Robustness of Zero-Shifting via Generalized Sampled-data Hold Functions

Jim Freudenberg[*] Rick Middleton[†] Julio Braslavsky[‡]

Department of Electrical and Computer engineering
The University of Newcastle
2308 New South Wales, Australia +61 +49 216033

Technical Report EE9431

August 1994[§]

Abstract

In this paper we study robustness and sensitivity properties of a sampled-data feedback system with a Generalized Sampled-data Hold Function (GSHF). We argue that shifting nonminimum phase zeros using GSHF control can lead to difficulties unless the zero is outside the closed loop bandwidth.

1 Introduction

Nonminimum phase (NMP) zeros of a linear time invariant plant impose inherent design limitations that cannot be overcome by any linear time invariant controller [8, 13]. This fact suggests that more general compensation schemes, such as periodic linear time-varying control, may prove useful in controlling NMP systems. Sampled-data control, wherein an analog plant is controlled by a digital computer through the use of periodic sample and hold, is one class of periodic controllers. Indeed, several authors have noted that the zeros of a discretized plant (unlike the poles) bear no straightforward relationship to the zeros of the original analog plant (e.g., [12, 2]). In particular, use of a Generalized Sampled-data Hold Function (GSHF) with a linear time invariant digital controller allows the zeros of the discretized plant to be placed arbitrarily [12, 2, 16]. Hence it is tempting to conclude that design limitations due to nonminimum phase zeros of an analog plant (cf. [8, 13]) may be circumvented by assigning the zeros of the discretized plant to be minimum phase [3].

On the other hand, several authors have pointed out potential disadvantages to the use of GSHF control. In ([2], p.75) the authors note that "the control signal may become highly irregular". In

[§]Revised August 26, 1994
[*]EECS Department, University of Michigan, Ann Arbor, MI 48109-2122 USA (313) 763-0586, *jfr@eecs.umich.edu*
[†]*rick@ee.newcastle.edu.au*
[‡]*brasky@ee.newcastle.edu.au*

[12], the author notes that systems with GSHF control can sometimes exhibit intersample ripple. Furthermore, the authors of [6] present analyses and simulations which suggest that systems with GSHF controllers are prone to robustness difficulties in addition to poor intersample behavior. Hence the potential utility of GSHF control in overcoming linear time invariant design limitations is still a matter of debate [6, 1].

In [10], the authors showed that design limitations imposed by NMP zeros of the analog plant remain present when the plant is discretized using a GSHF hold, *even if the discretized plant is minimum phase.* A brief summary of the results in [10] is contained in Section 2 of the present paper.

Let us now consider a common procedure by which a digital compensator is designed to control an analog plant. Namely, one discretizes the plant at an appropriate sample rate and designs the compensator so that the discretized feedback system has acceptable properties. As a consequence, the behavior of the analog signals in the resulting hybrid feedback system will be acceptable *at the sampling instants.* One then simulates the hybrid system to verify that the intersample behavior is also acceptable. If the plant is discretized with a ZOH, and if an appropriate sample rate and anti-aliasing filter are used, then this is very often the case.

As noted above, an interesting feature of GSHF control is its ability to locate the zeros of the discretized plant arbitrarily. Suppose that the analog plant has a NMP zero within the desired closed loop bandwidth, but that the discretized plant does not. Suppose also that the discrete closed loop system possesses feedback properties that would be unachievable if the discretized plant also had a problematic NMP zero. It follows as a straightforward corollary to the results of [10] that these feedback properties *cannot* also be present in the intersample behavior of the hybrid system.

A more intriguing question is whether the use of GSHF control to relocate zeros is responsible for sensitivity and robustness difficulties in the resulting feedback system above and beyond those due to the NMP zero of the analog plant. It was argued in [6] that the poor robustness properties of GSHF control are due to the way in which components of the high frequency plant response are aliased down into the baseband to form the frequency response of the discretized plant. In this paper we shall investigate this phenomenon in detail by developing a framework in which the robustness difficulties associated with zero-shifting may be studied quantitatively.

The remainder of this paper is organised as follows. In Section 2 we define notation, summarise relevant results from [10], and provide background material. In Section 3 we state and resolve the first of two *Gedanken* experiments. The first experiment shows that if one is concerned about the quality of the intersample response then GSHF control cannot circumvent design limitations due to an analog NMP zero. In Section 4 we state the second *Gedanken* experiment; unlike the first, this experiment is concerned only with performance at the sampling instants. Indeed, the analog plant enters only as a source of modelling uncertainty in the discretized system. We discuss differential sensitivity and robustness of the discretized feedback system and show that these properties are described by a so-called *fidelity function.* Interpolation constraints and a Poisson integral relation for the fidelity function are derived in Section 5. In Section 6 this integral relation is used to resolve the second *Gedanken* experiment. Conclusions are presented in Section 6.

2 Background

Consider the single-input single-output sampled-data feedback system depicted in Figure 1, where $P(s)$ and $F(s)$ are the transfer functions of the analog plant and anti-aliasing filter, $C_d(z)$ is the transfer function of the digital controller, $r(t), d(t)$, and $n(t)$ are the command, disturbance, and noise signals, $u(t)$ is the control input, and $y(t)$ is the system output. Denote the sampling period by T and the associated sampling and Nyquist frequencies by $\omega_s \triangleq 2\pi/T$, and $\omega_N \triangleq \pi/T$, respectively. We shall use the term *baseband* to denote the frequency range $\Omega_N \triangleq (-\omega_N, \omega_N]$.

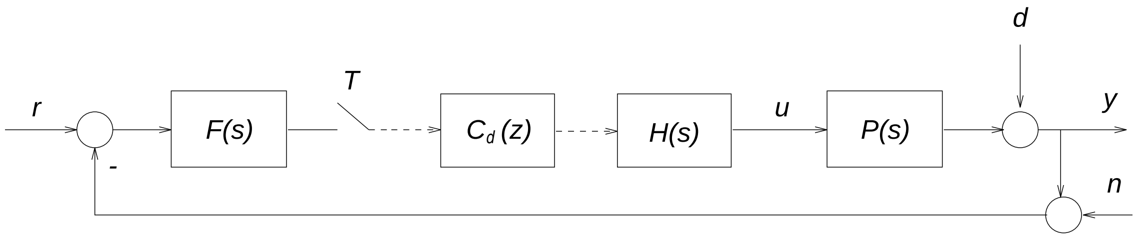

Figure 1: Sampled-data Feedback System

We denote the open and closed right halves of the s-plane by ORHP and CRHP, respectively. We denote the open and closed unit disks by $D \triangleq \{z : |z| < 1\}$, and $\bar{D} \triangleq \{z : |z| \leq 1\}$, respectively, and we denote their complements by D^c and $\bar{D}^c$. A rational function of s (respectively z) is *minimum phase* if it has no zeros in the ORHP (respectively, in $\bar{D}^c$). Otherwise, it is nonminimum phase, and the corresponding zeros are termed *nonminimum phase zeros.*

We shall assume that the plant, prefilter, and controller are each free of unstable hidden modes. In addition, suppose that $P(s) = P_0(s)e^{-s\tau}$, where $P_0(s)$ is rational and proper and $\tau \geq 0, F(s)$ is strictly proper, rational, and has no CRHP poles or zeros, and $C_d(z)$ is rational and proper. We furthermore assume that the hold function is a Generalized Sampled-data Hold Function (GSHF) as described in [12]:

$$u(t) = h(t - kT)u_k, \qquad kT \leq t < (k+1)T \tag{2.1}$$

We shall suppose that $h(t)$ satisfies the mild technical assumptions stated in [14], [10]. Then to this hold there is an associated frequency response function

$$H(s) = \int_0^T e^{-sT} h(t)dt. \tag{2.2}$$

For a zero-order hold (ZOH), it is well-known that $H(s) = (1 - e^{-sT})/s$. Response functions for other types of hold functions are derived in [14].

Next, let the transfer function of the discretized connection of plant, hold, and prefilter be denoted $(FPH)_d(z)$; we shall term this the discretized plant. The assumptions on $P(s), H(s)$, and $F(s)$ stated above suffice to guarantee that the discretized plant satisfies the following well-known formula [10]:

$$(FPH)_d(e^{sT}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F(s + jk\omega_s)P(s + jk\omega_s)H(s + jk\omega_s) \tag{2.3}$$

The formula (2.3) is crucial to analyses of frequency response properties of sampled-data systems. In particular (2.3) shows that the response of the discretized plant at a frequency $\omega \in \Omega_N$ depends upon the response of the analog plant, prefilter, and hold function at infinitely many frequencies $\omega + k\omega_s, k = 0, \pm 1, \pm 2, \ldots$ As we shall see, this fact has interesting implications for the response of the hybrid system to uncertainty, noise and disturbances.

