e^{-\frac{sT}{N}}}{s} \right) \sum_{k=0}^{N-1} \alpha_k e^{-\frac{skT}{N}} \tag{2.2.6}$$

(c) *the Fourier series expansion*

$$h(t) = \sum_{k=-\infty}^{\infty} \beta_k \, e^{jk\omega_s t}, \quad t \in [0, T), \tag{2.2.7}$$

where

$$\beta_k \triangleq \frac{1}{T} \int_0^T h(t) \, e^{-jk\omega_s t} \, dt \ .$$

Then

$$H(s) = (1 - e^{-sT}) \sum_{k=-\infty}^{\infty} \frac{\beta_k}{s - jk\omega_s} \tag{2.2.8}$$

□

It is straightforward to verify that both (2.2.3) and (2.2.5) satisfy Assumption 2.2.1 and include the ZOH as a special case.

The hold response function is useful in calculating the Laplace transform of the output of the hold device.

Lemma 2.2.2 *[28]*
Consider a hold defined by (2.2.1) and its associated response function (2.2.2). Then

$$U(s) = H(s) \, \tilde{U}(e^{sT}) \tag{2.2.9}$$

□

Using the hold response function we may derive a formula for the discrete transfer function of the series connection of hold, plant, prefilter, and sampler [28]:

$$(FPH)_d(z) \triangleq \mathcal{Z}\{\mathcal{S}_T\{\mathcal{L}^{-1}\{F(s)P(s)H(s)\}\}\} \ . \tag{2.2.10}$$

For brevity, we refer to (2.2.10) as the *discretized plant*, retaining the dependence on the hold and prefilter in the notation.

The following result is found in many digital control textbooks (*e.g.* [2, 16]).

Lemma 2.2.3 *Consider the discretized plant (2.2.10). Then*

$$(FPH)_d(e^{sT}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F(s + jk\omega_s) \, P(s + jk\omega_s) \, H(s + jk\omega_s). \tag{2.2.11}$$

Proof: Define $w(t) \triangleq \mathcal{L}^{-1}\{F(s)P(s)H(s)\}$, and note that $w(t)$ is the response of the series connection of hold, plant, and prefilter to a unit pulse sequence, $u_0 = 1$, $u_k = 0$, $k = 1, 2, \ldots$. It follows from the fact that $F(s)$ is rational and strictly proper that conditions (i) and (ii) of Lemma 2.1.1 are satisfied. We now show that $F(s)P(s)H(s)$ satisfies condition (iii) of Lemma 2.1.1. To see this, note first that using Assumption 2.2.1 in equation (2.2.2) and integrating by parts yields

$$s\,H(s) = \sum_{i=0}^{n} \left\{ -e^{-st} h(t)\Big|_{\tau_i}^{\tau_i+1} + \int_{\tau_i}^{\tau_i+1} e^{-st} h'(t)\, dt \right\}$$

For $s \in$ CRHP, Assumption 2.2.1 (c) yields that

$$|s\,H(s)| \leq 2(n+1)\bar{h} + T\bar{h}', \tag{2.2.12}$$

where $\bar{h} \triangleq \sup_{t\in(0,T)} |h(t)|$ and $\bar{h}' \triangleq \sup_{t\in(0,T)} |h'(t)|$. Together, (2.2.12) and the assumption that $F(s)$ is strictly proper imply the existence of a constant M such that $|F(s)P(s)H(s)| \leq M/|s|^2$ for all s to the right of all poles of $P(s)$ and $F(s)$. This fact implies, in turn, that condition (iii) is satisfied.

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It follows from (2.2.11) that the frequency response of the discretized system is periodic in ω, with period ω_s. Note also that the response of the discretized plant at frequency $e^{j\omega T}$ will depend upon the response of the analog plant, prefilter, and hold function at infinitely many frequencies $\omega + k\omega_s$, $k = 0, \pm 1, \pm 2, \ldots$. To emphasize this point, it will prove convenient to rewrite (2.2.11) as:

$$(FPH)_d(e^{sT}) = \left(\frac{1}{T}\right) F(s)P(s)H(s) + \left(\frac{1}{T}\right) \Delta_d(s) \ , \tag{2.2.13}$$

where

$$\Delta_d(s) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \Delta_k(s) \tag{2.2.14}$$

and

$$\Delta_k(s) = F(s + jk\omega_s)P(s + jk\omega_s)H(s + jk\omega_s). \tag{2.2.15}$$

Typically, the antialiasing filter has a monotonically decreasing Bode gain plot, and thus tends to diminish the contribution of the high frequency plant behavior to the discretized system response. The effect of the hold response in (2.2.11) is identical to that of the antialiasing filter, and shows that the hold response plays an equally important role in determining the effect of the high frequency plant response upon the discretized system. In this regard, it is instructive to consider the frequency response of the ZOH (Figure 2.2.1). This response is bounded by an envelope that decreases monotonically with frequency, and thus tends to diminish the effect of high frequency plant behavior in much the same way as does the antialiasing filter. Furthermore, the hold response has zeros that greatly reduce those high frequency components of the plant response that are aliased down to low frequencies. This fact is significant since it is precisely the low frequency range that is of greatest interest in typical feedback control problems. In particular, those components at $k\omega_s, k = \pm 1, \pm 2, \ldots$ that get mapped down to DC see zero gain; as discussed in Remark 5.3, this fact is responsible for the ability of a hybrid system with a ZOH and a discrete integrator to reject step disturbances. The response of a GSHF, on the other hand, need not have zeros at these frequencies, and thus high frequency plant behavior (including high frequency uncertainty) may have significant effect upon the response of the discrete system at low frequencies within the control bandwidth. For example, see the GSHF response plotted in Figure 2.2.1; this GSHF is taken from Example 2 of [23]. A thorough analysis of the attendant design implications remains to be worked out.

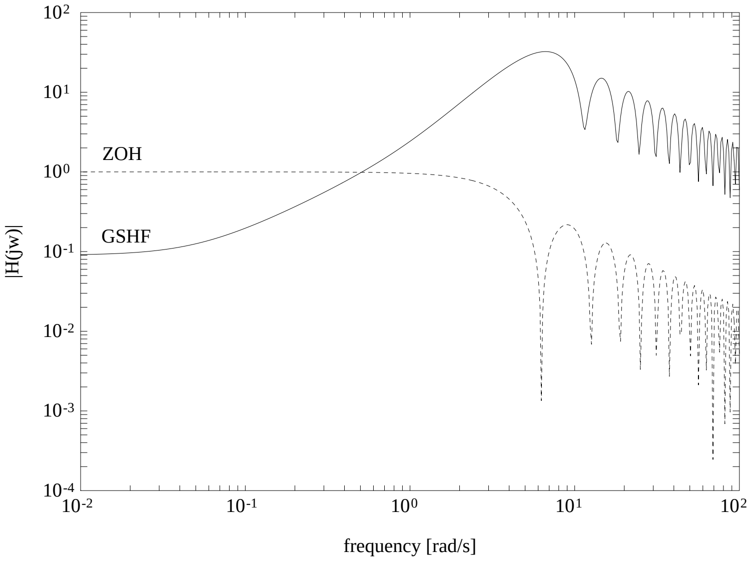

Figure 2.2.1: Frequency Response of Hold Functions

2.3 Closed Loop Stability

As with the case of a ZOH, closed loop stability is guaranteed by the assumptions that sampling is nonpathological and that the discretized system is closed loop stable.

Lemma 2.3.1 ***Nonpathological Sampling [28]***
Suppose that $P(s)$ and $F(s)$ satisfy (2.1.1)-(2.1.2) and assume further that

(i) if λ_i and λ_k are CRHP poles of $P(s)$, then

$$\lambda_i \neq \lambda_k + j\ell\omega_s, \qquad \ell = \pm 1, \pm 2, \cdots \tag{2.3.1}$$

(ii) if λ_i is a CRHP pole of $P(s)$, then

$$H(\lambda_i) \neq 0. \tag{2.3.2}$$

Then the discretized plant (2.2.10) is free of unstable hidden modes. □

Define the discrete sensitivity and complementary sensitivity functions

$$S_d(z) \triangleq \frac{1}{1 + (FPH)_d(z) C_d(z)} \tag{2.3.3}$$

and

$$T_d(z) \triangleq (FPH)_d(z)\, C_d(z)\, S_d(z)\ . \tag{2.3.4}$$

Lemma 2.3.2 *Suppose that $P(s)$, $F(s)$, $C_d(z)$, and $H(s)$ satisfy (2.1.1)-(2.1.2), that the nonpathological sampling conditions (2.3.1) - (2.3.2) are satisfied, that the product $(FPH)_d(z)\, C_d(z)$ has no pole-zero cancellations in D^c, and that all poles of $S_d(z)$ lie in D. Then the feedback system in Figure 2.1.1 is exponentially stable and $\mathcal{L}_2$ input-output stable.*

Proof: The proof may be obtained via straightforward modification of the proofs of Theorem 4 in [14] and Theorem 6 in [6].

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3 Frequency Response of a Hybrid System

The steady-state response of a stable hybrid feedback system to a complex sinusoidal input consists of a fundamental component at the frequency of the input as well as additional harmonics located at integer multiples of the sampling frequency away from the fundamental. This fact is discussed in well-known textbooks (*cf.* [2, 16]), and has been emphasized in several recent research papers[3] (*e.g.* [1, 19, 44, 13]). We now present expressions for the response of $y(t)$ in Figure 2.1.1 to disturbances and noise. Analogous expressions may be stated for the response to the reference input, and for the response of the control $u(t)$ to these signals. When evaluated along the $j\omega$-axis, the following expressions are identical to those derived in ([19], Theorem 2.1) using Fourier transform techniques.

