Sensitivity and Robustness of Sampled-data Control Systems: A frequency Domain Viewpoint[*][†]

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Abstract

In this paper we consider a frequency domain, input-output based approach to the analysis of hybrid sampled-data systems. Based on previous works, we expound a *frequency-domain lifting* technique as an alternative to the *time-domain* lifting framework for sampled-data systems. In some cases, the approach pursued here can provide simpler, more intuitive methods than time-domain lifting techniques to the analysis of the same problems. As applications, we consider the induced L_2 norms of the hybrid Sensitivity and Complementary Sensitivity Operators, and the stability robustness of the hybrid system to linear time-invariant perturbations. In particular, for the SISO case, "closed form" expressions arise for the induced norms, and the conditions for robust stability with multiplicative uncertainty reduce to a rank-one structured singular value test.

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1 Introduction.

A nice feature of linear time-invariant (LTI) systems is that an input-output operator can be represented as a multiplicative operator in the frequency domain, namely, a transfer function. This frequency domain approach has proven very valuable for analyzing sensitivity and robustness issues for analog control systems (*e.g.* Horowitz 1963, Freudenberg and Looze 1985, Doyle *et al.* 1992, Packard and Doyle 1993). However, this framework is hard to extend to the case where the controller is implemented digitally through sample and hold devices, as the system becomes periodically time-varying. This fact imposes a standing difficulty in generalizing concepts and techniques associated with frequency-response and H_∞ methods to sampled-data systems.

The periodic characteristics of hybrid analog/sampled-data systems allow us still to resort to transfer functions as input-output models, as long as we are only concerned with the sampled behavior. Intersample information is lost or hidden when using these models. This fact may be critical in the analysis of robustness and sensitivity, since *analog* uncertainties, disturbances and noise are frequently the issues of practical significance.

In view of this, several approaches to deal with continuous-time hybrid systems have been recently introduced.

One framework to handle hybrid systems is the so-called *lifting* technique (Bamieh *et al.* 1991, Bamieh and Pearson 1992, Toivonen 1992, Yamamoto 1993). The lifting technique transforms the sampled-data system into a discrete time-invariant equivalent system acting over infinite-dimensional signals. Time-invariance comes as a consequence of periodicity, but in contrast to the classical pure discrete approach, intersample behavior is built in the model, which is reflected in the infinite dimensionality of the transformed signals. Sampled-data H_∞-norm computation and optimization (Bamieh and Pearson 1992, Toivonen 1992, Kabamba and Hara 1993, Yamamoto 1993), robust stabilization to LTI perturbations (Dullerud and Glover 1993), and tracking (Yamamoto 1994) are some recent results obtained *via* lifting.

Another approach that incorporates intersample information is based on a *linear model with jumps* (Sivashankar and Khargonekar 1991, Sun *et al.* 1993, Tadmor 1991). Linear systems with jumps are represented by a standard set of differential equations whose states are periodically updated by the states of some difference equations. This models have been used to solve H_∞ optimal control and filtering problems for sampled-data systems (Sun *et al.* 1993, Tadmor 1991).

A third method is a pure frequency-domain technique based on the steady state response of a sampled-data system to sinusoidal inputs (Goodwin and Salgado 1994, Araki and Ito 1993, Araki *et al.* 1993, Freudenberg *et al.* 1993). It is known that the response of a hybrid system to a single sinusoid contains infinitely many *harmonics* shifted from the fundamental by integer multiples of the sampling frequency. Each of these harmonics is governed by a frequency response function, with many similar properties to those of a transfer function. Performance, sensitivity integrals and design tradeoffs have been considered using this approach (Leung *et al.* 1991, Goodwin and Salgado 1994, Araki and Ito 1993, Feuer and Goodwin 1994, Freudenberg *et al.* 1993, Freudenberg *et al.* 1994).