In [14], the well-known nonpathological sampling condition for plants discretized with a ZOH is generalized to the case of a GSHF. In particular, since the hold response function of a GSHF may have zeros in the right half plane, it is necessary to require that none of these coincides with an unstable pole of the analog plant. Under the nonpathological sampling hypothesis, it is straightforward to extend the exponential and $\mathcal{L}_2$ input-output stability results of [4] and [7] to the case of a GSHF.

Assuming that the hybrid system of Figure 1 is stable, one may use classical results from hybrid system theory to calculate the steady state response of the output $y(t)$ to a sinusoidal input signal. To state these formulas, define the *discrete sensitivity and complementary sensitivity functions*

$$S_d(z) \triangleq \frac{1}{1+(FPH)_d(z)C_d(z)} \quad \text{and} \quad T_d(z) := 1 - S_d(z), \tag{2.4}$$

the *fundamental sensitivity and complementary sensitivity functions*

$$S_{fun}(s) \triangleq 1 - \frac{1}{T}P(s)H(s)C_d(e^{sT})S_d e^{sT})F(s) \tag{2.5}$$

and

$$T_{fun}(s) \triangleq \frac{1}{T}P(s)H(s)C_d(e^{sT})S_d(e^{sT})F(s), \tag{2.6}$$

and, for $k = \pm 1, \pm 2, \ldots$, the *kth harmonic response function*

$$T_k(s) \triangleq \frac{1}{T}P(s+jk\omega_s)H(s+jk\omega_s)C_d(e^{sT})S_d(e^{sT})F(s). \tag{2.7}$$

Using these definitions, the steady state response of $y(t)$ to a disturbance $d(t) = e^{j\omega t}$ is given by

$$y_{dss}(t) = S_{fun}(j\omega)e^{j\omega t} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} T_k(j\omega)e^{j(\omega+k\omega_s)t}. \tag{2.8}$$

Similarly, the steady-state response to a sinusoidal noise input $n(t) = e^{j\omega t}$ is given by

$$y_{nss}(t) = T_{fun}(j\omega)e^{j\omega t} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} T_k(j\omega)e^{j(\omega+k\omega_s)t}. \tag{2.9}$$

From (2.8)-(2.9) we see that the steady state response of the hybrid system to a sinusoidal input with frequency ω consists of a fundamental component with frequency ω plus infinitely many harmonics at other frequencies, each separated from the fundamental by an integer multiple of the sampling frequency.

The functions (2.5)-(2.6) play a role similar to the usual sensitivity and complementary sensitivity functions for analog systems; see [10] for details. In particular, $S_{fun}(s)$ governs the fundamental component of the intersample response to disturbances, and $T_{fun}(s)$ plays this role for the response to noise. The harmonic response functions (2.7) have no counterpart in analog systems, and are thus associated with the cost of using a digital controller. Furthermore, $S_{fun}(s)$ and $T_{fun}(s)$ satisfy interpolation constraints at poles and zeros of the plant and compensator [10]. As in the case of analog systems, complex variable theory may be used to translate these constraints into integral relations imposed along the $j\omega$-axis. In [10] the authors derive analogues to the Poisson and Bode sensitivity integrals and the Poisson and Middleton complementary sensitivity integrals. Denote the set of all ORHP poles of $P(s)$ by $\{p_i; i = 1, \ldots, N_p\}$ and define the associated Blaschke product

$$B_p(s) \triangleq \prod_{i=1}^{N_p} \frac{p_i - s}{\bar{p}_i + s} \quad . \tag{2.10}$$

We shall need the following result [10].

Proposition 2.1 (Poisson Sensitivity Integral): *Assume that the system in Figure 1 is stable. Let $\xi = x + jy$ equal an ORHP zero of $P(s)$ or $H(s)$. Then*

$$\int_0^\infty \log |S_{fun}(j\omega)| W(\xi,\omega) d\omega \geq \pi \log |B_p^{-1}(\xi)|, \tag{2.11}$$

where

$$W(\xi,\omega) = \frac{x}{x^2 + (y-\omega)^2} + \frac{x}{x^2 + (y+\omega)^2} \quad . \tag{2.12}$$

□

It follows from (2.11) that NMP zeros of the analog plant impose a design tradeoff upon the fundamental component of the disturbance response. Specifically, requiring that $|S_{fun}(j\omega)| < 1$ over some frequency range implies that $|S_{fun}(j\omega)| > 1$ at other frequencies. As in the case of analog control, the tradeoff worsens if the plant has ORHP poles.

Also as in the analog case (cf. [8, 9]), the tradeoff between values of $|S_{fun}(j\omega)|$ in different frequency ranges depends upon the relative location of the NMP zeros. We now describe this dependence and present a bound on the peak in $|S_{fun}(j\omega)|$ associated with requiring $|S_{fun}(j\omega)|$ to be small at low frequencies.

Given a frequency interval $\Omega = [0, \omega_0)$ and a point $\xi = x + jy$ in the ORHP, define the weighted length of Ω by

$$\Theta(\xi, \Omega) \triangleq \int_0^{\omega_0} W(\xi,\omega) d\omega, \tag{2.13}$$

where $W(\xi,\omega)$ is given by (2.12). If ξ is real, then

$$\Theta(\xi, \Omega) = -\sphericalangle \frac{\xi - j\omega_s}{\xi + j\omega_s} \; ; \tag{2.14}$$

i.e. , the weighted length of the interval Ω equals the negative of the phase lag contributed by the Blaschke product $(\xi - s)/(\xi + s)$ at the upper end point of the interval. If ξ is complex, then

$$\Theta(\xi, \Omega) = -\frac{1}{2}\left(\sphericalangle \frac{\xi - j\omega_s}{\bar{\xi} + j\omega_s} + \sphericalangle \frac{\bar{\xi} - j\omega_s}{\xi + j\omega_s} \right). \tag{2.15}$$

with a similar phase lag interpretation.

Using this notation, the following result is a straightforward consequence of (2.11).

Corollary 2.2: *Suppose that*

$$|S_{fun}(j\omega)| \leq \alpha \;\; , \forall \omega \in \Omega = [0, \omega_0). \tag{2.16}$$

Then

$$\sup_{\omega > \omega_0} |S_{fun}(j\omega)| \geq (1/\alpha)^{\frac{\Theta(\xi,\Omega)}{\pi - \Theta(\xi,\Omega)}} \left| B_p^{-1}(\xi) \right|^{\frac{\pi}{\pi - \Theta(\xi,\Omega)}} . \tag{2.17}$$

□

The bound (2.17) shows that if disturbance attenuation is required throughout a frequency interval in which the NMP zero contributes significant phase lag, then disturbances will be greatly amplified at some higher frequency.

We shall also need to use the discrete Poisson integral to relate the value of the discrete sensitivity function at an arbitrary point in $\bar{D}^c$ to its value along the unit circle (*cf.* [15]). Denote the poles of

$(FPH)_d(z)$ lying in $\bar{D}^c$ by $\{\rho_i = e^{jp_iT}; i = 1, \ldots, N_p\}$ and define the associated Blaschke product by

$$B_\rho(z) \triangleq \prod_{i=1}^{N_p} \frac{z-\rho_i}{1-\bar{\rho}_i z} \tag{2.18}$$

The following result is adapted from [15].

Proposition 2.3: *Assume that $S_d(z)$ is stable. Let $\nu = re^{j\theta}$ lie in $\bar{D}^c$. Then*

$$\int_0^{\omega_N} \log |S_d(e^{j\omega T})| W_d(\nu, \omega) d\omega \geq \pi \log |B_\rho^{-1}(\nu)| + \pi \log |S_d(\nu)|, \tag{2.19}$$

where

$$W_d(\nu, \omega) = \frac{\frac{T}{2}(r^2-1)}{1-2r\cos(\omega T - \theta) + r^2} + \frac{\frac{T}{2}(r^2-1)}{1-2r\cos(\omega T + \theta) + r^2}\ . \tag{2.20}$$

□

Note that equality may be achieved in (2.19) by incorporating terms due to unstable compensator poles into the Blaschke product (2.18).