Lemma 3.1 *Denote the responses of $y(t)$ to each of $d(t)$ and $n(t)$ by $y_d(t)$ and $y_n(t)$, respectively. Then the Laplace transforms of these signals are given by*

$$\begin{aligned} Y_d(s) &= \left[1 - \frac{1}{T} P(s)\, H(s)\, C_d(e^{sT})\, S_d(e^{sT})\, F(s)\right] D(s) \\ &\quad - \sum_{\substack{\ell = -\infty \\ \ell \neq 0}}^{\infty} \left[\frac{1}{T} P(s)\, H(s)\, C_d(e^{sT})\, S_d(e^{sT})\, F(s+j\ell\omega_s)\right] D(s+j\ell\omega_s) \qquad (3.1) \\ Y_n(s) &= -\left[\frac{1}{T} P(s)\, H(s)\, C_d(e^{sT})\, S_d(e^{sT})\, F(s)\right] N(s) \\ &\quad - \sum_{\substack{\ell = -\infty \\ \ell \neq 0}}^{\infty} \left[\frac{1}{T} P(s)\, H(s)\, C_d(e^{sT})\, S_d(e^{sT})\, F(s+j\ell\omega_s)\right] N(s+j\ell\omega_s) \qquad (3.2) \end{aligned}$$

Proof: These formulas may be derived using standard techniques from sampled-data control theory (*cf.* [16, 2]). We present only a derivation of (3.1). Block diagram algebra and Lemma 2.2.2 yield

$$Y_d(s) = D(s) + P(s)\, H(s)\, \tilde{U}(e^{sT}) \qquad (3.3)$$

and

$$\tilde{U}(z) = -C_d(z)\, S_d(z) \tilde{Z}(z) \qquad (3.4)$$

where

$$\tilde{Z}(z) \triangleq \mathcal{Z}\{\mathcal{S}_T\{\mathcal{L}^{-1}\{F(s)D(s)\}\}\}.$$

The assumptions that $F(s)$ is strictly proper and that $d(t)$ satisfies Assumption 2.1.2 allow Lemma 2.1.1 to be applied, yielding

$$\tilde{Z}(e^{sT}) = \frac{1}{T} \sum_{\ell=-\infty}^{\infty} F(s+j\ell\omega_s) D(s+j\ell\omega_s)\ . \qquad (3.5)$$

Substituting (3.4) - (3.5) into (3.3) and rearranging yields the desired result.

[3] Similar results for systems with periodically time-varying *analog* controllers are found in [43].

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If the feedback system is stable, then the preceding formulas may be used to derive the steady state response of the system to a periodic input. As noted above, the response will be equal to the sum of infinitely many harmonics of the input frequency. The magnitude of each component is governed by a function analogous to the usual sensitivity or complementary sensitivity function.

Definition 3.1 ***Hybrid Sensitivity and Complementary Sensitivity Functions***
Define the fundamental sensitivity *and* complementary sensitivity functions *by*

$$S_{fun}(s) \triangleq 1 - \frac{1}{T} P(s)\, H(s)\, C_d(e^{sT})\, S_d(e^{sT})\, F(s) \tag{3.6}$$

and

$$T_{fun}(s) \triangleq \frac{1}{T} P(s)\, H(s)\, C_d(e^{sT})\, S_d(e^{sT})\, F(s) \tag{3.7}$$

respectively. For $k \neq 0$ define the kth harmonic response function *by*

$$T_k(s) \triangleq \frac{1}{T} P(s + jk\omega_s)\, H(s + jk\omega_s)\, C_d(e^{sT})\, S_d(e^{sT})\, F(s) \tag{3.8}$$

$\square$

These hybrid response functions are not transfer functions in the usual sense, because they do not equal the ratio of the transforms of output to input signals. However, as we shall now see, these functions do govern the steady state frequency response. We note that (3.6) and (3.7) are identical to the disturbance and reference gain functions defined in [19].

Lemma 3.2 ***Steady-State Frequency Response***
Suppose that the hypotheses of Lemma 2.3.2 are satisfied and assume that $d(t) = e^{j\omega t}$, $t \geq 0$, and $n(t) = e^{j\omega t}$, $t \geq 0$. Then as $t \to \infty$,

$$y_d(t) \to y_{dss}(t) \quad \text{and} \quad y_n(t) \to y_{nss}(t)$$

where

$$y_{dss}(t) = S_{fun}(j\omega) e^{j\omega t} - \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} T_k(j\omega) e^{j(\omega + k\omega_s)t} \tag{3.9}$$

and

$$y_{nss}(t) = -T_{fun}(j\omega) e^{j\omega t} - \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} T_k(j\omega) e^{j(\omega + k\omega_s)t} \tag{3.10}$$

$\square$

Proof: See Appendix A.

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Note that the fundamental component of the disturbance response can potentially be reduced through use of feedback. The fundamental component of the noise response, on the other hand, is only increased by using feedback. These facts are analogous to the continuous-time case. Two other properties of (3.9) - (3.10) are unique to hybrid systems. First is the existence of the harmonics at frequencies other than that of the input. The existence of these harmonics is due to the use of sampled-data feedback, and is a cost of feedback having no counterpart for analog systems. A second difference between analog and hybrid feedback systems is a limitation upon the ability of high gain feedback to reduce the magnitude of the fundamental component of the disturbance response. (By way of contrast recall that the disturbance response of an analog system can be made arbitrarily small at a given frequency provided that the plant gain is nonzero there.)

Lemma 3.3 ***High Compensator Gain***
Assume that $(FPH)_d(e^{j\omega T}) \neq 0$. Then, in the limit as $|C_d(e^{j\omega T})| \to \infty$, $S_{fun}(j\omega) \to S_{HG}(j\omega)$, where

$$S_{HG}(s) \triangleq \frac{\left(\frac{1}{T}\right)\Delta_d(s)}{(FPH)_d(e^{sT})} \tag{3.11}$$

and

$$S_d(e^{j\omega T}) \to 0 \tag{3.12}$$

Furthermore, the steady state responses of the system output and the sampler input to a disturbance $d(t) = e^{j\omega t}$, $t \geq 0$, satisfy

$$y_{dss}(kT) = \left[1 - \frac{F(j\omega)\left(\frac{1}{T}\right)\sum_{\ell=-\infty}^{\infty} P(j(\omega+\ell\omega_s))H(j(\omega+\ell\omega_s))}{(FPH)_d(e^{j\omega T})}\right] e^{j\omega kT} \tag{3.13}$$

and

$$v_{dss}(kT) = 0 \tag{3.14}$$

Proof: The formula (3.11) follows from Lemma 2.2.3 and the proof of Theorem 5.1(viii). The limit (3.12) is obvious. Equation (3.13) follows by setting $t = kT$ in (3.9). Finally, (3.14) follows by first showing that

$$v_{dss}(t) = F(j\omega)S_{fun}(j\omega)\, e^{j\omega T} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} T_{fun}(j(\omega+k\omega_s))F(j(\omega+k\omega_s))e^{j(\omega+k\omega_s)t}$$

yielding

$$v_{dss}(kT) = S_d(e^{j\omega T})F(j\omega)e^{j\omega kT}. \tag{3.15}$$

Together (2.12) and (2.15) yield (2.14).

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It follows from (3.11) that use of high gain in the digital controller *does not* in general diminish the fundamental component of the disturbance response arbitrarily closely to zero[4]. For a disturbance lying in the baseband, the obstruction to doing so is precisely the fact that the discrete frequency response depends upon the high frequency behavior of the plant, prefilter, and hold. It is true, of course, that the response of the sampler input may be reduced to zero at the sampling instants (*cf.* (3.14)). On the other hand, the sampled steady state output due to a baseband disturbance will be nonzero unless $F(s)$ is an ideal low pass filter: $F_{ideal}(j\omega) = 1, \forall\omega \in \Omega_N$, and $F_{ideal}(j\omega) = 0$ otherwise.

[4]However, see the remarks following Theorem 5.1 concerning the ZOH and use of integrators in $C_d(z)$.

4 Sensitivity and Robustness

We now describe how the fundamental sensitivity and complementary sensitivity functions, together with the harmonic response functions, may be used to describe differential sensitivity and robustness properties of a hybrid feedback system.

It is well-known [3], that the sensitivity function of a continuous-time feedback system governs the relative change in the command response of the system with respect to small changes in the plant. Derivations similar to those of Section 3 show that the steady state response of the system in Figure 2.1.1 to a command input $r(t) = e^{j\omega t}$, $t \geq 0$, is given by

$$y_{rss}(t) = T_{fun}(j\omega)e^{j\omega t} - \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} T_k(j\omega)e^{j(\omega+k\omega_s)t} \tag{4.1}$$

Since $T_{fun}(j\omega)$ depends upon $S_d(e^{j\omega T})$, it follows from Lemma 2.2.3 that the fundamental component of the command response at a particular frequency is sensitive to variations in the plant response at infinitely many frequencies.

Lemma 4.1 ***Differential Sensitivity***
At each frequency ω, the relative sensitivity of the steady state command response (4.1) to variations in $P(j(\omega + \ell\omega_s))$ is given by

(i) For $\ell = 0$,

$$\frac{P(j\omega)}{T_{fun}(j\omega)} \frac{\partial T_{fun}(j\omega)}{\partial P(j\omega)} = S_{fun}(j\omega) \; . \tag{4.2}$$

(ii) For all $\ell \neq 0$,

$$\begin{aligned} \frac{P(j(\omega+\ell\omega_s))}{T_{fun}(j\omega)} \frac{\partial T_{fun}(j\omega)}{\partial P(j(\omega+\ell\omega_s))} &= -T_{fun}(j(\omega+\ell\omega_s)) \\ &= -\frac{1}{T} P(s+j\ell\omega_s)\, H(s+j\ell\omega_s) F(s+j\ell\omega_s)\, C_d(e^{j\omega T})\, S_d(e^{j\omega T}) \end{aligned} \tag{4.3}$$

Proof: The proof is a straightforward calculation, keeping in mind the dependence of $S_d(e^{j\omega T})$ upon $P(s + j\ell\omega_s)$.

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These results may best be interpreted by considering frequencies in the baseband. Fix $\omega \in \Omega_N$. Then (4.2) states that the sensitivity of the fundamental component of the command response to small variations in the plant *at that frequency* is governed by the fundamental sensitivity function and hence may potentially be reduced through use of feedback (*cf.* the comments following Lemma 3.2). On the other hand, (4.3) states that the sensitivity of the fundamental component to *higher frequency* plant variations is governed by the fundamental complementary sensitivity function evaluated at the higher frequency, and thus *cannot* be reduced through the use of feedback. Further note that the sensitivity of the command response at a frequency in the baseband to higher frequency plant variations is proportional to the gain of the hold frequency response at the *higher* frequency, thus suggesting that the hold response should not be excessively large at high frequencies (*cf.* the remarks at the end of Section 2.2).

We now consider stability robustness. Suppose that the nominal plant in Figure 2.1.1 is subject to a multiplicative perturbation. As discussed in [11], such perturbations are useful in modelling high frequency plant uncertainty.