In this paper we introduce a mathematical framework motivated by the harmonic structure of a hybrid system, which we call *frequency-domain lifting*. This framework arises as a natural equivalent to the *time-domain* lifting technique when the sampled-data system is approached directly with frequency-domain techniques. As applications, we show how induced L_2 norms, and robust stability for LTI perturbations may be analyzed using this frequency lifting in a rather simple and intuitive way. In some particular cases of interest, this technique yields "closed form" expressions that can be numerically implemented as MATLAB functions.

2 Preliminaries.

In this section we describe the system under study and revise a well-known formula relating the z-transform and the Laplace transform of a continuous-time signal. Motivated by this formula, we introduce a lifting transformation that arises as a natural mathematical framework for a frequency domain approach to hybrid systems.

2.1 Sampled-data system.

We consider the multivariable sampled-data system depicted in Figure 1, where $P(s)$ and $F(s)$ are the transfer matrices of the analog plant and antialiasing filter, and $C_d(z)$ is the tranfer matrix of the digital controller. The signals are as follows:

- r reference input,
- d output disturbance,
- n sensor noise,
- y plant output,
- u, u_k analog and discrete control signals,
- v, v_k analog and sampled filter outputs.

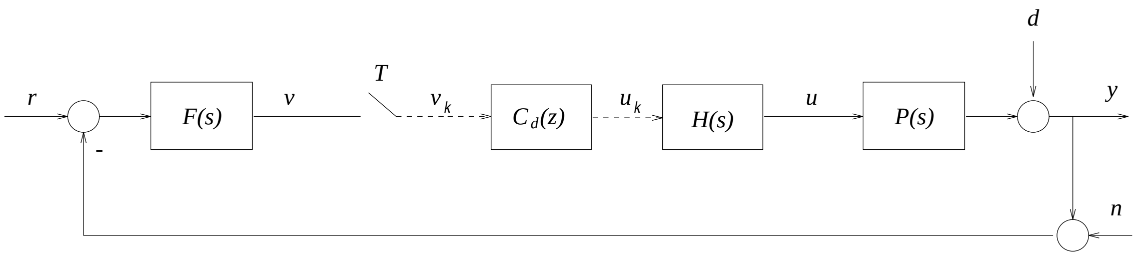

Figure 1: Sampled data feedback system considered.

Denote the sampling period by T, and the associated sampling and Nyquist frequencies by $w_s \triangleq 2\pi/T$ and $w_N \triangleq \pi/T$, respectively. We shall use the term *baseband* to denote the frequency range $\Omega_N \triangleq [-\omega_N, \omega_N]$. The sampling operation with period T is defined by

$$\mathcal{S}_T\, v = \{v_k\}_{k=-\infty}^{\infty},$$

where

$$v_k = v(kT), \quad kT \le t < (k+1)T, \quad k \in \mathbb{Z}.$$

The hold device $H(s)$ is a generalized sampled-data hold function (GSHF) (Kabamba 1987, Middleton and Freudenberg 1993), defined by

$$u(t) = h(t - kT)u_k, \quad kT \le t < (k+1)T, \quad k \in \mathbb{Z}.$$

The function $h(t)$, which characterizes the hold, is defined over the interval $[0, T]$, and satisfies some mild technical assumptions described in Freudenberg *et al.* (1993). We associate with this hold a *frequency response function*, given by the finite Laplace transform

$$H(s) = \int_0^T e^{-st}\, h(t)\, dt.$$

A rational transfer matrix of s (respectively z) is *stable* if it is analytic in the closed right-half plane (CRHP), $\mathrm{Re}\, s \geq 0$ (respectively outside the open unit disk, $|z| \geq 1$). We say that the rational transfer matrix G is *proper* if $\|G(j\infty)\|$ is finite, and *strictly proper* if $\|G(j\infty)\| = 0$, where $\|\cdot\|$ denotes the Euclidian norm for matrices. Given a stable, proper transfer matrix $G(s)$, we define $\|G\|_\infty \triangleq \sup_\omega \|G(j\omega)\|$.