We shall need the following discrete counterparts to the definition (2.13) and Corollary 2.2. Consider an interval $\Omega = [0, \omega_0)$, where $\omega_0 \leq \omega_N$, and a point $\xi = x + jy$ in the ORHP. The image of Ω under the mapping $z = e^{sT}$ is an arc, $\Omega_d = (1, e^{j\omega_0 T})$, of the unit circle, and the image of ξ is a point $e^{\xi T}$ in $\bar{D}^c$. Define the length of Ω_d, as weighted by $e^{\xi T}$, to be

$$\Theta_d(\xi, \Omega) = \int_0^{\omega_0} W_d(e^{\xi T}, \omega) d\omega, \tag{2.21}$$

where, from (2.20),

$$\begin{aligned} W_d(e^{\xi T}, \omega) &= \frac{\frac{T}{2}(e^{2xT}-1)}{1-2e^{xT}\cos((\omega - y)T) + e^{2xT}} \\ &+ \frac{\frac{T}{2}(e^{2xT}-1)}{1-2e^{xT}\cos((\omega + y)T) + e^{2xT}}\ . \end{aligned} \tag{2.22}$$

If ξ is real, then

$$\Theta_d(\xi, \Omega) = -\sphericalangle \prod_{k=-\infty}^{\infty} \frac{\xi - j(\omega_0 - k\omega_s)}{\xi + j(\omega_0 + k\omega_s)} \tag{2.23}$$

$$= -\sphericalangle \frac{\sinh\left((\xi - j\omega_0)\frac{T}{2}\right)}{\sinh\left((\xi + j\omega_0)\frac{T}{2}\right)}\ ; \tag{2.24}$$

i.e. , the weighted length of the interval Ω equals the negative of the *sum* of the phase lags contributed by the Blaschke product $(\xi - s)/(\xi + s)$ at each of the points $\omega_0 + k\omega_s$, $k = 0, \pm 1, \pm 2, \ldots$, that are mapped to the upper end point of the interval. It is straightforward to verify that the length of the discrete arc Ω_d weighted by the point $e^{\xi T}$ is *greater* than that of the corresponding analog interval $\Omega = [0, \omega_0)$ as weighted by the point ξ. As an example, see Figure 2, which contains plots of $\Theta(\xi, \Omega)$ and $\Theta_d(\xi, \Omega)$ for the point $\xi = \frac{1}{T}$ and values of ω_0 ranging from 0 to ω_N. Similar remarks apply to the case of a complex ξ.

The following result is a straightforward consequence of (2.19).

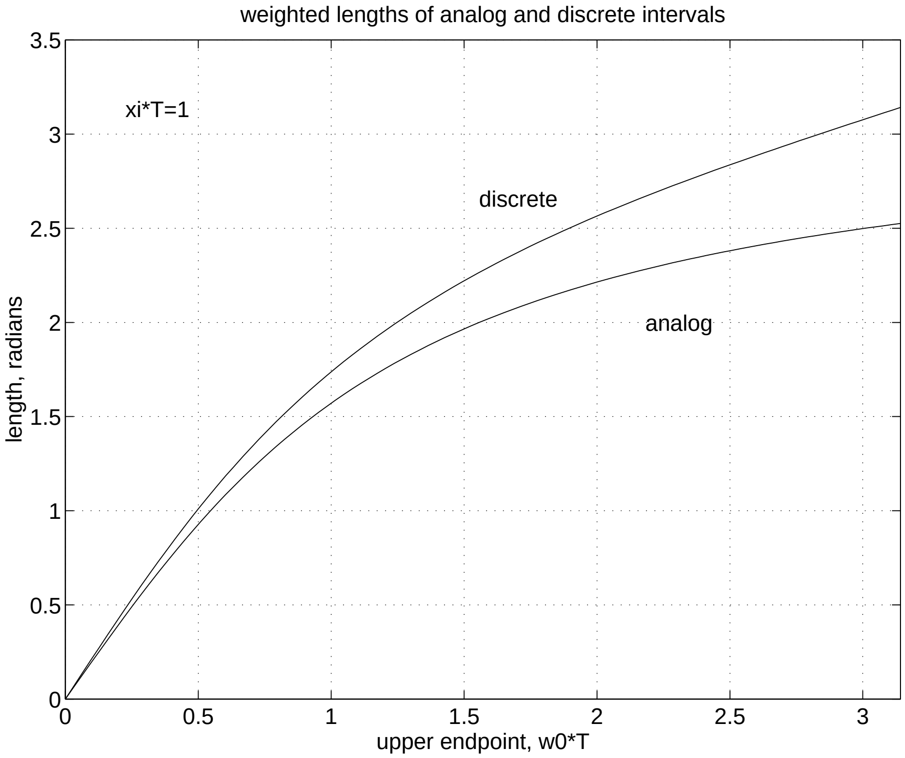

Figure 2: Weighted lengths of analog and discrete intervals

Corollary 2.4: *Suppose that*

$$\left|S_d(e^{j\omega T})\right| \leq \alpha, \forall \omega \in \Omega = [0, \omega_0), \tag{2.25}$$

where $\omega_0 \leq \omega_N$, and let $\nu = e^{\xi T}$, where ξ lies in the ORHP. Then

$$\sup_{\omega \in [\omega_0, \omega_N)} \left|S_d(e^{j\omega T})\right| \geq (1/\alpha)^{\frac{\Theta_d(\xi,\Omega)}{\pi - \Theta_d(\xi,\Omega)}} \left|B_\rho^{-1}(\nu)\right|^{\frac{\pi}{\pi - \Theta_d(\xi,\Omega)}} \left|S_d(\nu)\right|^{\frac{\pi}{\pi - \Theta_d(\xi,\Omega)}} \tag{2.26}$$

□

If ν is a NMP zero of the discretized plant, then $S_d(\nu) = 1$ and $|S_d(e^{j\omega T})|$ is guaranteed to have a peak greater than one. Since $\Theta_d(\xi, \omega_0) \geq \Theta(\xi, \omega_0)$ it follows that the lower bound on this peak is *guaranteed* to be greater than that given by (2.24) in the analog case.

Finally, we shall need the discrete version of the Bode sensitivity integral [15]. For a fixed sampling period, this integral implies a nontrivial sensitivity tradeoff even if no bandwidth constraint is imposed.

Lemma 2.5: *Assume that $S_d(z)$ is stable and that $(FPH)_d(z)C_d(z)$ is strictly proper. Suppose in addition that*

$$|S_d(e^{j\omega T})| \leq \beta, \quad 0 \leq \omega \leq \omega_0 < \omega_N. \tag{2.27}$$

Then necessarily

$$\sup_{\omega_0 < \omega < \omega_N} |S_d(e^{j\omega T})| \geq \left(\frac{1}{\beta}\right)^{\frac{\omega_0}{\omega_N - \omega_0}} \left(\prod_{i=1}^{N_p} e^{p_i T}\right)^{\frac{\omega_N}{\omega_N - \omega_0}} \tag{2.28}$$

□

It follows from (2.28) that if $\beta < 1$, then there necessarily exists a peak in sensitivity at higher frequencies.

3 *Gedanken* Experiment No.1: Analog Performance

Consider the following scenario. We wish to design a digital compensator for an analog plant having a problematic NMP zero. Suppose that a GSHF is used so that the discretized plant is minimum phase, or has NMP zeros only at less problematic locations. Then one can design a digital controller so that the discrete sensitivity function satisfies the specification

$$\left|S_d(e^{j\omega T})\right| \le \beta_1 \ , \ 0 \le \omega \le \omega_1 \tag{3.1}$$

$$\left|S_d(e^{j\omega T})\right| \le \gamma \ , \ \omega_1 < \omega \le \omega_N , \tag{3.2}$$

where $\beta_1 > 1$ and γ satisfies the lower bound (2.28) imposed by the discrete Bode sensitivity integral. On the other hand, the intersample behavior of the hybrid system must satisfy constraints due to the analog NMP zero. We now present a *Gedanken* experiment whose result shows that these constraints manifest themselves as limitations upon the ability of the analog response to approximate that of the discretized system.

***Gedanken* Experiment No.1**: Suppose that we wish to design a digital controller for an analog plant. Then the following three questions (among others) are of interest:

(A1) Is the nominal response of the discretized system satisfactory? Equivalently, is the response of the sampled-data system satisfactory *at the sampling instants*?

(A2) Does the nominal analog response approximate that of the discrete system, so that a satisfactory discrete response will correspond to satisfactory intersample behavior?

(A3) Is the analog response *insensitive* to plant uncertainty, disturbances, and sensor noise?

Clearly it is desirable that the answers to all three questions be affirmative. *The proposed experiment is to determine whether affirmative answers to all three of these questions can be obtained simultaneously.*

We shall consider that the answer to (A1) is affirmative if the discrete sensitivity and complementary sensitivity functions are well behaved. Specifically, we require that $S_d(e^{j\omega T})$ satisfy bounds of the form (3.1)-(3.2). It follows from the identity

$$S_d(e^{sT}) + T_d(e^{sT}) = 1 \tag{3.3}$$

that if the bounds (3.1)-(3.2) are satisfied, then $|T_d(e^{j\omega T})|$ is also bounded.