Theorem 4.1 *Consider the feedback system of Figure 2.1.1, and assume that the hypotheses of Lemma 2.3.2 are satisfied. Suppose that the nominal plant is replaced by*

$$P'(s) \triangleq P(s)\,(1 + W(s)\,\Delta(s)), \tag{4.4}$$

where $W(s)$ is rational with no CRHP poles or zeros, and $W(s)\,P(s)$ is proper. Assume further that the perturbation $\Delta(s)$ has real coefficients, is stable and proper, and satisfies $\|\Delta(s)\|_\infty < 1$. Then a necessary *condition for the perturbed system to remain stable for all such Δ's is that*

$$\|W(s)\,T_{fun}(s)\|_\infty \le 1\ . \tag{4.5}$$

Proof: It is necessary for closed loop stability that

$$S'_d(z) \triangleq \frac{1}{1 + (FP'H)_d(z)\,C_d(z)}$$

have no poles in D^c. Rearranging yields

$$S'_d(z) = \frac{S_d(z)}{1 + (FPW\Delta H)_d(z)\,S_d(z)\,C_d(z)} \tag{4.6}$$

The proof now proceeds by contradiction. Suppose that (4.5) is violated. Then $\exists\ \omega_0$ such that $\|W(j\omega_0)\,T_{fun}(j\omega_0)\|_\infty > 1$. Assume for the moment that there exists a $\Delta(s)$ satisfying the stated assumptions, and such that

$$\Delta(\pm j\omega_0) = -\frac{1}{W(\pm j\omega_0)\,T_{fun}(\pm j\omega_0)} \tag{4.7}$$

and

$$\Delta(j(\omega_0 + k\omega_s)) = 0, \quad k = \pm 1, \pm 2, \cdots \quad k \neq -\frac{2\omega_0}{\omega_s} \tag{4.8}$$

The assumptions on $W(s)$, and $\Delta(s)$ imply that Lemma 2.1.1 may be used to calculate $(FPW\Delta H)_d(e^{j\omega_0 T})$. Using (4.7) yields

$$(FPW\Delta H)_d(e^{j\omega_0 T}) = -\frac{\frac{1}{T}F(j\omega_0)P(j\omega_0)H(j\omega_0)}{T_{fun}(j\omega_0)} \tag{4.9}$$

Substituting (4.9) into (4.6) shows that $S'_d(z)$ has an unstable pole at $z = e^{j\omega_0 T}$.

It remains to show that a $\Delta(s)$ satisfying the required properties exists. We do this following a construction in [5]. Consider

$$\Delta(s) = -\frac{1}{r}\,z(s)\,f(s)^k\,\beta(s), \qquad k = 1, 2, \cdots \tag{4.10}$$

where

$$\begin{aligned}
f(s) &= \frac{s\omega_0}{(s^2+\omega_0^2)q + s\omega_0}, \qquad q > 0 \\
\beta(s) &= -(s/\omega_0)\sin\theta + \cos\theta \\
z(s) &= \frac{1 - e^{-sT}2\cos\omega_0 T + e^{-2sT}}{s^2 + \omega_0^2} \\
r &= |z(j\omega_0)\,W(j\omega_0)\,T_{fun}(j\omega_0)| \\
\theta &= \angle z(j\omega_0)\,W(j\omega_0)\,T_{fun}(j\omega_0)
\end{aligned}$$

It is straightforward to verify that (4.10) satisfies (4.7)-(4.8) for all k and q. Furthermore, as noted in [5], as k and q become large, $\Delta(j\omega) \to 0$ for all $\omega \neq \pm\omega_0$, and thus $\|\Delta\|_\infty < 1$ is satisfied.

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It follows that if $|T_{fun}(j\omega)|$ is very large at any frequency, then the hybrid system will exhibit poor robustness to uncertainty in the analog plant model at that frequency. For further results on robustness for sampled-data systems, see [12].

5 Interpolation Constraints

It is well known that the sensitivity and complementary sensitivity functions of a stable, continuous-time feedback system must satisfy certain interpolation constraints at the CRHP poles and zeros of the plant and compensator. Specifically, the sensitivity function must equal zero at the CRHP poles, and the complementary sensitivity function must equal zero at the CRHP zeros.

As shown in [18], these constraints may be used in conjunction with the Poisson integral to describe frequency dependent design tradeoffs. An entirely analogous set of interpolation constraints and design tradeoffs applies to the discrete-time part of a hybrid feedback system ([38, 31, 30, 29]).

We now present a set of interpolation constraints that must be satisfied by the hybrid sensitivity and complementary sensitivity functions (3.6) - (3.7). As we shall see, a significant difference between the hybrid case and the continuous-time only or discrete-time only cases is that the poles and zeros of the compensator yield different constraints than do those of the plant.

Theorem 5.1 ***Interpolation Constraints for the Fundamental Sensitivity and Complementary Sensitivity Functions.***
Assume that $P(s), F(s), H(s)$ and $C_d(z)$ satisfy all conditions stated in Section 2 and that the hybrid feedback system of Figure 2.1.1 is stable. Then the following conditions are satisfied:

(i) $S_{fun}(s)$ and $T_{fun}(s)$ have no poles in the CRHP.

(ii) Let p be a CRHP pole of $P(s)$. Then

$$S_{fun}(p) = 0 \quad \text{and} \quad T_{fun}(p) = 1. \tag{5.1}$$

(iii) Let ζ be a CRHP zero of $P(s)$. Then

$$T_{fun}(\zeta) = 0 \quad \text{and} \quad S_{fun}(\zeta) = 1. \tag{5.2}$$

(iv) Let γ be a CRHP zero of $H(s)$. Then

$$T_{fun}(\gamma) = 0 \quad \text{and} \quad S_{fun}(\gamma) = 1. \tag{5.3}$$

(v) Let a be a zero of $C_d(z)$ with $a \in D^c$. Define

$$a_k \triangleq \frac{1}{T}\log(a) + jk\omega_s, \quad k = 0, \pm 1, \pm 2, \cdots \tag{5.4}$$

Then

$$T_{fun}(a_k) = 0 \quad \text{and} \quad S_{fun}(a_k) = 1. \tag{5.5}$$

(vi) Let p be a CRHP pole of $P(s)$. Define

$$p_k \triangleq p + jk\omega_s, \quad k = \pm 1, \pm 2, \cdots \tag{5.6}$$

Then

$$T_{fun}(p_k) = 0 \quad \text{and} \quad S_{fun}(p_k) = 1, \tag{5.7}$$

(vii) $T_{fun}(s)$ *has no CRHP zeros other than those given by (iii) - (vi) above.*

(viii) Let b *be a pole of* $C_d(z)$ *with* $b \in D^c$*. Define*

$$b_k \triangleq \frac{1}{T}\log(b) + jk\omega_s, \quad k = 0, \pm 1, \pm 2, \cdots \tag{5.8}$$

Then

$$S_{fun}(b_k) = 1 - \frac{\frac{1}{T}P(b_k)\,H(b_k)\,F(b_k)}{(FPH)_d(b)} \tag{5.9}$$

and

$$T_{fun}(b_k) = \frac{\frac{1}{T}P(b_k)\,H(b_k)\,F(b_k)}{(FPH)_d(b)}. \tag{5.10}$$

Proof: Introduce factorizations

$$P(s)\,F(s) = e^{-s\tau}\frac{N(s)}{M(s)}, \tag{5.11}$$

where $N(s)$ and $M(s)$ are coprime rational functions with no poles in the CRHP, and

$$(FPH)_d(z) = \frac{N_d(z)}{M_d(z)}, \tag{5.12}$$

where $N_d(z)$ and $M_d(z)$ are coprime rational functions with no poles in D^c. By the Youla parameterization, all controllers $C_d(z)$ that stabilize (5.12) have the form [5]

$$C_d = \frac{Y_d + M_d Q_d}{X_d - N_d Q_d}, \tag{5.13}$$

where Q_d, X_d, and Y_d are stable, and X_d and Y_d satisfy the Bezout identity

$$M_d X_d + N_d Y_d = 1. \tag{5.14}$$

It follows that $S_d = M_d(X_d - N_d Q_d)$ and

$$C_d S_d = M_d(Y_d + M_d Q_d). \tag{5.15}$$

Using (5.15) in (3.6) yields

$$S_{fun}(s) = 1 - \frac{1}{T}e^{-s\tau}\frac{N(s)H(s)}{M(s)}M_d(e^{sT})\left[Y_d(e^{sT}) + M_d(e^{sT})Q_d(e^{sT})\right] \tag{5.16}$$

and

$$T_{fun}(s) = \frac{1}{T}e^{-s\tau}\frac{N(s)H(s)}{M(s)}M_d(e^{sT})\left[Y_d(e^{sT}) + M_d(e^{sT})Q_d(e^{sT})\right]. \tag{5.17}$$

(i) $T_{fun}(s)$ is stable because each factor in the numerator of (5.17) is stable, and because the assumption of nonpathological sampling guarantees that any unstable pole of $1/M(s)$ must be cancelled by a corresponding zero of $M_d(e^{sT})$.

[5] We suppress dependence on the transform variable when convenient; the meaning will always be clear from context.

(ii) It follows from (5.14) that $Y_d(e^{pT}) = 1/(N_d\, e^{pT})$. Using this fact, and evaluating (5.16) in the limit as $s \longrightarrow p$ yields

$$S_{fun}(s) \longrightarrow 1 - \lim_{s \to p} \frac{\frac{1}{T} H(s)P(s)F(s)}{(FPH)_d(e^{sT})} \tag{5.18}$$

Apply Lemma 2.2.3 to $(FPH)_d(e^{sT})$:

$$S_{fun}(s) \longrightarrow 1 - \lim_{s \to p} \frac{H(s)P(s)F(s)}{\sum_{k=-\infty}^{\infty} F(s + jk\omega_s)\, P(s + jk\omega_s)\, H(s + jk\omega_s)} \tag{5.19}$$

By the assumptions that $F(s)$ is stable and that sampling is nonpathological, $P(s)$ and $F(s)$ have no poles at $p + jk\omega_s$, $k \neq 0$. Using this fact, and the fact that $H(s)$ has no finite poles, yields that each term in the denominator of (5.19) remains finite as $s \longrightarrow p$ except the term $k = 0$. The result follows.

To prove (iii) - (vi), observe first that (5.16) implies $T_{fun}(s)$ can have CRHP zeros only at the CRHP zeros of $N(s)$, $H(s)$, $M_d(e^{sT})$, or $[Y_d(e^{sT}) + M_d(e^{sT})Q_d(e^{sT})]$

(iii) - (vii) By our assumptions, $P(s)$, $F(s)$, and $P(s)F(s)$ are each free of unstable hidden modes. Hence $N(s)$ and $M(s)$ can have no common CRHP zeros and (iii) follows. By the assumption of nonpathological sampling, neither can $H(s)$ and $M(s)$. Hence (iv) follows. Note next that the zeros of $M_d(e^{sT})$ lie at $p + jk\omega_s$, $k = 0, \pm 1, \pm 2, \cdots$, where p is any CRHP pole of $P(s)$ and hence a zero of $M(s)$. It follows from this fact that the ratio $M_d(e^{sT})/M(s)$ can have zeros only for $k = \pm 1, \pm 2, \cdots$. By the assumption of nonpathological sampling, no other cancellations occur, and all these zeros are indeed zeros of $T_{fun}(s)$. This proves (vi). By (5.13), the CRHP zeros of $[Y_d(e^{sT}) + M_d(e^{sT})Q_d(e^{sT})]$ are identical to the CRHP zeros of $C_d(e^{sT})$. By the hypotheses of Lemma 2.3.2, none of these zeros can coincide with those of $M_d(e^{sT})$, and thus with those of $M(s)$. This proves (v). Statement (vii) now follows because (iii)-(vi) exhaust all possibilities for $T_{fun}(s)$ to have CRHP zeros.