Throughout the paper we shall assume that the transfer matrix of the analog plant can be written as $P(s) = P_0(s)e^{-s\tau}$, where $P_0(s)$ is rational and proper and $\tau \geq 0$, $F(s)$ is strictly proper, rational and stable, and $C_d(z)$ is rational and proper. The assumption that the filter is strictly proper is standard and guarantees that the sampling operation is well-defined (*cf.* Sivashankar and Khargonekar 1993, Dullerud and Glover 1993).

We now introduce a well-known result (*e.g.* Åström and Wittenmark 1990, Franklin *et al.* 1990) that describes the harmonic structure of the sampling process, and that will play a central role in our subsequent discussion. Denote by $\mathcal{Z}$ the z-transform of a sequence, $\mathcal{Z}\{u_k\} \triangleq \sum_{k=-\infty}^{\infty} u_k z^{-k}$, and by $\mathcal{L}$ the Laplace transform of a signal $\mathcal{L}\{u(t)\} = U(s)$. The following statement is taken from Freudenberg *et al.* (1993).

Lemma 2.1
Consider $G(s)$, the Laplace transform of the function $g(t)$ defined for $t \geq 0$. Let $x + jy$, $-\infty < y < \infty$, be a vertical line in the domain of convergence of $G(s)$. Assume that the following conditions are satisfied:

(i) $\lim_{t\to 0^+} g(t) = 0$

(ii) $g(t)$ is continuous in the neighborhood of $t = kT$, $k = 1, 2, \ldots$

(iii) the infinite sum $\sum_{k=-\infty}^{\infty} G(s + jk\omega_s)$ is uniformly convergent for $s = x + jy$, $y \in (-\omega_n, \omega_n)$.

Define

$$G_d(z) \triangleq \mathcal{Z}\{\mathcal{S}_T\{\mathcal{L}^{-1}\{G(s)\}\}\}.$$

Then

$$G_d(e^{sT}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G(s + jk\omega_s). \tag{2.1}$$

□

This result tells us how the sampled signal is build upon the repeated addition of shifted versions of the original signal, yielding a $j\omega_s$-periodic function. We shall call the original signal, say $G(s)$, the *fundamental*, and the aliases, $G(s + jk\omega_s)$, the *harmonics*. We shall use for $G(s + jk\omega_s)$ the abbreviated notation

$$G_k(s) \triangleq G(s + jk\omega_s)\ . \tag{2.2}$$

The assumptions on $P(s)$, $H(s)$ and $F(s)$ stated above suffice to guarantee that the *discretized plant*, $(FPH)_d(z) \triangleq \mathcal{Z}\{\mathcal{S}_T\{\mathcal{L}^{-1}\{F(s)P(s)H(s)\}\}\}$ satisfies the formula

$$(FPH)_d(e^{j\omega T}) \triangleq \frac{1}{T} \sum_{k=-\infty}^{\infty} H_k(j\omega)P_k(j\omega)F_k(j\omega). \tag{2.3}$$

Conditions i, ii and iii follow since F is strictly proper, P is proper, and the assumptions on H.

In particular, (2.3) shows that the response of the discretized plant at a frequency $\omega \in \Omega_N$ depends upon the response of the analog plant, prefilter, and hold function at infinitely many frequencies $\omega + k\omega_s, k = 0, \pm 1, \pm 2, \ldots$ Some implications of this fact for the response of the hybrid system to uncertainty, noise and disturbances have been comunicated in Freudenberg *et al.* (1994).

In (Middleton and Freudenberg 1993), the well-known nonpathological sampling condition for plants discretized with a ZOH is generalized to the case of a GSHF. In particular, since the hold response function of a GSHF may have zeros in the right half plane, it is necessary to require that none of these coincides with an unstable pole of the analog plant. Under the nonpathological sampling hypothesis, it is straightforward to extend the exponential and L_2 input-output stability results of (Chen and Francis 1991) and (Francis and Georgiou 1988) to the case of a GSHF. We shall assume throughout this paper that the system considered in Figure 1 is L_2 input-output stable.