To quantify the answer to (A2), define the *fidelity function*.

$$\begin{aligned} S_{fid}(s) &\triangleq S_{fun}(s) - S_d(e^{sT}) && (3.4)\\ &= -T_{fun}(s) + T_d(e^{sT}) && (3.5) \end{aligned}$$

If $|S_{fid}(e^{j\omega T})| \ll 1$, then at frequency ω the fundamental component of the analog response to disturbances, noise, and commands will closely approximate that of the discretized system.

Since the discrete frequency response is periodic in ω, it is clearly not possible (nor desirable) that $S_{fun}(j\omega)$ and $T_{fun}(j\omega)$ closely approximate the discrete responses at all frequencies. Hence we shall consider that the answer to (A2) is affirmative if fidelity is achieved over a low frequency range:

$$|S_{fid}(j\omega)| \leq \beta_2\ ,\ 0 \leq \omega \leq \omega_2 < \omega_N. \tag{3.6}$$

Finally, as discussed in [10], it is necessary to keep the fundamental sensitivity and complementary sensitivity functions bounded at all frequencies to prevent large intersample response to disturbances and noise, as well as to keep differential sensitivity and stability robustness from being poor. Hence an affirmative answer to (A3) will require that $S_{fun}(j\omega)$ and $T_{fun}(j\omega)$ satisfy upper bounds of the form

$$|S_{fun}(j\omega)| \leq M_S(\omega) \tag{3.7}$$

and

$$|T_{fun}(j\omega)| \leq M_T(\omega) \tag{3.8}$$

at all frequencies.

It follows immediately from (2.11) that the analog NMP zero imposes a limitation upon our ability to achieve affirmative answers to all of questions (A1)-(A3).

Corollary 3.1: *Suppose that the hybrid feedback system is stable and that (3.1) and (3.6) both hold. Define* $\omega^* \triangleq \min\{\omega_1, \omega_2\}$*, and* $\Omega^* = [0, \omega^*)$*. If the analog plant has a NMP zero at* ξ*, it follows that*

$$\sup_{\omega > \omega^*} |S_{fun}(j\omega)| \geq (1/(\beta_1 + \beta_2))^{\frac{\Theta(\xi,\Omega^*)}{\pi - \Theta(\xi,\Omega^*)}} \left|B_p^{-1}(\xi)\right|^{\frac{\pi}{\pi - \Theta(\xi,\Omega^*)}} \tag{3.9}$$

Proof: It follows immediately from Corollary 2.2. □

The peak in $|S_{fun}(j\omega)|$ associated with making both β_1 and β_2 small will tend to violate the bounds (3.7)-(3.8) *unless* ω^* is sufficiently small that the NMP zero does not contribute significant phase lag at this frequency.

4 *Gedanken* Experiment No.2: Discrete Response

We argued in the preceding section that GSHF control is ineffective at removing the design limitations due to analog NMP zeros. Indeed, there exists a tradeoff between the quality of the response at sampling instants and that of the intersample behavior. Specifically, if $|S_d(e^{j\omega T})|$ is made small over a wide frequency band relative to the location of the NMP zero, then $|S_{fun}(j\omega)|$ cannot closely approximate the discrete response over this band without incurring large peaks at higher frequencies.

In the present section, we shall argue that use of GSHF control to shift zeros so that the discrete sensitivity function can be made small over a wide frequency range may lead to unacceptable robustness difficulties *even if no requirement is imposed upon the analog response.* The source of these difficulties is the necessity to maintain stability robustness against the contribution of high frequency aliases to the discrete plant response.

Consider again the formula (2.3), which shows that the response of the discretized plant at a frequency $\omega \in \Omega_N$ depends upon the response of the analog plant, prefilter, and hold function at infinitely many frequencies $\omega + k\omega_s, k = 0, \pm 1, \pm 2, \ldots$.

Typically, the anti-aliasing filter has a monotonically decreasing Bode gain plot, and thus tends to diminish the contribution of the high frequency plant behavior to the discretized system response.

The effect of the hold response in (2.3) is identical to that of the anti-aliasing filter, and shows that the hold response plays an equally important role in determining the effect of the high frequency plant behavior upon the discretized system. In this regard, it is instructive to compare the responses of the ZOH and a GSHF taken from Example 2 in [12] Figure 3. Note that the

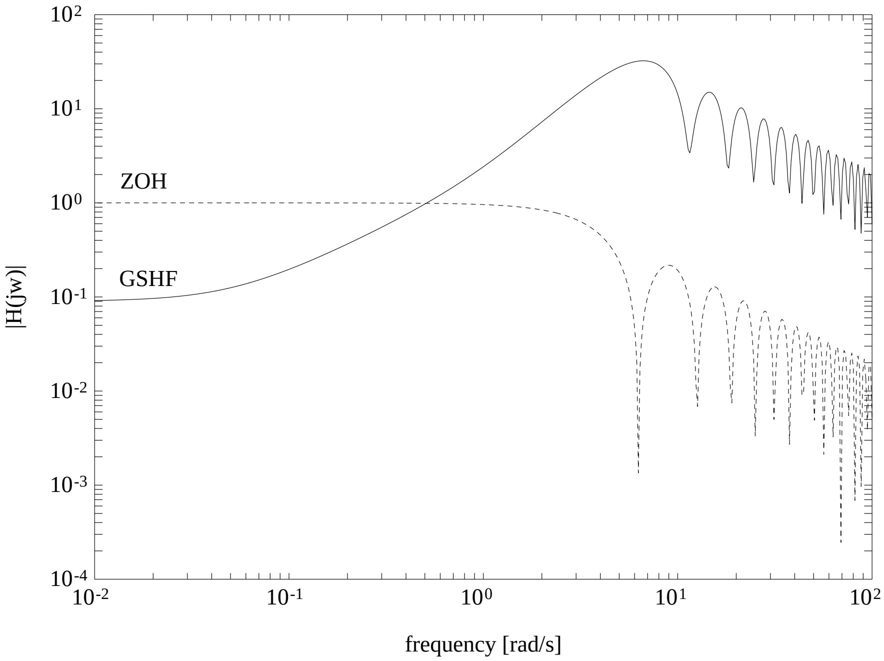

Figure 3: Frequency Responses of a ZOH and a GSHF

frequency response of *this particular* GSHF has larger gain at high frequencies than does that of the ZOH. Hence it follows that the frequency response of a plant discretized with this GSHF will depend more heavily upon the high frequency characteristics of the analog plant than if the plant were discretized with a ZOH.

To explore this phenomenon further, let us rewrite (2.3) as

$$(FPH)_d(e^{sT}) = \left(\frac{1}{T}\right)F(s)P(s)H(s) + \left(\frac{1}{T}\Delta_d(s)\right), \tag{4.1}$$

where

$$\Delta_d(s) \triangleq \sum_{k\neq 0} \Delta_k(s) \text{ and } \Delta_k(s) \triangleq F(s+jk\omega_s)P(s+jk\omega_s)H(s+jk\omega_s) \tag{4.2}$$

It follows from (4.1)-(4.2) that if for some value of $s, P(s) = 0$ but $(FPH)_d(e^{sT}) \neq 0$, then *necessarily* the response of the discretized plant at $z = e^{sT}$ must depend upon the response of the analog plant at one or more of the frequencies $s + jk\omega_s, k \neq 0$. As a corollary, the response of the discretized system will be potentially sensitive to uncertainty in the analog plant at these frequencies. This fact is significant in that uncertainty in the plant model generally increases at higher frequencies. Hence if a strong dependence upon high frequency plant behavior is required to shift a zero, then one might suspect that the sensitivity and robustness of the resulting design would be poor. We now propose another *Gedanken* experiment whose result should serve to clarify this issue.

***Gedanken* Experiment No.2**: Suppose that we wish to design a digital controller for an analog plant. Then the following two questions (among others) are of interest:

(D1) Is the nominal response of the discrete system satisfactory?

(D2) Is the *discrete* response insensitive to uncertainty in the analog plant?

Clearly, it is desirable that the answers to both questions be affirmative. *The proposed experiment is to determine whether affirmative answers to both of these questions can be obtained simultaneously.*

By way of contrast with *Gedanken* Experiment No.1, we are now concerned solely with the response of the system *at the sampling instants* and are imposing no requirement that the nominal or robust intersample behavior be satisfactory. The only requirement concerning the analog system is that the behavior at sampling instants be robust against uncertainty in the analog plant.

Let us now consider the problems of achieving small differential sensitivity and robust stability against linear time invariant uncertainty in the *analog* plant. As the source of uncertainty is modeling error in the analog plant, these problems are more interesting from an engineering standpoint than are their discrete counterparts.