(viii) It follows from (5.15) that $C_d(b_k)S_d(b_k) = 1/(FPH)_d(b)$. Substitution of this identity into (3.6)-(3.7) yields (5.10)-(5.11).

$\bigtriangledown\bigtriangledown\bigtriangledown$

There are a number of differences between the interpolation constraints for the hybrid and the continuous-time cases; we now describe these in detail.

Remark 5.1 ***CRHP Plant Zeros***
Each CRHP plant zero yields constraints (5.2) directly analogous to the continuous-time case. Note in particular that these constraints are present independently *of the choice of the hold function. The zeros of the discretized plant lying in D^c, on the other hand, do not impose any inherent constraints on $S_{fun}(s)$. Indeed, suppose that $\nu \in D^c$ is a zero of $(FPH)_d(z)$. Then for each $\nu_k \triangleq \frac{1}{T}\log(\nu) + jk\omega_s$, $k = 0, \pm 1, \pm 2, \cdots$, it follows that*

$$S_{fun}(\nu_k) = 1 - \frac{1}{T} P(\nu_k)H(\nu_k)C_d(\nu)F(\nu_k), \tag{5.20}$$

and thus the size of $S_{fun}(\nu_k)$ is not independent of the choice of compensator.

Remark 5.2 ***CRHP Plant Poles***
Each CRHP plant pole yields constraints (5.1) directly analogous to the continuous-time case. Furthermore, each of these poles yields the additional constraints (5.7), which arise from the periodically spaced zeros of $S_d(e^{sT})$ and the fact that nonpathological sampling precludes all but one of these zeros from being cancelled by a pole of $P(s)$.

The additional interpolation constraints imposed upon $S_{fun}(s)$ and $T_{fun}(s)$ by unstable plant poles have interpretations in terms of steady state disturbance rejection properties of the feedback systems. We now state these properties for purposes of later comparison to those associated with unstable compensator poles.

Corollary 5.1 *Assume that $P(s)$ has a pole at $s = j\omega$, $\omega \in \Omega_N$. Then the steady state response to a disturbance $d(t) = e^{j(\omega+\ell\omega_s)t}$, $t \geq 0, \omega \in \Omega_N$, $\ell = 0, \pm1, \pm2, \ldots$ is given as follows.*

(a) *If $\ell = 0$, then*

$$y_{dss}(t) = 0 \ . \tag{5.21}$$

(b) *If $\ell \neq 0$, then*

$$y_{dss}(t) = e^{j(\omega+\ell\omega_s)t} - \frac{F(j(\omega+\ell\omega_s))}{F(j\omega)} e^{j\omega t}. \tag{5.22}$$

Proof: By (2.9) the steady state response is given by

$$y_{dss}(t) = S_{fun}(j(\omega+\ell\omega_s))e^{j(\omega+\ell\omega s)t} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} T_k(j(\omega+\ell\omega_s))e^{j(\omega+(k+\ell)\omega_s)t}$$

(a) By (5.1), $S_{fun}(j\omega) = 0$. Furthermore, by (2.8) and the fact that $S_d(e^{j\omega T}) = 0$, it follows that $T_k(j\omega) = 0$ also.

(b) By (5.7), $S_{fun}(j(\omega+\ell\omega_s)) = 1$. Next, suppose that $k \neq -\ell$. Then by (3.8) and the fact that $S_d(e^{j\omega T}) = 0$ it follows that $T_k(j(\omega+\ell\omega_s)) = 0$. In the case that $k = -\ell$ one may evaluate a limit similar to that used to obtain (5.19) to show that $T_{-\ell}(j(\omega+\ell\omega_s)) = F(j(\omega+\ell\omega_s))/F(j\omega)$ and the result follows.

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It follows from (5.21) that a disturbance of frequency $\omega \in \Omega_N$ will be asymptotically rejected if the plant has a pole at ω. (If necessary, the pole may be augmented to the plant via an *analog* precompensator.) However, any high frequency disturbance of frequency $\omega + \ell\omega_s$, $\omega \in \Omega_N$, $\ell \neq 0$ will be passed directly through to the output along with an alias of frequency ω whose amplitude is determined by the ratio of the gains of the antialiasing filter evaluated at the two frequencies.

Remark 5.3 ***Unstable Compensator Poles***
For analog systems, unstable plant and compensator poles yield identical constraints on the sensitivity and complementary sensitivity functions; namely, when evaluated at such a pole, sensitivity must equal zero and complementary sensitivity must equal one. Comparing (viii) and (ii) in Theorem 5.1, we see that in a hybrid system unstable plant and compensator poles will generally yield different constraints on sensitivity and complementary sensitivity. In particular, unstable compensator poles will yield corresponding zeros of $S_{fun}(s)$ only in special cases. Moreover, unstable compensator poles do not in general yield asymptotic disturbance rejection. We formalize this observation in the following corollary, which is a counterpart to Corollary 5.1 for unstable plant poles.

Corollary 5.2 *Assume that $C_d(z)$ has a pole at $z = e^{j\omega T}$, and that $P(s)$ has no poles at $s = j(\omega + k\omega_s)$, $k = 0, \pm 1, \pm 2, \ldots$. Then the steady state response to a disturbance input $d(t) = e^{j(\omega+\ell\omega_s)t}, t \geq 0, \omega \in \Omega_N, \ell = 0, \pm 1, \pm 2, \ldots$ satisfies*

$$y_{dss}(t) = S_{fun}(j(\omega + \ell\omega_s))e^{j(\omega+\ell\omega s)t} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} T_k(j(\omega + \ell\omega_s))e^{j(\omega+(k+\ell)\omega_s)t} \,, \tag{5.23}$$

where

$$S_{fun}(j\omega) = 1 - \frac{\frac{1}{T}P(j(\omega + \ell\omega_s))\, H(j(\omega + \ell\omega_s))\, F(j(\omega + \ell\omega_s))}{(FPH)_d(e^{j\omega T})} \tag{5.24}$$

and

$$T_k(j(\omega + \ell\omega_s)) = \frac{\frac{1}{T}P(j(\omega + (k+\ell)\omega_s))\, H(j(\omega + (k+\ell)\omega_s))\, F(j(\omega + \ell\omega_s))}{(FPH)_d(e^{j\omega T})} \tag{5.25}$$

□

Note that even for $\ell = 0$ the steady state disturbance response is in general *nonzero*. For continuous-time systems, it is well known that a periodic disturbance may be asymptotically rejected by incorporating the dynamics of the disturbance into the system. For a hybrid system, Corollaries 5.1 and 5.2 show that for asymptotic disturbance rejection to be present, the dynamics should be augmented to the plant using an analog precompensator. Including a discretized version of these dynamics in the digital compensator will not, in general, achieve the desired result. Exceptions to this statement may be obtained by imposing additional structure on the hold response function.

Corollary 5.3 *Let the hypotheses of Corollary 5.2 be satisfied. Choose $\omega \in \Omega_N$. Assume that*

$$P(j\omega) \neq 0 \tag{5.26}$$

$$H(j\omega) \neq 0 \tag{5.27}$$

and

$$H(j(\omega + k\omega_s)) = 0, \text{ for all } k = \pm 1, \pm 2, \cdots \tag{5.28}$$

Then

(i) the steady state response to an input $d(t) = e^{j\omega t}$, $t \geq 0$ satisfies

$$y_{dss}(t) = 0 \tag{5.29}$$

(ii) the steady state response to an input $d(t) = e^{j(\omega+\ell\omega_s)t}$, $t \geq 0$, $\ell = \pm 1, \pm 2, \cdots$ satisfies

$$y_{dss}(t) = e^{j(\omega+\ell\omega_s)t} - \frac{F(j(\omega + \ell\omega_s))}{F(j\omega)}e^{j\omega t} \tag{5.30}$$

Proof: (i) The steady state response to $e^{j\omega t}$ is given by (3.9). Hypotheses (5.26)-(5.27) imply that $(FPH)_d(e^{j\omega t}) = \frac{1}{T}F(j\omega)P(j\omega)H(j\omega) \neq 0$, and it follows that the coefficient of $e^{j\omega t}$ in (3.9) equals one. Using (5.28) in (5.25) shows that the coefficients of the higher frequency terms in (3.9) all equal zero.

(ii) Hypotheses (5.26)-(5.28) imply that (4.28) equals one, (4.25) equals zero for $k + \ell \neq 0$, and that

$$T_{-\ell}(j(\omega - \ell\omega_s)) = \frac{F(j(\omega + \ell\omega_s))}{F(j\omega)}.$$

These facts, together with (4.23) yield (5.30).

▽▽▽

A consequence of Corollary 5.3 is the well known fact that a discrete integrator may be used in conjunction with a ZOH to achieve asymptotic rejection of constant disturbances. Related results for hybrid systems with a ZOH are found in [15, 42].

Remark 5.4 ***Zeros of Hold Response***
By (5.3) zeros of $H(s)$ lying in the CRHP impose constraints on the sensitivity function identical to those imposed by CRHP zeros of the plant. A ZOH has CRHP zeros only on the $j\omega$-axis. As discussed in [28], GSHF response functions may have zeros in the open right half plane.

Remark 5.5 ***Zeros of*** $S_{fun}(s)$
Our list of zeros of CRHP zeros for $T_{fun}(s)$ is exhaustive; however, our list for $S_{fun}(s)$ is not. It is interesting to contrast this situation with the analog case. For analog systems, the CRHP zeros of the sensitivity function consist precisely of the union of the CRHP poles of the plant and compensator. On the other hand, by Theorem 5.1 (ii) and (viii), unstable plant poles yield zeros of $S_{fun}(s)$ while unstable compensator poles generally do not. Furthermore, as the following example shows, $S_{fun}(s)$ may have CRHP zeros even if both plant and compensator are stable.

Example 5.1
Consider the plant $P(s) = 1/(s+1)$. Discretizing with a ZOH, sample period $T = 1$, and no antialiasing filter (i.e. $F(s) = 1$) yields $(HPF)_d(z) = .6321/(z - .3670)$. A stabilizing discrete controller for this plant is

$$C_d(z) = \frac{(4.8158)(z^2 + .1z + 0.3988)}{(z^2 - 1.02657z + 0.9025)}.$$

Both plant and compensator are stable; yet it may be verified that $S_{fun}(s)$ has zeros at $s = 0.2 \pm j$.

Remark 5.6 ***Zeros of*** $C_d(z)$
Each zero of the compensator lying in D^c imposes infinitely many interpolation constraints (5.5) upon the continuous-time system because there are infinitely many points in the s-plane that map to the location of the zero in the z-plane. These constraints are due to the fact that a pole at any of these points will lead to an unstable discrete pole-zero cancellation.