In the next subsection, we introduce a transformation motivated by (2.3), which we call the *frequency-domain lifting.* Such transformation is a natural correlate to the *time-domain lifting* for sampled-data systems under a frequency domain formulation.

2.2 A frequency domain lifting.

A well-known feature of a sampled-data system is its multi-frequency response to a single sinusoidal input (Åström and Wittenmark 1990, Franklin *et al.* 1990). Based on this distinctive characteristic, frequency-domain approaches have been proposed to extend the ideas of sensitivity functions and frequency response to hybrid systems (Araki and Ito 1993, Araki *et al.* 1993, Goodwin and Salgado 1994).

We introduce here a lifting transformation that provides a mathematical framework for such frequency-domain approaches. The idea is to rearrange the way we represent signals to explicit the presence of the harmonics generated by the sampling process.

Consider a signal $Y(j\omega)$ in $L_2(-\infty, \infty)$, and define the following sequence of functions,

$$Y_k(j\omega) = Y(j(\omega + k\omega_s)), \tag{2.4}$$

for ω in the baseband Ω_N and k in $\mathbb{Z}$. We arrange the sequence in a vector, and we denote it by

$$\boldsymbol{y}(\omega) \triangleq \begin{bmatrix} \vdots \\ Y_1(j\omega) \\ Y_0(j\omega) \\ Y_{-1}(j\omega) \\ \vdots \end{bmatrix}. \tag{2.5}$$

The function $\boldsymbol{y}(\omega)$ is defined over Ω_N taking values in ℓ_2. Moreover, the space of such functions is a Hilbert space under the norm

$$\|\boldsymbol{y}\| \triangleq \left(\int_{\Omega_N} \|\boldsymbol{f}(\omega)\|_{\ell_2}^2 d\omega \right)^{1/2}, \tag{2.6}$$

and inner product

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle \triangleq \int_{\Omega_N} \langle \boldsymbol{f}(\omega), \boldsymbol{g}(\omega) \rangle_{\ell_2} \, d\omega \,. \tag{2.7}$$

We denote this space by $L_2(\Omega_N; \ell_2)$ (See Appendix A for a more precise description). Since the signals in $L_2(\Omega_N; \ell_2)$ are basically rearrangements of signals in $L_2(-\infty, \infty)$, it is not surprising that both spaces are essentially the same, as stated in the following result.

Lemma 2.2
The space $L_2(\Omega_N;\ell_2)$ is isometrically isomorphic to $L_2(-\infty,\infty)$.

Proof: See Appendix A. □

We saw how to construct a function in $L_2(\Omega_N;\ell_2)$ from the original signal in $L_2(-\infty,\infty)$. The result above tells us that *every* vector in $L_2(\Omega_N;\ell_2)$ corresponds to a frequency signal in $L_2(-\infty,\infty)$. We formalize this relationship by defining the *frequency domain lifting operator*, $\mathcal{F}$, mapping

$$\begin{aligned} \mathcal{F} : L_2(-\infty,\infty) &\rightarrow L_2(\Omega_N;\ell_2) \\ Y(j\omega) &\mapsto \boldsymbol{y}(\omega)\,. \end{aligned}$$

By Lemma 2.2, $\mathcal{F}$ is invertible, and moreover, $||\mathcal{F}|| = 1 = ||\mathcal{F}^{-1}||$. In particular, if $\mathcal{T}$ is a bounded linear operator mapping $L_2(-\infty,\infty) \rightarrow L_2(-\infty,\infty)$, then the lifted operator $\boldsymbol{\mathcal{T}} = \mathcal{F}\mathcal{T}\mathcal{F}^{-1}$ is a bounded linear operator mapping $L_2(\Omega_N;\ell_2) \rightarrow L_2(\Omega_N;\ell_2)$ with the same operator norm.