Motivated by the discussion surrounding (4.1)-(4.2), we shall consider separately uncertainty in the two terms on the right hand side of (4.1). In particular since uncertainty in the analog plant tends to increase with frequency, it follows that for $\omega \in \Omega_N$ uncertainty in the term $F(j\omega)P(j\omega)H(j\omega)$ will tend to be dominated by uncertainty in the term $\Delta_d(j\omega)$ due to the high frequency aliases.

To state the desired formulas for differential sensitivity and stability robustness, we shall need the following definitions.

Definition 4.1 (High gain sensitivity and complementary sensitivity functions): *In the limit as $|C_d(e^{sT})| \to \infty$, $S_{fun}(s) \to S_{HG}(s)$ and $T_{fun}(s) \to T_{HG}(s)$, where*

$$S_{HG}(s) \triangleq \begin{cases} \dfrac{\left(\frac{1}{T}\Delta_d(s)\right)}{(FPH)_d(e^{sT})}, & (FPH)_d(e^{sT}) \neq 0 \\ 1, & (FPH)_d(e^{sT}) = 0 \end{cases} \tag{4.3}$$

and

$$T_{HG}(s) \triangleq 1 - S_{HG}(s). \tag{4.4}$$

□

At a given frequency ω, not necessarily in the baseband, $S_{HG}(j\omega)$ is a measure of the contribution of aliases from other frequencies to the frequency response of the discretized plant. By (4.1), it is therefore a measure of the difference between the analog and discrete responses. Further design interpretations are presented in Section 3 of [10]. For now, we use $S_{HG}(s)$ in the following result.

Proposition 4.2: *At each value of s*

$$S_{fid}(s) = S_{HG}(s)T_d(e^{sT}) \tag{4.5}$$

Proof: It is straightforward to show that the *relative difference* between the discrete command response and the fundamental component of the analog response is given by

$$\frac{T_{fun}(s) - T_d(e^{sT})}{T_d(e^{sT})} = -S_{HG}(s). \tag{4.6}$$

Together, (3.5) and (4.6) yield the desired result. □

We now use $S_{fid}(s), S_{HG}(s)$, and $T_{HG}(s)$ to describe the relative differential sensitivity of the discrete command and control response to uncertainty in the analog plant. (Recall that the discrete control response is governed by the transfer function $C_d(z)S_d(z)$.) For purposes of comparison, we first state the corresponding result for uncertainty in the *discretized* plant.

Lemma 4.3: (Differential sensitivity to the discretized plant): *For each* $\omega \in \Omega_N$,

$$\frac{(FPH)_d(e^{j\omega T})}{T_d(e^{j\omega T})}\frac{\partial T_d(e^{j\omega T})}{\partial (FPH)_d(e^{j\omega T})} = S_d(e^{j\omega T}) \tag{4.7}$$

and

$$\frac{(FPH)_d(e^{j\omega T})}{C_d(e^{j\omega T})S_d(e^{j\omega T})}\frac{\partial C_d(e^{j\omega T})S_d(e^{j\omega T})}{\partial (FPH)_d(e^{j\omega T})} = -T_d(e^{j\omega T}). \tag{4.8}$$

Proof: Straightforward calculation. □

Whether $S_d(e^{j\omega T})$ and $T_d(e^{j\omega T})$ also correctly describe sensitivity with respect to analog plant variations depends upon how closely the discrete and analog responses approximate one another.

Proposition 4.4 (Differential sensitivity to the analog plant): *For each* $\omega \in \Omega_N$ *and* $k = 0, \pm 1, \pm 2, \ldots$

$$\begin{aligned}
\frac{P\left(j(\omega + k\omega_s)\right)}{T_d(e^{j\omega T})}\frac{\partial T_d(e^{j\omega T})}{\partial P\left(j(\omega + k\omega_s)\right)} &= T_{HG}\left(j(\omega + k\omega_s)\right) S_d(e^{j\omega T}) && (4.9)\\
\frac{\Delta_d(j\omega)}{T_d(e^{j\omega T})}\frac{\partial T_d(e^{j\omega T})}{\partial \Delta_d(e^{j\omega T})} &= S_{HG}(j\omega)S_d(e^{j\omega T}) && (4.10)\\
\frac{P\left(j(\omega + k\omega_s)\right)}{C_d(e^{j\omega T})S_d(e^{j\omega T})}\frac{\partial C_d(e^{j\omega T})S_d(e^{j\omega T})}{\partial P\left(j(\omega + k\omega_s)\right)} &= -T_{HG}(j\omega)T_d(e^{j\omega T}) \\
&= -T_{fun}(j\omega) && (4.11)\\
\frac{\Delta_d(j\omega)}{C_d(e^{j\omega T})S_d(e^{j\omega T})}\frac{\partial C_d(e^{j\omega T})S_d(e^{j\omega T})}{\partial \Delta_d(j\omega)} &= -S_{HG}(j\omega)T_d(e^{j\omega T}) \\
&= -S_{fid}(j\omega). && (4.12)
\end{aligned}$$

Proof: Straightforward calculation. □

Consider a frequency $\omega \in \Omega_N$ and suppose that $S_{HG} \simeq 0$ and $T_{HG}(j\omega) \simeq 1$, so that the *net* contribution of high frequency aliases to the discretized response is relatively small (*cf.* (4.3)). It then follows from (4.9) and (4.11) that $S_d(e^{j\omega T})$ and $T_d(e^{j\omega T})$ accurately describe sensitivity to baseband ($k = 0$) variations in the analog plant.

Furthermore, from (4.10) and (4.12) it follows that sensitivity to the *net* contribution of the high frequency aliases will be small. On the other hand, as $\omega \to \infty, T_{HG}(j\omega) \to 0$ and thus sensitivity to variations in individual components of the high frequency plant response ($k \neq 0$) will also become small. It is important to note, however, that large peaks in $S_{HG}(j\omega)$ and $T_{HG}(j\omega)$ will cause sensitivity to analog variations to be much worse than that to discrete variations. Finally, note that differential sensitivity of the control response is governed by $T_{fun}(s)$ and $S_{fid}(s)$. As we now show, these functions are also related to stability robustness.

Consider uncertainty in the discretized plant due to analog plant uncertainty of the form

$$P'(s) \triangleq P(s)\left(1 + W(s)\Delta(s)\right), \tag{4.13}$$

where $\Delta(s)$ is stable and proper and $W(s)$ is a stable, minimum phase weighting function used to represent frequency dependence of the modeling error. A method for calculating the stability margin of a sampled-data system to uncertainty of the form (4.13) is presented in [5]; however, we shall not need this framework for the present analysis. Instead, we cite the following result from [10].

Lemma 4.5 (Theorem 4.1, [10]): *Suppose that the feedback system of Figure 1 is nominally stable. Assume that $W(s)P(s)F(s)$ is proper. Then a* necessary *condition for the system to remain stable for all $\Delta(s)$ satisfying*

$$|\Delta(j\omega)| < 1\ ,\ \forall \omega \in \mathbb{R}, \tag{4.14}$$

is that

$$|W(j\omega)T_{fun}(j\omega)| \leq 1\ ,\ \forall \omega \in \mathbb{R}. \tag{4.15}$$

□

Typically, $|W(j\omega)|$ becomes unbounded at high frequencies, and so it is necessary that $|T_{fun}(j\omega)| \to 0$ sufficiently rapidly as $\omega \to \infty$. An inspection of the proof of this result in [10] reveals that the effects of aliases in (4.1) are ignored in deriving (4.15). We now develop a stronger necessary condition that does take aliases into account.

Let the uncertainty in the discrete plant due to (4.13) be written as

$$\begin{aligned}(FPH)'_d(e^{sT}) &= (\frac{1}{T})F(s)P(s)H(s)\left(1 + W_1(s)\Delta_1(s)\right) \\ &+ (\frac{1}{T})\Delta_d(s)\left(1 + W_2(s)\Delta_2(s)\right),\end{aligned} \tag{4.16}$$

where

$$W_1(s)\Delta_1(s) = W(s)\Delta(s) \tag{4.17}$$

and

$$\Delta_d(s)W_2(s)\Delta_2(s) = \sum_{k\neq 0} F(s+jk\omega_s)P(s+jk\omega_s)W(s+jk\omega_s)\Delta(s+jk\omega_s)H(s+jk\omega_s). \tag{4.18}$$

Theorem 4.6: *Assume that the feedback system in Figure 1 is nominally stable, and that*

$$|\Delta_i(j\omega)| < 1\ ,\ \forall \omega \in \Omega_N, i = 1, 2. \tag{4.19}$$

Then a sufficient *condition for the system to remain stable is that*

$$|W_1(j\omega)T_{fun}(j\omega)| + |W_2(j\omega)S_{fid}(j\omega)| \leq 1, \forall \omega \in \Omega_N. \tag{4.20}$$

If the functions $\Delta_i(j\omega), i = 1, 2$, are arbitrary *save for the bound (4.19), then condition (4.20) is also* necessary.