6 Integral Relations

The interpolation constraints derived in the preceding section fix the values of the hybrid response functions at points of the CRHP. For continuous-time systems the Poisson integral may be used to translate the interpolation constraints into equivalent integral relations that the sensitivity and complementary sensitivity functions must satisfy along the $j\omega$ - axis [18]. In this section, we show that similar integral relations must be satisfied by the fundamental sensitivity and complementary sensitivity functions. We also show that these functions must satisfy generalizations of the Bode sensitivity integral [3] and its dual for complementary sensitivity [29, 30]).

6.1 Notation

Denote the nonmiminum phase zeros of $P(s)$ by

$$\{\zeta_i; i = 1, \ldots, N_\zeta\}, \tag{6.1.1}$$

the nonminimum phase zeros of $H(s)$ by

$$\{\gamma_i; i = 1, \ldots, N_\gamma\}, \tag{6.1.2}$$

the nonminimum phase zeros of $C_d(z)$ by

$$\{a_i; i = 1, \ldots, N_a\} \tag{6.1.3}$$

and the ORHP poles of $P(s)$ by

$$\{p_i; i = 1, \ldots, N_p\}, \tag{6.1.4}$$

including multiplicities in each case. To each a_i and p_i, denote the associated NMP zeros of $T_{fun}(s)$ by

$$\{a_{ik} = \frac{1}{T} log(a_i) + jk\omega_s, k = 0, \pm 1, \pm 2, \ldots\} \tag{6.1.5}$$

and

$$\{p_{ik} = p_i + jk\omega_s, k = \pm 1, \pm 2, \ldots\}, \tag{6.1.6}$$

respectively. As discussed in [28], it is possible that the hold function has a countable infinity of NMP zeros, and thus that N_γ in (6.1.2) equals infinity.

Denote the Blaschke products of NMP zeros of $P(s)$ and $H(s)$ by

$$B_\zeta(s) = \prod_{i=1}^{N_\zeta} \left(\frac{\zeta_i - s}{\bar{\zeta}_i + s} \right) \tag{6.1.7}$$

and[6]

$$B_\gamma(s) = \prod_{i=1}^{N_\gamma} \left(\frac{\gamma_i - s}{\bar{\gamma}_i + s} \right). \tag{6.1.8}$$

Denote the Blaschke product of ORHP plant poles by

$$B_p(s) = \prod_{i=1}^{N_p} \left(\frac{p_i - s}{\bar{p}_i + s} \right) \tag{6.1.9}$$

For each NMP zero of $C_d(z)$ and for each ORHP pole of $P(s)$, denote the Blaschke products of associated NMP zeros of $T_{fun}(s)$ by[4]

$$B_{a_i}(s) = \prod_{k=-\infty}^{\infty} \left(\frac{a_{ik} - s}{\bar{a}_{ik} + s} \right) \tag{6.1.10}$$

and

$$B_{p_i}(s) = \prod_{\substack{k = -\infty \\ k \neq 0}}^{\infty} \left(\frac{p_{ik} - s}{\bar{p}_{ik} + s} \right). \tag{6.1.11}$$

[6] That the Blaschke products (6.1.8) and (6.1.10)-(6.1.11) converge follows from [21].

6.2 Poisson Sensitivity Integral

We now derive an integral *inequality* that must be satisfied by $\log|S_{fun}(j\omega)|$.

Theorem 6.2.1 *Assume that the hypotheses of Lemma 2.3.2 are satisfied. Let $\xi = x + jy$ equal one of (6.1.1)-(6.1.2) or (6.1.5)-(6.1.6). Then*

$$\int_0^\infty \log|S_{fun}(j\omega)|\, W(\xi,\omega) d\omega \geq \pi\, log|B_p^{-1}(\xi)|, \tag{6.2.1}$$

where

$$W(\xi,\omega) = \begin{cases} \frac{2x}{x^2+\omega^2} & , y = 0 \\ \frac{x}{x^2+(y-\omega)^2} + \frac{x}{x^2+(y+\omega)^2} & , y \neq 0 \end{cases} \tag{6.2.2}$$

Proof: Denote the NMP zeros of $S_{fun}(s)$ by μ_1, $\mu_2, \cdots$ and define the Blaschke product

$$B_\mu(s) = \prod_i \frac{\mu_i - s}{\bar{\mu}_i + s}.$$

Then $S_{fun}(s) = S_m(s)B_\mu(s)$ where $S_m(s)$ has no poles or zeros in the ORHP. The Poisson integral (p. 225 of [27]) implies that (6.2.1) holds with *equality* if $B_p(\xi)$ is replaced by $B_\mu(\xi)$. Since the set of NMP zeros of $S_{fun}(s)$ due to the ORHP poles of $P(s)$ is generally a *proper* subset of all such zeros (*cf.* Remark 5.5) inequality (6.2.1) follows.

$\bigtriangledown\bigtriangledown\bigtriangledown$

Theorem 6.2.1 has several design implications, which we describe in a series of remarks.

Remark 6.2.1 ***NMP Plant Zeros***
As in the continuous time case, if the plant is nonminimum phase, then requiring that $|S_{fun}(j\omega)| < 1$ over a frequency range Ω implies that, necessarily, $|S_{fun}(j\omega)| > 1$ at other frequencies. The severity of this tradeoff depends upon the relative location of the NMP zero and the frequency range Ω; a thorough discussion of this dependence, and a lower bound upon the peak in $S_{fun}(j\omega)$ are given in [18, 17]. The term due to the Blaschke product in (6.2.1) shows that plants with approximate ORHP pole-zero cancellations yield particularly sensitive feedback systems.

Remark 6.2.2 ***NMP Hold Zeros***
*A nonminimum phase zero of the hold response imposes precisely the same tradeoff as does a zero of the plant in the same location. This tradeoff is exacerbated if the NMP hold zero is near an unstable plant pole. Poor sensitivity in this case is to be expected, as an exact pole-zero cancellation yields an unstable hidden mode in the discretized plant (*cf.* Lemma 2.3.1).*

Remark 6.2.3 ***Unstable Plant Poles***
Using an analog controller, the sensitivity function of a system with an unstable, but minimum phase, plant can be made arbitrarily small over an arbitrarily wide frequency range [47] while maintaining sensitivity bounded outside this range. This is no longer true for digital controllers and the fundamental sensitivity function. The following result is an immediate consequence of (6.2.1).

Corollary 6.2.1

(i) Assume that the plant has a real ORHP pole, $p = x$. Then

$$\|S_{fun}\|_\infty \geq \sqrt{1 + \left(\frac{x}{\omega_N}\right)^2} \tag{6.2.3}$$

(ii) Assume that the plant has an ORHP complex conjugate pole pair, $p = x + jy, \bar{p} = x - jy$. Then for $k = \pm 1, \pm 2 \ldots$

$$\|S_{fun}\|_\infty \geq \sqrt{1 + \left(\frac{x}{k\omega_N}\right)^2}\sqrt{1 + \left(\frac{x}{y - k\omega_N}\right)^2} \tag{6.2.4}$$

□

In either case, the fundamental sensitivity function necessarily has a peak strictly greater than one.

For a real pole, achieving good sensitivity requires that the sampling rate be sufficiently fast with respect to the time constant of the pole; *e.g.* achieving $\|S_{fun}\|_\infty < 2$ requires that $\omega_N > x/\sqrt{3}$. This condition is also necessary for a complex pole pair. Furthermore sensitivity will be poor if $y \approx k\omega_N$ for some $k \neq 0$. The reason for poor sensitivity in this case is clear; if $y = k\omega_N$, then the complex pole pair violates the nonpathological sampling condition (2.3.1), and the discretized plant will have an unstable hidden mode.

More generally, we have

Corollary 6.2.2 *Assume that the plant has unstable poles p_i and p_ℓ with $p_i \neq \bar{p}_\ell$.*

Then

$$\|S_{fun}\|_\infty \geq \max_{k \neq 0} \left| \frac{\bar{p}_i + p_\ell + jk\omega_s}{p_i - p_\ell - jk\omega_s} \right| \tag{6.2.5}$$

and

$$\|S_{fun}\|_\infty \geq \max_{k \neq 0} \left| \frac{p_i + p_\ell + jk\omega_s}{\bar{p}_i - p_\ell - jk\omega_s} \right| \tag{6.2.6}$$

□

It follows that if sampling is "almost pathological", in that $p_i - p_\ell \approx jk\omega_s$, or $\bar{p}_i - p_\ell \approx jk\omega_s$, then sensitivity will be large.

Remark 6.2.4 ***Approximate Discrete Pole Zero Cancellations***
Suppose that the discrete compensator has an NMP zero a_i. Then (6.2.1) holds with ξ equal to one of the points a_{ik} (6.1.5) in the s-plane that map to a_i in the z-plane. If the plant has an unstable pole near one of these points, then the right hand side of (6.2.1) will be large, and S_{fun} will have a large peak. Poor sensitivity is plausible, because this situation corresponds to an approximate pole-zero cancellation between a NMP zero of the compensator and a pole of the discretized plant.

6.3 Poisson Complementary Sensitivity Integral

We now derive a result for $T_{fun}(s)$ dual to that for $S_{fun}(s)$ obtained in the previous section. An important difference is that we can characterize *all* NMP zeros of $T_{fun}(s)$, and thus obtain an integral *equality*.

First, we note some additional properties of the hold response function.

Lemma 6.3.1

(a) *The hold response function (2.2.2) may be factored as*

$$H(s) = H_m(s)e^{-s\tau_H} B_\gamma(s), \tag{6.3.1}$$

where $\tau_H \geq 0$, $B_\gamma(s)$ *is given by (6.1.8), and* $\log|H_m(s)|$ *satisfies the Poisson integral relation.*

(b) *If* $H(s)$ *is the response of an FDLTI hold (2.2.3), then* $\tau_H = 0$.

(c) *Suppose that* $H(s)$ *is the response of a piecewise constant hold (2.2.5) that is not identically zero. Let* $\bar{k}$ *denote the smallest value of* k *for which* $\alpha_k \neq 0$. *Then* $\tau_H = \bar{k}T/N$.

Proof: Part (a) follows from pp. 132-133 of [21]. Parts (b) and (c) are immediate from (2.2.4) and (2.2.6).

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Explicit expressions for the zeros of a piecewise constant hold with $\alpha_0 \neq 0$ and approximations to the zeros of an FDLTI hold are given in Lemma 2.4 of [28]. We remark that, in each case, $H(s)$ possesses infinitely many zeros which approach infinity along well defined paths which may lie in the ORHP. See [28] for further discussion.