The advantage of the transformation is that, in the new space, operators of the hybrid system are represented as multiplicative operators described by infinite-dimensional transfer matrices. In other words, the action of $\mathcal{T}$ viewed in $L_2(\Omega_N;\ell_2)$ is given by $(\boldsymbol{\mathcal{T}}\boldsymbol{y})(\omega) = \boldsymbol{T}_\omega \boldsymbol{y}(\omega)$, where $\boldsymbol{T}_\omega$ is a bounded linear operator in ℓ_2 for each fixed ω in Ω_N. An important consequence of this fact is that the operator norm

$$||\mathcal{T}|| \triangleq \sup_{Y \in L_2} \left\{ \frac{||\mathcal{T}Y||_{L_2}}{||Y||_{L_2}} : Y \neq 0 \right\}, \tag{2.8}$$

can be computed as

$$||\boldsymbol{\mathcal{T}}|| = \sup_{\omega \in \Omega_N} ||\boldsymbol{T}_\omega||_{\ell_2}. \tag{2.9}$$

Notice the similarity of (2.9) to the familiar expression of the L_2-induced norm of an operator in a LTI system, *i.e.,* the H_∞-norm of a transfer function.

In the following sections, we show how this approach can be used to study two example problems, the computation of L_2-induced norms, and the derivation of necessary and sufficient conditions for robust stability for the hybrid system of Figure 1.

3 Induced L_2 norms.

In this section we study the L_2-induced norms of the Sensitivity and Complementary Sensitivity Operators, mapping $d(t)$ and $n(t)$ to $y(t)$, respectively, of the hybrid system of Figure 1. The norms are characterized in the frequency-lifted domain as the supremum over frequencies in the baseband of the maximum singular value of the associated infinity transfer matrices.

In the particular case of a scalar system, we compute closed forms for these expressions. By application of Lemma 2.1, we reduce the problem to the calculus of the H_∞ norm of rational discrete transfer functions. We finish with an illustrative numerical example.

3.1 Hybrid Sensitivity Operators

The steady-state response of a stable hybrid system to a sinusoidal input consists of a fundamental component and infinitely many aliases shifted by multiples of the sampling frequency (Åström and Wittenmark 1990, Franklin *et al.* 1990). Analogous expressions are obtained for the response to more general inputs, such as noise $n(t)$ and output disturbance $d(t)$ (Freudenberg *et al.* 1993, Goodwin and Salgado 1994). Suppose all the inputs in $L_2(-\infty,\infty)$. If $N(s)$ is the Laplace transform of the noise $n(t)$, then the corresponding response is expressed in frequency domain by

$$Y(j\omega) = -P(j\omega)H(j\omega)S_d(e^{j\omega T})C_d(e^{j\omega T})\,\frac{1}{T}\sum_{k=-\infty}^{\infty} F_k(\omega)N_k(\omega)\,, \tag{3.1}$$

where

$$S_d(z) \triangleq [I + C_d(z)(FPH)_d(z))]^{-1} \tag{3.2}$$

is the discrete Sensitivity Function. Similarly, for the response to a disturbance given by $D(s)$, we have

$$Y(j\omega) = D(j\omega) - P(j\omega)H(j\omega)S_d(e^{j\omega T})C_d(e^{j\omega T})\,\frac{1}{T}\sum_{k=-\infty}^{\infty} F_k(j\omega)D_k(\omega). \tag{3.3}$$

Equations (3.1) and (3.3) define, in frequency domain, the action of the Sensitivity and Complementary Sensitivity Operators, $\mathcal{S}$ and $\mathcal{T}$ respectively, of the hybrid system. Measuring the norm of these operators we can quantify important robustness properties of the system (Sivashankar and Khargonekar 1993, Kabamba and Hara 1993), and, in particular, the degradation of LTI designs under digital implementations (Leung *et al.* 1991, Chen and Francis 1990).