Proof: Substituting (4.16) into the definition of the perturbed sensitivity function and rearranging yields

$$S'_d(e^{sT}) = \frac{S_d(e^{sT})}{1 + W_1(s)T_{fun}(s)\Delta_1(s) + W_2(s)S_{fid}(s)\Delta_2(s)}, \tag{4.21}$$

from which the desired results follow. □

In point of fact, $\Delta_1(s)$ and $\Delta_2(s)$ are not arbitrary, but are related to the analog uncertainty via (4.17)-(4.18).

Corollary 4.7: *Assume that $\Delta(s)$ in (4.13) is arbitrary save for the bound (4.14). Define*

$$\underline{w} = \inf_{\omega \notin \Omega_N} |W(j\omega)| \tag{4.22}$$

Then a necessary *condition for robust stability is that*

$$|W(j\omega)T_{fun}(j\omega)| + \underline{w}|S_{fid}(j\omega)| \leq 1, \ , \ \forall \omega \in \Omega_N. \tag{4.23}$$

□

Since relative uncertainty in the analog plant (4.13) typically becomes large at high frequencies, and since the Nyquist frequency is usually chosen to be around 5 times the desired closed loop bandwidth, it is not unreasonable to assume that $\underline{w}$ in (4.22) is greater than 1. Hence (4.23) requires that $|S_{fid}(j\omega)| < 1$ over the baseband. This fact is significant since, as we shall see in the next section, $S_{fid}(s)$ must satisfy a Poisson integral relation.

5 Interpolation Constraints and an Integral Relation

We now develop a set of interpolation constraints and an integral relation that must be satisfied by the fidelity function, $S_{fid}(s)$. We first require an additional assumption that will hold generically.

Assumption 5.1: *If ξ is a CRHP zero of $P(s)$ or $H(s)$, then $e^{\xi T}$ is* not *a zero of $(FPH)_d(z)$.* □

Proposition 5.2 (Interpolation Constraints): *Suppose that the sampled-data feedback system is stable, and that $P(s), F(s), H(s)$, and $C_d(z)$ satisfy all assumptions stated in Section 2 as well as Assumption 5.1. Then the following conditions are satisfied:*

(i). Let ζ be a CRHP zero of $P(s)$. Then

$$S_{fid}(\zeta) = T_d(e^{\zeta T}). \tag{5.1}$$

(ii). Let γ be a CRHP zero of $H(s)$. Then

$$S_{fid}(\gamma) = T_d(e^{\gamma T}). \tag{5.2}$$

(iii). Let a be a zero of $C_d(z)$ with $a \in D^c$. Define

$$a_k \triangleq \frac{1}{T}\log(a) + jk\omega_s \ \ , k = 0, \pm 1, \pm 2, \ldots. \tag{5.3}$$

Then

$$S_{fid}(a_k) = 0 \ \ , \forall k. \tag{5.4}$$

(iv). Let p be a CRHP pole of $P(s)$. Define

$$p_k \triangleq p + jk\omega_s \ \ , k = \pm 1, \pm 2, \ldots. \tag{5.5}$$

Then

$$S_{fid}(p) = 0 \ \ and \ \ S_{fid}(p_k) = 1 \tag{5.6}$$

(v). Let δ be a CRHP zero of $\Delta_d(s)$. Then

$$S_{fid}(\delta) = 0. \tag{5.7}$$

(vi). $S_{fid}(s)$ has no CRHP zeros other than those given in (iii), (iv), and (v).

Proof: Conditions (i) and (ii) follow from Assumption 5.1, and the identity

$$S_{fid}(s) = \frac{\frac{1}{T}\Delta_d(s)}{(FPH)_d(e^{sT})} T_d(e^{sT}) \tag{5.8}$$

Condition (iii) follows from the identity

$$S_{fid}(s) = \frac{1}{T}\Delta_d(s) S_d(e^{sT}) C_d(e^{sT}). \tag{5.9}$$

Condition (iv) follows from (5.8). Condition (v) follows from (5.9). Finally, (5.9) shows that the zeros of $S_{fid}(s)$ are restricted to those of $\Delta_d(s), S_d(e^{sT})$ and $C_d(e^{sT})$, and condition (vi) follows. $\square$

Denote the NMP zeros of $P(s)$ by

$$\{\zeta_i; i = 1, \ldots, N_\zeta\}, \tag{5.10}$$

the NMP zeros of $H(s)$ by

$$\{\gamma_i; i = 1, \ldots, N_\gamma\}, \tag{5.11}$$

the NMP zeros of $C_d(s)$ by

$$\{a_i; i = 1, \ldots, N_a\}, \tag{5.12}$$

the NMP zeros of $\Delta_d(s)$ by

$$\{\delta_i; i = 1, \ldots, N_\delta\}, \tag{5.13}$$

the ORHP poles of $P(s)$ by

$$\{p_i; i = 1, \ldots, N_p\}, \tag{5.14}$$

and the corresponding poles of $(FPH)_d(z)$ by

$$\{\delta_i; i = 1, \ldots, N_p\}. \tag{5.15}$$

Define associated Blaschke products

$$B_a(s) \triangleq \prod_{i=1}^{N_a} \prod_{k=-\infty}^{\infty} \frac{a_i - s - jk\omega_s}{\bar{a}_i + s + jk\omega_s} \tag{5.16}$$

$$= \prod_{i=1}^{N_a} \frac{\sinh\left((a_i - s)\frac{T}{2}\right)}{\sinh\left((\bar{a}_i + s)\frac{T}{2}\right)} \tag{5.17}$$

$$B_p(s) \triangleq \prod_{i=1}^{N_p} \frac{p_i - s}{\bar{p}_i + s} \tag{5.18}$$

$$B_\rho(s) \triangleq \prod_{i=1}^{N_p} \frac{\sinh\left((p_i - s)\frac{T}{2}\right)}{\sinh\left((\bar{p}_i + s)\frac{T}{2}\right)} \tag{5.19}$$

$$B_\delta(s) \triangleq \prod_{i=1}^{N_\delta} \frac{\delta_i - s}{\bar{\delta} + 2} \tag{5.20}$$

Finally, note that we may factor

$$\Delta_d(s) = \Delta_m(s) B_\delta(s) e^{-s\tau_\Delta} \tag{5.21}$$

for some $\tau_\Delta \geq 0$, where $\Delta_m(s)$ satisfies the Poisson integral relation [11].

Using these definitions yields

Theorem 5.3 *Let $\xi = x + jy$ equal one of (5.10)-(5.11). Then*

$$\begin{aligned} \int_0^\infty \log |S_{fid}(j\omega)| W(\xi,\omega) d\omega \quad &= \quad \pi x (N_c T + \tau_\Delta) \\ &+ \quad \pi \log \left| B_a^{-1}(\xi) \right| \\ &+ \quad \pi \log \left| B_p^{-1}(\xi) \right| \\ &+ \quad \pi \log \left| B_\delta^{-1}(\xi) \right| \\ &+ \quad \pi \log \left| T_d(e^{\xi T}) \right| \end{aligned} \tag{5.22}$$

where $W(\xi, \omega)$ is given by (2.12).

Proof: Note that $S_{fid}(s) = S_m(s) B_a(s) B_p(s) B_\rho(s) B_\delta(s) e^{-s\tau_\Delta} e^{-sN_cT}$, where $S_m(s)$ satisfies the Poisson integral relation [11]. This fact, together with the identities $|S_{fid}(j\omega)| = |S_m(j\omega)|$ and $S_{fid}(\xi) = T_d(\xi)$, yield the desired results. $\square$

The integral (5.22) imposes a constraint upon values of $|S_{fid}(j\omega)|$. Analysis of design implications is deferred to the next section.

6 Result of *Gedanken* Experiment No.2

Suppose that the analog plant has at least one NMP zero and is subject to large modeling uncertainty at high frequencies. Using the Poisson integrals $S_d(e^{j\omega T})$ and $S_{fid}(j\omega)$ ((2.11) and (5.22) respectively) we now show that there exists a limitation upon the ability of a sampled-data feedback system to satisfy, with affirmative answers, questions (D1) and (D2) of the second *Gedanken* experiment. Specifically, we shall show that there exists a tradeoff between (i) requiring high performance in the discrete system and (ii) stability robustness with respect to the analog plant uncertainty. The severity of this tradeoff is determined by the relative location of the analog NMP zero, and is *independent of whether or not the discretized plant is minimum phase.* Furthermore, unlike the limitations revealed in *Gedanken* experiment No.1, the tradeoff exists *even if no performance requirements are imposed upon intersample behavior.*

We shall state performance requirements in terms of bounds upon the discrete sensitivity function:

$$|S_d(e^{j\omega T})| \quad \leq \quad \beta, \qquad \forall \omega \in \Omega_\epsilon \triangleq [0, \epsilon\omega_N) \tag{6.1}$$

$$|S_d(e^{j\omega T})| \quad \leq \quad \gamma, \qquad \epsilon\omega_N < \omega \leq \omega_N,\ \epsilon < 1, \tag{6.2}$$

where $\beta < 1$ and γ satisfies the lower bound (2.28).