Theorem 6.3.1 *Assume that the hypotheses of Lemma 2.3.2 are satisfied. Let* $p_\ell = x + jy$ *be an ORHP pole of* $P(s)$. *Then*

$$\begin{aligned}\int_0^\infty \log|T_{fun}(j\omega)|\, W(p_\ell,\omega)\, d\omega \;&=\; \pi x\tau_P + \pi x\tau_H + \pi x N_C T + \\ &\quad \pi\log|B_\zeta^{-1}(p_\ell)| + \pi\log|B_\gamma^{-1}(p_\ell)| + \\ &\quad \pi\sum_{i=1}^{N_p}\log|B_{p_i}^{-1}(p_\ell)| + \pi\sum_{i=1}^{N_a}\log|B_{a_i}^{-1}(p_\ell)|\end{aligned} \tag{6.3.2}$$

where

$$W(p_\ell,\omega) = \begin{cases} \frac{2x}{x^2+\omega^2} & , y = 0 \\ \frac{x}{x^2+(y-\omega)^2} + \frac{x}{x^2+(y+\omega)^2} & , y \neq 0 \end{cases} \tag{6.3.3}$$

Proof: Note that $T_{fun}(s)$ has an inner-outer factorization

$$T_{fun}(s) = T_m(s)\, e^{-s\tau_P}\, e^{-s\tau_H}\, e^{-s\tau N_C T}\, B_\zeta(s)\, B_\gamma(s) \prod_{i=1}^{N_p} B_{p_i}(s) \prod_{i=1}^{N_a} B_{a_i}(s)$$

where $\log|T_m(s)|$ satisfies the Poisson integral [21]. Since $\log|T_m(j\omega)| = \log|T_{fun}(j\omega)|$, the result follows.

$\nabla\nabla\nabla$

We comment on the design implications of Theorem 6.3.1 in a series of remarks.

Remark 6.3.1
The first three terms on the right hand side of (6.3.2) show that $|T_{fun}(j\omega)|$ will display a large peak if there is a long time delay in the plant, digital controller, or hold function.

Remark 6.3.2
The fourth and fifth terms on the right hand side of (6.3.2) show that $|T_{fun}(j\omega)|$ will display a large peak if there is an approximate unstable pole-zero cancellation in the plant, or between the plant and the hold function. By the nonpathological sampling condition (2.3.2), the latter peak corresponds to an approximate unstable pole-zero cancellation in the discretized *plant.*

The following result is an analogue of Corollary 6.2.2 for $T_{fun}(s)$.

Corollary 6.3.1

(i) Assume that $p_\ell = x$, a real pole. Then

$$\|T_{fun}\|_\infty \geq \frac{\sinh\left(\frac{\pi x}{\omega_N}\right)}{\left(\frac{\pi x}{\omega_N}\right)} . \tag{6.3.4}$$

(ii) Assume that $p_\ell = x + jy$, a complex pole. Then

$$\|T_{fun}\|_\infty \geq \frac{\sinh\left(\frac{\pi x}{\omega_N}\right)}{\left(\frac{\pi x}{\omega_N}\right)} \left|\frac{\sinh\left(\frac{\pi p_\ell}{\omega_N}\right)}{\left(\frac{\pi p_\ell}{\omega_N}\right)}\right| \left|\frac{\left(\frac{\pi y}{\omega_N}\right)}{\sin\left(\frac{\pi y}{\omega_N}\right)}\right| . \tag{6.3.5}$$

Proof: By rearranging definition (6.1.11), we have

$$B_{p_i}(s) = \prod_{k=1}^{\infty} \frac{1 - \left(\frac{p_i - s}{jk\omega_s}\right)^2}{1 - \left(\frac{\bar{p}_i + s}{jk\omega_s}\right)^2}$$

Using the identities ([27], p. 387)

$$\frac{\sin \pi\alpha}{\pi\alpha} = \prod_{k=1}^{\infty}\left(1 - \frac{\alpha^2}{k^2}\right)$$

and $\sin j\alpha = j\sinh\alpha$ yields

$$B_{p_i}(s) = \frac{\sinh \pi\left(\frac{p_i - s}{\omega_s}\right)}{\pi\left(\frac{p_i - s}{\omega_s}\right)} \frac{\pi\left(\frac{\bar{p}_i + s}{\omega_s}\right)}{\sinh \pi\left(\frac{\bar{p}_i + s}{\omega_s}\right)} \tag{6.3.6}$$

Note that the first factor on the right hand side of (6.3.6) converges to one as $s \rightarrow p_i$. It follows that

$$B_{p_\ell}(p_\ell) = \frac{\left(\frac{\pi x}{\omega_N}\right)}{\sinh\left(\frac{\pi x}{\omega_N}\right)} \tag{6.3.7}$$

Inverting yields (6.3.4). Furthermore

$$B_{\bar{p}_\ell}(p_\ell) = \frac{\sin\left(\frac{\pi y}{\omega_N}\right)}{\left(\frac{\pi y}{\omega_N}\right)} \frac{\left(\frac{\pi p_\ell}{\omega_N}\right)}{\sinh\left(\frac{\pi p_\ell}{\omega_N}\right)} \tag{6.3.8}$$

Together (6.3.7)-(6.3.8) yield (6.3.5).

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Plots of the bounds (6.3.4)-(6.3.5) versus $\mathrm{Re}(p_\ell)$ for various values of $\mathrm{Im}(p_\ell)$ are given in Figure 6.3.1; the pole location has been normalized by the Nyquist frequency. Note that for a complex pole $\|T_{fun}\|_\infty$ will become arbitrarily large as $y \rightarrow k\omega_N$, $k = \pm 1, \pm 2 \ldots$, because sampling becomes pathological at such frequencies. It follows from these plots that to achieve good robustness the Nyquist frequency should be chosen several times larger than the radius of any unstable pole.

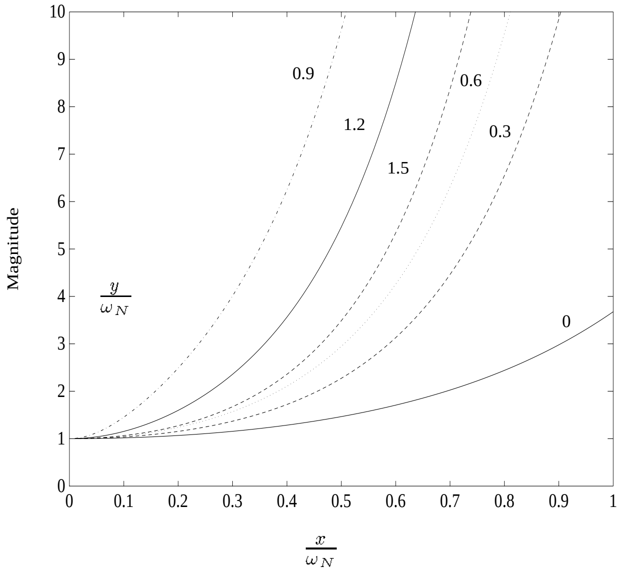

Figure 6.3.1: Bounds on T_{fun}

6.4 Bode Sensitivity Integral

The following is a generalization of the classical sensitivity integral theorem of Bode [Bod 45]. As in the case of the Poisson sensitivity integral, we only obtain an *inequality*, because $S_{fun}(s)$ may possess NMP zeros in addition to those associated with ORHP plant poles.

Theorem 6.4.1 *Assume that the hypotheses of Lemma 2.3.2 are satisfied.*

$$\int_0^\infty \log |S_{fun}(j\omega)| d\omega \geq \pi \sum_{i=1}^{N_p} Re(p_i) \tag{6.4.1}$$

Proof: The assumption that $h(t)$ satisfies Assumption 2.2.1 implies that $H(s)$ satisfies (2.2.2). This fact, together with the assumption that $F(s)$ is strictly proper, imply that

$$\lim_{\substack{s \to \infty \\ \mathrm{Re}(s) \geq 0}} |s T_{fun}(s)| = 0.$$

Hence the technique used in [18] to derive the continuous-time version of (6.4.1) may be applied.

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The sensitivity integral states that if $|S_{fun}(j\omega)| < 1$ over some frequency range, then necessarily $|S_{fun}(j\omega)| > 1$ at other frequencies. Hence there is a tradeoff between reducing and amplifying the fundamental component of the response to disturbances at different frequencies. This tradeoff is exacerbated if the plant has ORHP poles. As in the analog cases, (6.4.1) does not impose a meaningful design limitation unless an additional bandwidth constraint is imposed ([18, 17]). The need to prevent aliasing in a sampled-data system implies that bandwidth constraints are potentially more severe than in the analog case. Design implications remain to be worked out, although it should be noted that the frequency response of the hold function as well as that of the antialiasing filter will need to be considered.

6.5 Middleton Complementary Sensitivity Integral

In this section we derive an integral relation for $T_{fun}(s)$ that is dual to the Bode sensitivity integral obtained for $S_{fun}(s)$ in the preceding section. This result is a generalization to hybrid systems of the complementary sensitivity integral introduced in [29, 30]. As in the case of the Poisson complementary sensitivity integral, exhaustive knowledge of the zeros of $T_{fun}(s)$ yields an integral equality.

Theorem 6.5.1 *Assume that the hypotheses of Lemma 2.3.2 are satisfied. Suppose also that $T_{fun}(0) \neq 0$. Define $T'_{fun}(0) = dT_{fun}(s)/ds|_{s=0}$. Then*

$$\begin{aligned}
\int_0^\infty \log \left| \frac{T_{fun}(j\omega)}{T_{fun}(0)} \right| \frac{d\omega}{\omega^2} \quad = \quad & \frac{\pi}{2}(\tau_P + \tau_H + N_C T) + \\
& \pi \sum_{k=1}^{N_\zeta} \frac{1}{\zeta_k} + \pi \sum_{k=1}^{N_\gamma} \frac{1}{\gamma_k} - \pi \sum_{k=1}^{N_p} \frac{1}{p_k} + \\
& \pi \sum_{k=1}^{N_p} \frac{T}{2} \coth(\frac{p_k T}{2}) + \pi \sum_{k=1}^{N_a} \frac{T}{2} \coth(\frac{a_k T}{2}) + \frac{\pi}{2} \frac{T'_{fun}(0)}{T_{fun}(0)} \; . \quad (6.5.1)
\end{aligned}$$

Proof: See Appendix B.

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This result states that if the ratio $|T_{fun}(j\omega)/T_{fun}(0)| < 1$ over some frequency range, then necessarily this ratio must exceed one at other frequencies. Hence there exists a tradeoff between reducing and amplifying the fundamental component of the response to noise in different frequency ranges. Further comments are found in the following series of remarks.

Remark 6.5.1 The first three terms on the right hand side of (6.5.1) show that the tradeoff worsens if the plant, hold, or compensator has a time delay. The second three terms show that the tradeoff worsens if the plant, hold, or compensator has nonminimum phase zeros. It is easy to verify that the sum of the seventh and eighth terms is positive, and thus the tradeoff also worsens if the plant has ORHP poles. For an interpretation of the ninth term, see Remark 6.5.3 below.