From the definition of frequency lifting in §2, it is straightforward to alternatively write (3.1) and (3.3) as

$$\boldsymbol{y} = -\boldsymbol{T}_\omega \boldsymbol{n}\,, \tag{3.4}$$

and

$$\boldsymbol{y} = \boldsymbol{S}_\omega \boldsymbol{n}\,, \tag{3.5}$$

where $\boldsymbol{T}_\omega$ and $\boldsymbol{S}_\omega$ are the following infinite-dimensional transfer matrices defined on Ω_N

$$\boldsymbol{T}_\omega = \begin{bmatrix} \ddots & \vdots & \vdots & \\ \cdots & G_k(j\omega)F_k(j\omega) & G_k(j\omega)F_{k-1}(j\omega) & \cdots \\ \cdots & G_{k-1}(j\omega)F_k(j\omega) & G_{k-1}(j\omega)F_{k-1}(j\omega) & \cdots \\ & \vdots & \vdots & \ddots \end{bmatrix}, \tag{3.6}$$

$$\boldsymbol{S}_\omega = \begin{bmatrix} \ddots & \vdots & \vdots & \\ \cdots & 1 - G_k(j\omega)F_k(j\omega) & -G_k(j\omega)F_{k-1}(j\omega) & \cdots \\ \cdots & -G_{k-1}(j\omega)F_k(j\omega) & 1 - G_{k-1}(j\omega)F_{k-1}(j\omega) & \cdots \\ & \vdots & \vdots & \ddots \end{bmatrix}, \tag{3.7}$$

with

$$G(j\omega) \triangleq \frac{1}{T} P(j\omega)H(j\omega)S_d(e^{j\omega T})C_d(e^{j\omega T}). \tag{3.8}$$

From (2.9), the norms of the operators $\mathcal{T}$ and $\mathcal{S}$, are then given by

$$\|\boldsymbol{\mathcal{T}}\| = \sup_{\omega \in \Omega_N} \|\boldsymbol{T}_\omega\|_{\ell_2} \quad \text{and} \quad \|\boldsymbol{\mathcal{S}}\| = \sup_{\omega \in \Omega_N} \|\boldsymbol{S}_\omega\|_{\ell_2}. \tag{3.9}$$

Similar expressions have been obtained in (Araki *et al.* 1993).

Remark 3.1 Note that the Complementary Sensitivity Operator always has finite rank, while the Sensitivity Operator always has full rank (infinite). This can be seen as follows: from Figure 1, if n_v and n_y are the dimensions of signals $v(t)$ and $y(t)$, respectively, then $F(s)$ takes values in $\mathbb{C}^{n_v \times n_y}$ and $G(s)$, from (3.8), in $\mathbb{C}^{n_y \times n_v}$. By assumption, $F(s)$ is stable and strictly proper, and so is $G(s)$, as any unstable pole of $P(s)$ or $C_d(e^{sT})$ will be cancelled by a corresponding zero of $S_d(e^{sT})$ and the hold response is an entire function. The strict causality of $G(s)$ follows from the hold response, which rolls off as $1/\omega$ on the $j\omega$-axis (Middleton and Freudenberg 1993). Then, both $F(\omega)$ and $G(j\omega)$ are in $L_2(-\infty, \infty)$, and we can write alternatively

$$\boldsymbol{T}_\omega \boldsymbol{n} = \sum_{k=1}^{n_y} \boldsymbol{g}_k \langle \boldsymbol{n}, \bar{\boldsymbol{f}}_k \rangle_{\ell_2},$$