If the discretized plant is minimum phase, then any such specification can be achieved. Note that the bounds (6.1) - (6.2) imply that $|T_d(e^{j\omega T})|$ is also bounded. We now show that requiring $S_d(e^{j\omega T})$ to satisfy bounds such as (6.1) - (6.2) imposes constraints upon the value of $T_d(e^{sT})$ off the unit circle.

Lemma 6.1: *Assume that $S_d(z)$ is stable and that $S_d(e^{j\omega T})$ satisfies the bounds (6.1) - (6.2). Consider $\nu = e^{\xi T}$, where $\xi \in$ ORHP. Then*

$$|T_d(e^{\xi T})| \geq F_T(\xi, \beta, \gamma, \epsilon), \tag{6.3}$$

where

$$F_T(\xi, \beta, \gamma, \epsilon) \triangleq 1 - \beta^{\frac{\Theta_d(\xi, \Omega_\epsilon)}{\pi}} \gamma^{\frac{\pi - \Theta_d(\xi, \Omega_\epsilon)}{\pi}} \tag{6.4}$$

Proof: It follows from Corollary 2.4 that

$$|T_d(e^{\xi T}) - 1| \leq \beta^{\frac{\Theta_d(\xi, \Omega_\epsilon)}{\pi}} \gamma^{\frac{\pi - \Theta_d(\xi, \Omega_\epsilon)}{\pi}} |B_\xi(e^{\xi T})|, \tag{6.5}$$

from which the result follows. □

This result shows that if the specification imposed upon $|S_d(e^{j\omega T})|$ is very stringent (*i.e.* if $\beta \ll 1$) then the value of $T_d(e^{\xi T})$ will nearly equal one. Proximity to one is determined by the relative location of the zero with respect to the interval Ω_ϵ. This point is illustrated for a real zero, $\xi = x$, in Figure 4, wherein we plot $F_T(\xi, \beta, \gamma, \epsilon)$ *vs.* the ratio $x/\epsilon\omega_N$ for $\gamma = 2.0$ and various values of β. For each β, the lower bound on $\left|T_d(e^{xT})\right|$ increases monotonically as the zero location decreases relative to the frequency interval Ω_ϵ.

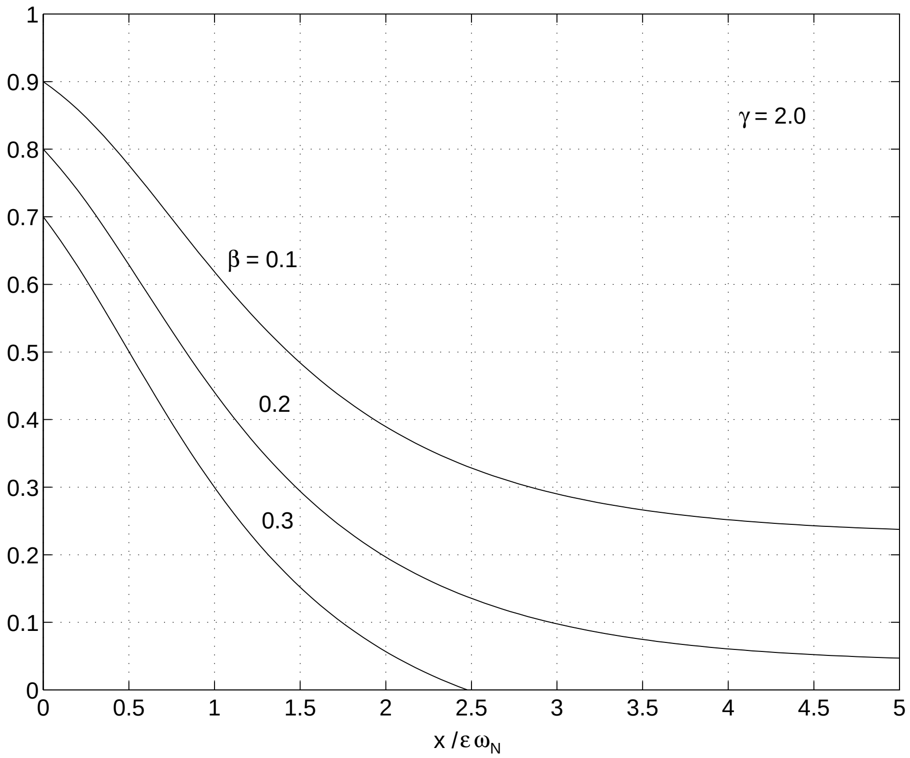

Figure 4: Lower Bound on $|T_d(e^{\xi T})|$

We shall suppose that the analog plant is subject to uncertainty of the form (4.13), where $\Delta(s)$ is arbitrary save for the bound (4.14), and $|W(j\omega)| \to \infty$ as $\omega \to \infty$. Let $\underline{w}$ be given by (4.22). It follows from Lemma 4.5 and Corollary 4.7 that *necessary* conditions for robust stability are that

$$|S_{fid}(j\omega)| < \frac{1}{\underline{w}}, \quad \forall \omega \in \Omega_N \tag{6.6}$$

and

$$|T_{fun}(j\omega)| < \frac{1}{\underline{w}}, \quad \forall \omega \notin \Omega_N \tag{6.7}$$

Our next result is an immediate corollary to Theorem 5.3.

Corollary 6.2: *Assume that the feedback system of Figure1 is stable. Let ζ denote a NMP zero of $P(s)$. Suppose that $S_{fid}(s)$ satisfies the bound (6.6). Then*

$$\sup_{\omega > \omega_N} |S_{fid}(j\omega)| \geq F_S(\underline{w}, \zeta)|T_d(e^{\zeta T})|^{\frac{\pi}{\pi - \Theta(\zeta, \omega_N)}} \tag{6.8}$$

where

$$F_S(\underline{w}, \zeta) \triangleq (\underline{w})^{\frac{\Theta(\zeta, \omega_N)}{\pi - \Theta(\zeta, \omega_N)}}. \tag{6.9}$$

Furthermore,

$$\sup_{\omega > \omega_N} |T_{fun}(j\omega)| \geq F_S(\underline{w}, \zeta)|T_d(e^{\zeta T})|^{\frac{\pi}{\pi - \Theta(\zeta, \omega_N)}} + \left||S_d(e^{j\omega T})| - 1\right| \tag{6.10}$$

Proof: The first four terms on the right hand side of (5.22) are nonnegative, and thus may be ignored. The bound (6.8) follows by imposing (6.6), exponentiating both sides, and rearranging the result. The bound (6.10) follows from (3.5). □

Consider Figure 5, which contains plots of $F_S(\underline{w}, \zeta)$ *vs.* the location of a real zero, $\zeta = X$, for various values of $\underline{w}$. These plots show that, for a fixed value of $\underline{w}$, $F(\underline{w}, \zeta)$ is large when ζ is at a relatively low frequency with respect to ω_N. In this case $|S_{fid}(j\omega)|$ will have a large peak outside the baseband *unless* the value of $|T_d(e^{\zeta T})|$ is sufficiently small. If $|S_d(e^{j\omega T})|$ satisfies bounds as (6.1)-(6.2), then (6.10) shows that there exists a corresponding peak in $|T_{fun}(j\omega)|$. The latter peak will imply, in turn, that (6.7) will be violated, and thus that the system will not be robustly stable. These remarks imply that to satisfy the stability robustness requirements (6.6)-(6.7) it is *necessary* that $|T_d(e^{\zeta T})|$ be small. However, as Lemma 6.1 shows, requiring $|T_d(e^{\zeta T})|$ to be small imposes a limitation upon the response of the discretized system *even if the discrete plant is minimum phase.* It follows that design limitations due to analog NMP zeros *must* be taken into account in the design of the discretized feedback system even if no performance requirement is imposed upon the intersample behaviour. Neglecting such limitations will lead to poor robustness properties against uncertainty in the analog plant.

We summarize by combining Lemma 6.1 and Corollary 6.2 to obtain the following result.