Remark 6.5.2 One difference between (6.5.1) and the analogous results in [29, 30] is that the latter references assume the presence of an integrator in the system. We avoid this requirement by instead assuming that $T_{fun}(0) \neq 0$ and normalizing $T_{fun}(s)$ by its DC value. This approach could also have been taken in [29, 30]; we have adopted it here to obtain a more general result. If there is indeed an integrator in the system, as was assumed in ([29, 30]), then the following corollary, which follows from Corollaries 5.1 and 5.2, shows that the normalization factor is unnecessary.

Corollary 6.5.1 *Assume that*

(i) $P(s)$ contains at least one integrator and/or

(ii) $C_d(z)$ contains at least one integrator and $H(s)$ is a ZOH.

Then $T_{fun}(0) = 1$. □

Remark 6.5.3 We now provide an interpretation for the ninth term in (6.5.1). To do this, we consider the hybrid system depicted in Figure 6.5.1.

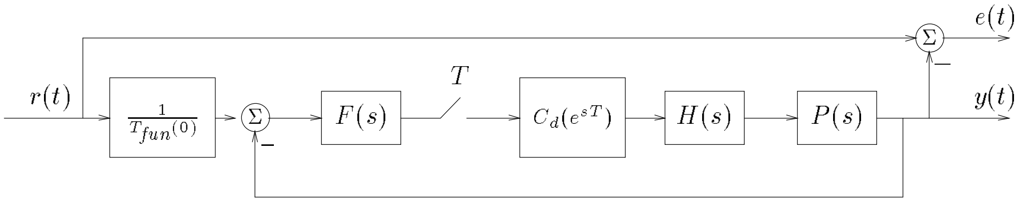

Figure 6.5.1: Hybrid system.

The proof of the following result is straightforward, and hence omitted.

Lemma 6.5.1 *Suppose that the hypotheses of Lemma 2.3.2 are satisfied.*

(i) Consider the response of the system to a unit step input. Then, as $t \to \infty$, $e(t) \to e_{ss}(t)$, where

$$e_{ss}(t) = \sum_{\substack{\ell=-\infty \\ \ell\neq 0}}^{\infty} -\frac{P(j\ell\omega_s)H(j\ell\omega_s)}{P(0)H(0)} e^{j\ell\omega_s t} \tag{6.5.2}$$

Furthermore, if $H(s)$ is a ZOH and/or if $P(s)$ contains an integrator, then $e_{ss}(t) = 0$.

(ii) Consider the response of the system to a unit ramp input. Then as $t \to \infty$, $e(t) \to e_{ss}(t)$, where

$$e_{ss}(t) = -\frac{T'_{fun}(0)}{T_{fun}(0)} - \sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{\infty} [\alpha_\ell + \beta_\ell t] \, e^{j\ell\omega_s t} \, , \tag{6.5.3}$$

with

$$\alpha_\ell = \frac{d}{ds}\left[\frac{T_{fun}(s)}{T_{fun}(0)} \frac{F(s+j\ell\omega_s)}{F(s)}\right] |_{s=-j\ell\omega_s} \tag{6.5.4}$$

and

$$\beta_\ell = \frac{P(-j\ell\omega_s)H(-j\ell\omega_s)}{P(0)H(0)} \, . \tag{6.5.5}$$

Furthermore, if $H(s)$ is a ZOH and/or if $P(s)$ contains an integrator, then $\beta_\ell = 0$ and $e_{ss}(t)$ is bounded. If both these conditions are satisfied, then $\alpha_\ell = 0$ and

$$e_{ss}(t) = -\frac{T'_{fun}(0)}{T_{fun}(0)} \, . \tag{6.5.6}$$

□

It follows from part (ii) of Lemma 6.5.1 that the constant $T'_{fun}(0)/T_{fun}(0)$ plays a role similar to that played by the reciprocal of the velocity constant in a Type-1 analog feedback system[7]. Hence the ninth term on the right hand side of (6.5.1) can ameliorate the severity of the design tradeoff *only* if the steady state error to a ramp input is large and positive, so that the output lags the reference input significantly.

7 Conclusions

We conclude with a brief summary of inherent design limitations for hybrid feedback systems.

Perhaps most important is the fact that those plant properties such as nonminimum phase zeros, unstable poles, and time delays that pose design difficulty for analog feedback systems continue to pose difficulty when the controller is implemented digitally. Furthermore, the existence of such a difficulty is independent of the type of hold function used. It is important, however, that the intersample behavior be examined if the problems are to be detected. Examining system response only at the sampling instants may be misleading.

There are also a number of design limitations unique to digital controller implementations. First, there are limits upon the ability of high compensator gain to achieve disturbance rejection unless the hold function satisfies additional constraints. Second, there are design limitations due to potential nonminimum phase zeros of the hold function. Perhaps most interesting are the design limitations due to unstable plant poles. If the sample rate is "almost pathological" and/or is slow with respect to the time constant of the pole, then sensitivity, robustness, and response to exogeneous inputs will all be poor.

A number of issues need further research. In particular, the harmonic response functions also satisfy interpolation constraints and integral relations; design interpretations remain to be investigated. Furthermore, it is apparent from the results in [1, 26, 40] that the fundamental and harmonic response functions are related to the induced $\mathcal{L}_2$ norm of the system. A thorough analysis remains to be carried out.

[7] For an analog version of (6.5.6), see p. 286 of [39]

Perhaps most interesting is the observation that the hold response function plays a role identical to that of the antialiasing filter in mapping high frequency plant behavior, including uncertainty, into the response of the discretized plant.

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Appendix A: Proof of Lemma 3.2

We consider only the disturbance response, calculations for the noise response being entirely analogous. To evaluate the steady-state response to $d(t) = e^{j\omega t}$, we must first evaluate the inverse Laplace transform of $Y_d(s)$, and then discard all terms due to those poles lying in the OLHP. Inverting the Laplace transform requires that we evaluate the Bromwich integral [27]

$$y_d(t) = \frac{1}{2\pi j}\int_{\gamma - j\infty}^{\gamma + j\infty} e^{st} Y_d(s)\ ds \ , \tag{A.1}$$

where $\gamma > 0$. This integral may be evaluated using the residue theorem.

It follows from (3.3)-(3.4) that $Y_d(s)$ has poles due to the disturbance located along the imaginary axis at $s = j(\omega + k\omega_s)$, $k = 0, \pm 1, \pm 2, \ldots$. By the assumption of closed loop stability all other poles of $Y_d(s)$ lie in the OLHP. Using (3.3)-(3.4), it may be shown that these OLHP poles have the following properties: (i) they all lie to the right of some vertical line $Re[s] = c < 0$, (ii) there are finitely many poles due to $P(s)$ and no poles due to $H(s)$, and (iii) there are finitely many sequences of poles due to $C_d(e^{sT})$, $S_d(e^{sT})$, and $F(s + jk\omega_s)$, $k = 0, \pm 1, \pm 2, \ldots$ lying on vertical lines and spaced at intervals equal to ω_s.

Next, it is straightforward to verify that the residues of $e^{st}Y_d(s)$ at the $j\omega$-axis poles are given by

$$\lim_{s\to j(\omega+k\omega_s)} (s-j(\omega+k\omega_s))e^{st}Y_d(s) = \begin{cases} S_{fun}(j\omega)e^{j\omega t}, & k=0 \\ -T_k(j\omega)e^{j(\omega+k\omega_s)t}, & k\neq 0 \ . \end{cases} \tag{A.2}$$

We need not calculate explicitly the residues at the other poles; as we shall show, they do not contribute to the steady-state response.

Consider the contours of integration C_n, $n=1,2,3,\dots$ depicted in Figure A.1, and chosen so that (i) C_1 encloses only that $j\omega$-axis pole lying in Ω_N, (ii) the horizontal line $Im[s]=R_1$ does not contain any OLHP poles of $Y_d(s)$, and (iii) $R_{n+1}=R_n+\omega_s$.

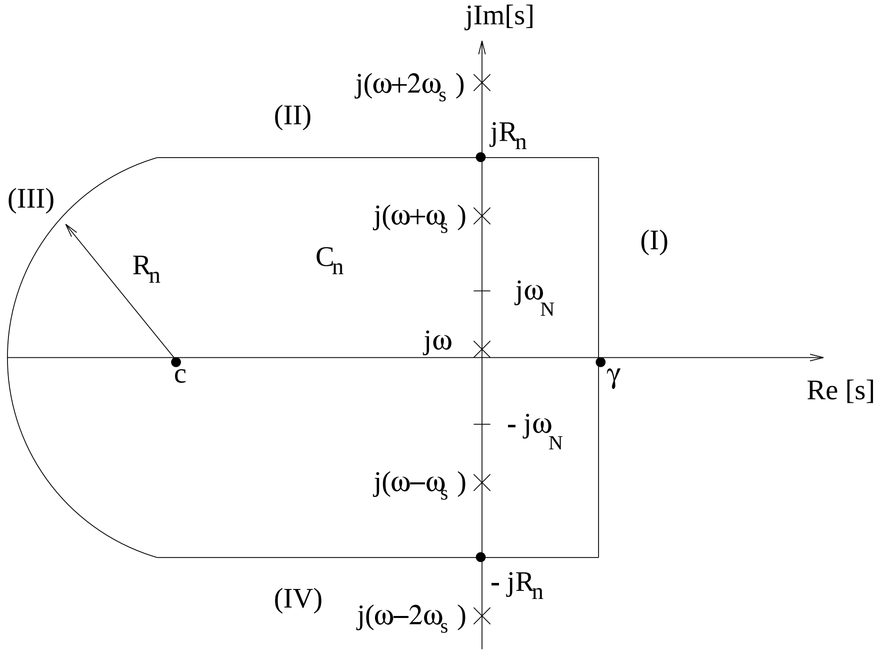

Figure A.1: Contours of Integration

(Figure A.1 and subsequent calculations are appropriate for the case that $|\omega|<\omega_N$. Modifications to the general case are straightforward.) Our construction of the contour of integration guarantees that for n sufficiently large no poles of $Y_d(s)$ will lie on C_N. Hence the residue theorem may be applied to yield

$$\begin{aligned} &\frac{1}{2\pi j}\left\{\int_I e^{st}Y_d(s)\,ds + \int_{II} e^{st}Y_d(s)\,ds + \int_{III} e^{st}Y_d(s)\,ds + \int_{IV} e^{st}Y_d(s)\,ds\right\} \\ &= \ S_{fun}(j\omega)e^{j\omega t} - \sum_{\substack{k=-N \\ k\neq 0}}^{N} T_k(j\omega)e^{j(\omega+k\omega_s)t} + OLHP(t)\ , \end{aligned} \tag{A.3}$$

where $OLHP(t)$ denotes the contribution of the $OLHP$ poles.