where $\boldsymbol{g}_k(\omega)$ and $\boldsymbol{f}_k(\omega)$ are from the lifted $\mathcal{F}G\mathcal{F}^{-1} = [\boldsymbol{g}_1(\omega), \boldsymbol{g}_2(\omega), \ldots, \boldsymbol{g}_{n_y}(\omega)]$ and $\mathcal{F}F\mathcal{F}^{-1} = [\boldsymbol{f}_1(\omega), \boldsymbol{f}_2(\omega), \ldots, \boldsymbol{f}_{n_y}(\omega)]$, respectively. Hence $\boldsymbol{T}_\omega$ is the sum of n_y rank-one operators, and therefore it has at most rank n_y. On the other hand, notice that as $\boldsymbol{S}_\omega = \boldsymbol{I} - \boldsymbol{T}_\omega$, it will always have infinite rank since $\boldsymbol{T}_\omega$ is finite rank. □

The norms in (3.9) can be numerically computed by truncating the matrices at some harmonic, and evaluating the maximum singular value (Araki and Ito 1993). Note, however, that the convergence could be slow.

3.2 Closed form expressions: scalar case.

In the particular case of a scalar system, *i.e.*, n_y, $n_v = 1$, we can derive closed form expressions for the norms given by (3.9). The Complementary Sensitivity Operator $\mathcal{T}$ is then of rank one, and so the computation of the norms in (3.9) is reduced to a single-eigenvalue problem.

Using $G(s)$ defined in (3.8), we write (3.1) as

$$Y(j\omega) = -G(j\omega) \sum_{k=-\infty}^{\infty} F_k(j\omega)N_k(j\omega), \tag{3.10}$$

or, in lifted domain (*cf.* Remark 3.1),

$$\begin{aligned} \boldsymbol{y}(\omega) &= -\boldsymbol{T}_\omega \, \boldsymbol{n}(\omega) \\ &= -\boldsymbol{g}(\omega) \, \langle \boldsymbol{n}(\omega), \bar{\boldsymbol{f}}(\omega) \rangle_{\ell_2}. \end{aligned} \tag{3.11}$$

We then have the following result.

Theorem 3.1 (Induced L_2 norm of the Complementary Sensitivity Operator)
Assume the hybrid system in Figure 1 to be L_2-input-output stable. Then

$$\|\mathcal{T}\| = \sup_{\omega\in\Omega_N} \left(|C_d(e^{j\omega T})||S_d(e^{j\omega T}))| \sqrt{\left(|PH|^2\right)_d(e^{j\omega T}))} \sqrt{\left(|F|^2\right)_d(e^{j\omega T}))} \right) , \tag{3.12}$$

where

$$\left(|PH|^2\right)_d(e^{j\omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} |P_k(j\omega)H_k(j\omega)|^2, \tag{3.13}$$

and

$$\left(|F|^2\right)_d(e^{j\omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} |F_k(j\omega)|^2, \tag{3.14}$$

Proof: From (2.8) and (2.9), the induced L_2 norm of the Complementary Sensitivity Operator is computed as

$$\|\mathcal{T}\| = \sup_{\omega\in\Omega_N} \|\boldsymbol{T}_\omega\|_{\ell_2}, \tag{3.15}$$

where $\boldsymbol{T}_\omega$ is the infinite matrix defined as the "outer product" of vectors $\boldsymbol{g}(\omega)$ and $\bar{\boldsymbol{f}}(\omega)$ in (3.11). To simplify notation, we shall omit the dependence on ω in vectors in $L_2(\Omega_N;\ell_2)$, assuming a fixed value in Ω_N.

The ℓ_2 norm of $\boldsymbol{T}_\omega$ is the square root of the maximum eigenvalue of the associated self-adjoint operator $\boldsymbol{T}_\omega^*\boldsymbol{T}_\omega$. But applying $\boldsymbol{T}_\omega^*\boldsymbol{T}_\omega$ to $\bar{\boldsymbol{f}}$ yields

$$\begin{aligned} \boldsymbol{T}_\omega^*\boldsymbol{T}_\omega\bar{\boldsymbol{f}} &= \bar{\boldsymbol{f}}\langle\boldsymbol{g}\langle\bar{\boldsymbol{f}},\bar{\boldsymbol{f}}\rangle_{\ell_2},\boldsymbol{g}\rangle_{\ell_2} \\ &= \|\boldsymbol{g}\|_{\ell_2}^2\,\|\bar{\boldsymbol{f}}\|_{\ell_2}^2\,\bar{\boldsymbol{f}}\ . \end{aligned}$$