Corollary 6.3: *Assume that the feedback system of Figure 2.1 is stable. Let ζ denote a NMP zero of $P(s)$. Suppose that $S_d(e^{sT})$ satisfies the bounds (6.1)-(6.2) and that $S_{fid}(s)$ satisfies the bound (6.6). Then necessarily*

$$\sup_{\omega > \omega_N} |S_{fid}(j\omega)| \geq F_S(\underline{w}, \zeta) F_T(\zeta, \beta, \gamma, \epsilon)$$

and

$$\sup_{\omega > \omega_N} |T_{fun}(j\omega)| \geq F_S(\underline{w}, \zeta) F_T(\zeta, \beta, \gamma, \epsilon) + \gamma + 1$$

To illustrate, Figure 6 contains plots of $F_S(\underline{w}, \zeta)F_T(\zeta, \beta, \gamma\epsilon)$ versus $\zeta = x$ for $\epsilon = 0.1, \beta = 0.1, \gamma = 2$, and various values of $\underline{w}$. In all cases, the bound is large if the zero is at a relatively low frequency and becomes vacuous if the zero is at a relatively high frequency.

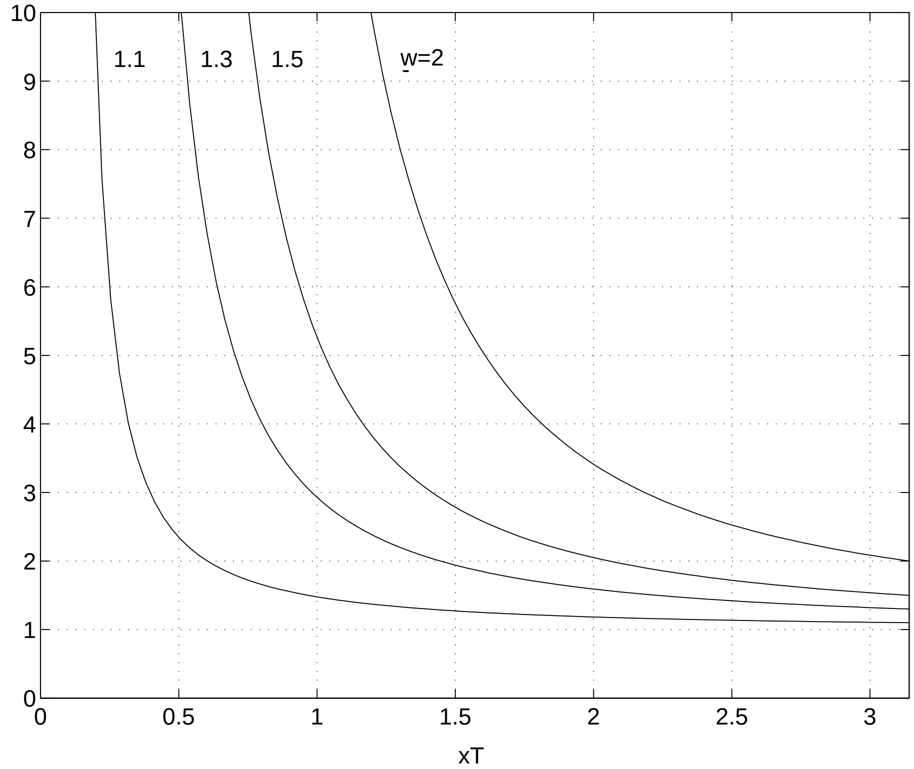

Figure 5: Lower bound Figure S_{fid} without T_d

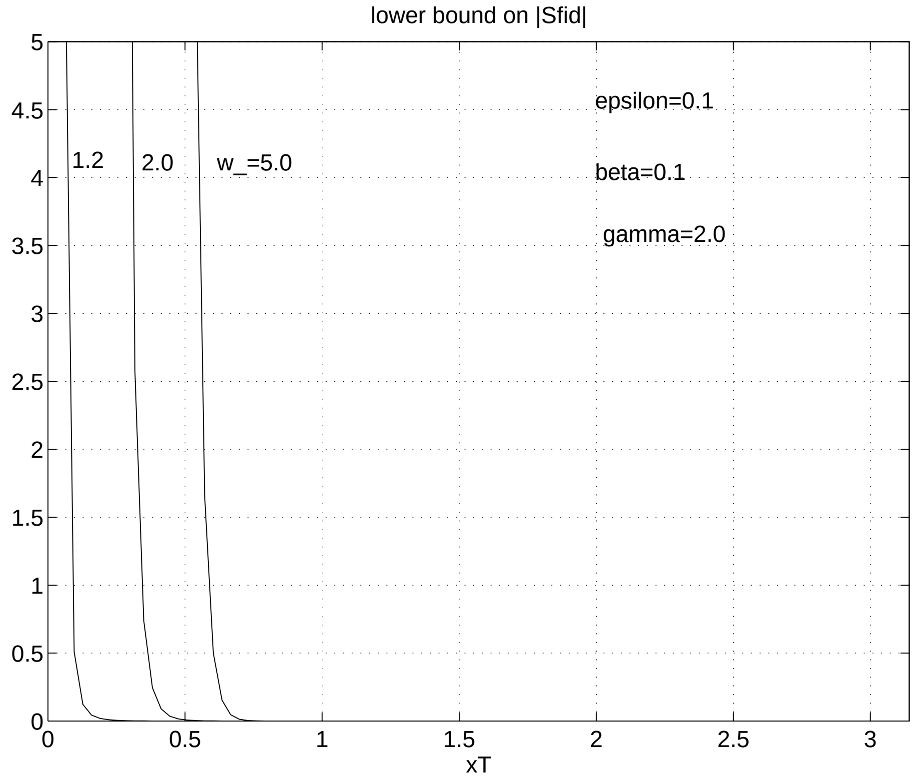

Figure 6: $F_S(\underline{w}, \zeta) F_T(\zeta, \beta, \gamma\epsilon)$ *vs.* $\zeta = x$

References

[1] M. Araki. Recent developments in digital control theory. In *Proceedings of the 12th World Congress of the IFAC*, pages IX–251, 1993.

[2] K. J. Åström and B. Wittenmark. *Computer-Controlled Systems: Theory and Design*. Prentice Hall, Englewood Cliffs NY, 2nd edition, 1990.

[3] E.-W. Bai and S. Dasgupta. A note on generalized hold functions. *Systems and Control Letters*, 14:363–368, 1990.

[4] T. Chen and B. A. Francis. Input-output stability of sampled-data systems. *IEEE Transactions on Automatic Control*, AC-36(1):50–58, January 1991.

[5] G. Dullerud and K. Glover. Robust stabilization of sampled-data systems to structured lti perturbations. *IEEE Transactions on Automatic Control*, AC-38(10):1497–1508, October 1993.

[6] A. Feuer and G. C. Goodwin. Generalized sample hold functions: Facts and fallacies. In *1992 CDC*, Tucson, AZ, 1992. IEEE.

[7] B. A. Francis and T. T. Georgiou. Stability theory for linear time-invariant plants with periodic digital controllers. *IEEE Transactions on Automatic Control*, AC-33(9):820–832, January 1988.

[8] J. S. Freudenberg and D. P. Looze. Right half plane poles and zeros and design tradeoffs in feedback systems. *IEEE Transactions on Automatic Control*, AC-30(6):555–565, June 1985.

[9] J. S. Freudenberg and D. P. Looze. *Frequency Domain Properties of Scalar and Multivariable Feedback Systems*, volume 104 of *Lecture Notes in Control and Information Sciences*. Srpinger-Verlag, 1988.

[10] J. S. Freudenberg, R. H. Middleton, and J. H. Braslavsky. Inherent design limitations for linear sampled-data feedback systems. Technical Report EE9347, Department of Electrical and Computer Engineering, The University of Newcastle, Australia, October 1993. To appear in the International J. of Control.

[11] K. Hoffman. *Banach spaces of analytic functions*. Dover, New York, 1962.

[12] P. T. Kabamba. Control of linear systems using generalized sampled-data hold functions. *IEEE Transactions on Automatic Control*, pages 772–783, September 1987.

[13] R. H. Middleton. Trade-offs in linear control systems design. *Automatica*, 27(2):281–292, March 1991.

[14] R. H. Middleton and J. S. Freudenberg. Zeros and non-pathological sampling for generalised sampled-data hold functions. Technical Report EE9318, Department of Electrical and Computer Engineering, The University of Newcastle, Australia, May 1993.

[15] H.-K. Sung and S. Hara. Properties of sensitivity and complementary sensitivity functions in single-input single-output digital control systems. *International J. of Control*, 48(6):2429–2439, 1988.

[16] W. Yan, B.D.O. Anderson, and R.R. Bitmead. On the gain margin improvement using dynamic compensation based on generalized sampled-data hold functions. Technical report, Department of Systems Engineering, Australian National University, Canberra, Australia, 1989.