We now sketch a proof that as $t\to\infty$, $OLHP(t)\to 0$. First, it is clear that the contribution to $OLHP(t)$ from each pole of $P(s)$ converges to zero. Consider next the contribution of one of the finitely many sequences of poles described in (iii) above. Let this sequence be denoted $\rho_k=\rho+jk\omega_s$, $k=0,\pm1,\pm2,\dots$, and $Re(\rho)<0$. We shall assume that ρ is real for notational simplicity, and shall also assume for simplicity that each ρ_k is a simple pole. Then, for any fixed value of t, the contribution to $OLHP(t)$ from this sequence of poles is given by

$$y_\rho(t) := e^{\rho t} \lim_{K\to\infty} \sum_{k=-K}^{K} Res(\rho_k) e^{jk\omega_s t} \ , \tag{A.4}$$

where $Res(\rho_k) = \lim_{s\to\rho_k}(s-\rho_k)Y_d(s)$. By the Riesz-Fischer Theorem ([36], p. 70), if it may be shown that the sequence $\{Res(\rho_k)\}$ is square-summable, then the series in (A.4) will converge to a bounded periodic function of t. Since $r < 0$, it thus follows that $y_\rho(t) \to 0$ as $t \to \infty$. Since there are only finitely many sequences of the form (A.4), we then have that $OLHP(t) \to 0$.

We now show that the sequence $\{Res(\rho_k)\}$ is square-summable. ¿From the proof of Lemma 3.1, we have that

$$Y_d(s) = D(s) - P(s)H(s)C_d(e^{sT})S_d(e^{sT})\tilde{Z}(e^{sT}) \ , \tag{A.5}$$

where $\tilde{Z}(e^{sT})$ is given by (3.3). Hence

$$Res(\rho_k) = -P(\rho_k)H(\rho_k) \lim_{s\to\rho_k}(s-\rho_k)C_d(e^{sT})S_d(e^{sT})\tilde{Z}(e^{sT}) \ . \tag{A.6}$$

Because $C_d(e^{sT})$, $S_d(e^{sT})$, and $\tilde{Z}(e^{sT})$ are each periodic in s along vertical lines, it may be shown that the limit on the right hand side of (A.6) is independent of k. Denote the common value of this limit by L_ρ. Then (A.4) becomes

$$y_\rho(t) := e^{\rho t} L_\rho \lim_{K\to\infty} \sum_{k=-K}^{K} P(\rho_k)H(\rho_k) e^{jk\omega_s t} \ . \tag{A.7}$$

By Assumption 2.1.1, $|P(\rho_k)|$ converges to a finite constant as $k \to \infty$. Next, the proof of Lemma 2.2.3 may be modified to show that for $s = \rho + jy, \rho < 0$,

$$|sH(s)| \le e^{-\rho T}\{2(n+1)\bar{h} + T\bar{h}'\} \tag{A.8}$$

It follows that the sequence $\{P(\rho_k)H(\rho_k)\}$ is square summable, thus completing the proof that $OLHP(t) \to 0$.

The desired result (3.9) will hold if it may be shown that the last three integrals in (A.3) converge to zero as $N \to \infty$. We now show that the integral (II) converges to zero; similar calculations apply to (IV). Consider values of s such that $s = x + jR_n$, $C \le x \le \gamma$, and R_n is sufficiently large that $R_n > \omega$ and that C_n encloses all poles of $P(s)$. It may be shown that there exists constants M and M_P, independent of n, such that

$$|C_d(e^{sT})S_d(e^{sT})\tilde{Z}(e^{sT})| \le M \text{ and } |P(s)| \le M_P$$

for all such s. Furthermore, it follows from Lemma A.1 of [28] that there exists a constant M_γ with $|se^{st}H(s)| \le M_\gamma$, $\forall\ s$ with $Re[s] \le \gamma$. Using these bounds in (A.4) yields.

$$|e^{st}Y_d(s)| \le e^{\gamma t}(R_n - \omega)^{-1} + MM_pM_\gamma(R_n-\omega)^{-1} \tag{A.9}$$

Using this bound in (II) yields that the integral converges to zero as $R_n \to \infty$.

It remains to show that the integral (III) converges to zero. This follows, as in the proof of Lemma 2.8 in [28], by applying Jordan's Lemma (p. 199, [27]). □

Appendix B: Proof of Theorem 6.5.1

We need the following preliminary result.

Lemma B.1 *Assume that $f(s)$ is analytic and bounded in the CRHP and that $f(0) = 1$. Then*

$$\lim_{x\to 0}\int_0^\infty \frac{\log|f(j\omega)|}{x^2+\omega^2}d\omega = \int_0^\infty \frac{\log|f(j\omega)|}{\omega^2}d\omega \tag{B.1}$$

Proof: The result follows from the Lebesgue Dominated Convergence Theorem (*cf.* [36], p. 37). To apply this result, it suffices to (a) note that $|\log|f(j\omega)|/(x^2+\omega^2)| \le |\log|f(j\omega)|/\omega^2|$ for all x and ω and (b) show that the integral on the right hand side of (B.1) is finite. The latter follows by noting that

(i) $|\log|f(j\omega)||$ is bounded on the $j\omega$-axis except at zeros of $f(j\omega)$,

(ii) these zeros, including a possible zero at infinity, are removable singularities [27] and thus do not cause the integral to become unbounded, and

(iii) the integral approaches a finite limit as $\omega \to 0$.

Statement (iii) follows by using L'Hospital's Rule to show that

$$\lim_{\omega\to 0}\frac{\log f(j\omega)}{\omega^2} = \frac{f'(0)^2 - f''(0)}{2},$$

where $f'(0) = df(s)/ds|_{s=0}$ and $f''(0) = d^2f(s)/ds|_{s=0}$. $\square$

Proof of Theorem 6.5.1:

We begin by applying the Poisson integral to the fundamental complementary sensitivity function for an arbitrary real $x > 0$. Subtracting $\log|T_{fun}(0)|$ from both sides yields

$$\begin{aligned}
&\frac{2}{\pi}\int_0^\infty \log\left|\frac{T_{fun}(j\omega)}{T_{fun}(0)}\right|\frac{x}{x^2+\omega^2}d\omega \\
&\quad= x(\tau_P+\tau_H+N_cT) + \log|B_\zeta^{-1}(x)| + \log|B_\gamma^{-1}(x)| \\
&\qquad+\sum_{k=1}^{N_p}\log|B_{p_k}^{-1}(x)| + \sum_{k=1}^{N_a}\log|B_{a_k}^{-1}(x)| + \log\left|\frac{T_{fun}(x)}{T_{fun}(0)}\right|,
\end{aligned} \tag{B.2}$$

where the terms on the right hand side are as defined in Section 6.1. Dividing both sides by x, taking the limit as $x \to 0$, and applying Lemma B.1 yields

$$\frac{2}{\pi}\int_0^\infty \log\left|\frac{T_{fun}(j\omega)}{T_{fun}(0)}\right|\frac{1}{x^2+\omega^2}d\omega = \tau_P+\tau_H+N_cT$$

$$+\lim_{x\to 0}\frac{1}{x}\log|B_\zeta^{-1}(x)| + \lim_{x\to 0}\frac{1}{x}\log|B_\gamma^{-1}(x)| \tag{B.3}$$

$$+\lim_{x\to \omega}\frac{1}{x}\sum_{k=1}^{N_p}\log|B_{p_k}^{-1}(x)| + \lim_{x\to 0}\frac{1}{x}\sum_{k=1}^{N_a}\log|B_{a_k}^{-1}(x)| + \lim_{x\to 0}\frac{1}{x}\log\left|\frac{T_{fun}(x)}{T_{fun}(0)}\right|. \tag{B.4}$$

We now use L'Hospital's rule and the fact that the zeros and poles (6.1.1)-(6.1.6) must occur in complex conjugate pairs to evaluate the various limits on the right hand side of (B.4):

(i)

$$\begin{aligned}
\lim_{x\to 0}\frac{1}{x}\log|B_{\zeta}^{-1}(x)| &= \lim_{x\to 0}\frac{1}{x}\sum_{k=1}^{N_\zeta}\log|\frac{\bar{\zeta}_k+x}{\zeta_k-x}| \\
&= \sum_{k=1}^{N_p}\lim_{x\to 0}\frac{d}{dx}\left(\frac{1}{x}\log\left[\frac{\bar{\zeta}_k+x}{\zeta_k-x}\right]\right) \\
&= \sum_{k=1}^{N_p}\frac{2Re(\zeta_k)}{|\zeta x|^2} \\
&= 2\sum_{k=1}^{N_p}\frac{1}{\zeta_k}\ .
\end{aligned} \tag{B.5}$$

(ii) A calculation similar to (i) applies to the fifth term in the RHS of (B.4) if it may be shown that the possibly infinite sum $\sum_{k=1}^{N_p} 1/\gamma_k$ converges. Convergence of this series follows from: (i) the fact that $T_{fun}(0)\neq 0 \Rightarrow H(0)\neq 0$ and (ii) applying arguments based on properties of zeros of functions analytic in the CRHP (*cf.* p. 132 of [21]).

(iii)

$$\lim_{x\to 0}\frac{1}{x}\sum_{k=1}^{N_a}\log|B_{a_k}^{-1}(x)| = \lim_{x\to 0}\frac{1}{x}\sum_{k=1}^{N_a}\log\prod_{\ell=-\infty}^{\infty}\left|\frac{\bar{a}_{k_\ell}+x}{\zeta_\ell-x}\right|\ . \tag{B.6}$$

It follows from p. 175 of [7] that

$$\prod_{\ell=-\infty}^{\infty}\left|\frac{\bar{a}_{k\ell}+x}{a_{k\ell}-x}\right| = \left|\frac{\sinh((\bar{a}_k+x)\frac{T}{2})}{\sinh((a_k-x)\frac{T}{2})}\right| \tag{B.7}$$

Substituting (B.7) into (B.6) and applying L'Hospital's rule yields the desired result.

(iv) A calculation similar to (iii) applies to the sixth term on the RHS of (B.4), keeping in mind that the factor $\ell=0$ is not present in the infinite product that corresponds to (B.7).

(v) Applying L'Hospital's rule yields

$$\begin{aligned}
\lim_{x\to 0}\frac{1}{x}\log\left|\frac{T_{fun}(x)}{T_{fun}(0)}\right| &= \lim_{x\to 0}\frac{d}{dx}\left(\frac{1}{2}\log\left(\frac{T_{fun}(x)}{T_{fun}(0)}\right)^2\right) \\
&= \frac{T'_{fun}(0)}{T_{fun}(0)}
\end{aligned}$$

$\bigtriangledown\bigtriangledown\bigtriangledown$