Hence $\bar{\boldsymbol{f}}$ is an eigenvector of $\boldsymbol{T}_\omega^*\boldsymbol{T}_\omega$ corresponding to the – only – positive eigenvalue $\|\boldsymbol{g}\|^2\,\|\bar{\boldsymbol{f}}\|^2$. Therefore,

$$\begin{aligned} \|\boldsymbol{T}_\omega\|_{\ell_2} &= \|\boldsymbol{g}\|_{\ell_2}\,\|\bar{\boldsymbol{f}}\|_{\ell_2} \\ &= |C_d(e^{j\omega T})||S_d(e^{j\omega T}))| \sqrt{\left(|PH|^2\right)_d(e^{j\omega T}))} \sqrt{\left(|F|^2\right)_d(e^{j\omega T}))}\ , \end{aligned}$$

which replaced in (3.15) completes the proof. □

Similarly to the previous case, the Sensitivity Operator is also represented in lifted domain by a multiplicative operator $\boldsymbol{S}_\omega$ in ℓ_2 for fixed ω in Ω_N. Equation (3.3) is then rewritten as

$$\boldsymbol{y} = \boldsymbol{d} - \boldsymbol{T}_\omega\,\boldsymbol{d}\ , \tag{3.16}$$

so $\boldsymbol{S}_\omega = \boldsymbol{I} - \boldsymbol{T}_\omega$, where $\boldsymbol{I}$ is the identity operator in ℓ_2. Although $\boldsymbol{S}_\omega$ has infinite rank, its norm is still determined by the eigenvalue of a finite dimensional matrix. This is so, because $\boldsymbol{T}_\omega$ is finite rank and $\boldsymbol{S}_\omega = \boldsymbol{I} - \boldsymbol{T}_\omega$ (*cf.* Yamamoto and Khargonekar 1994).

Theorem 3.2 (Induced L_2 norm of the Sensitivity Operator)
Assume the hybrid system in Figure 1 to be L_2-input-output stable. Then

$$\begin{aligned} \|\mathcal{S}\| = \sup_{\omega\in\Omega_N} \frac{1}{2} \Bigg(& \sqrt{\|\boldsymbol{T}_\omega\|^2 - |T_d(e^{j\omega T})|^2 + (|S_d(e^{j\omega T})|+1)^2} \\ + & \sqrt{\|\boldsymbol{T}_\omega\|^2 - |T_d(e^{j\omega T})|^2 + (|S_d(e^{j\omega T})|-1)^2} \Bigg)\ . \end{aligned} \tag{3.17}$$

Proof: Similarly to the previuos case, the norm of the Sensitivity Operator is computed from (2.8) and (2.9) as

$$\|\mathcal{S}\| = \sup_{\omega \in \Omega_N} \|\boldsymbol{S}_\omega\|_{\ell_2}. \tag{3.18}$$

Recall that $\boldsymbol{S}_\omega = \boldsymbol{I} - \boldsymbol{T}_\omega$, with $\boldsymbol{T}_\omega$ the "outer product" of vectors $\boldsymbol{g}$ and $\bar{\boldsymbol{f}}$. We shall make a suitable transformation to reduce the computation of the ℓ_2-norm of the matrix $\boldsymbol{S}_\omega$ to the computation of the norm of a 2×2 matrix.

Assume ω fixed in Ω_N and let $\mathcal{G}$ and $\mathcal{F}$ denote the linear spaces in ℓ_2 spanned by $\boldsymbol{g}$ and $\bar{\boldsymbol{f}}$. Then, consider the spaces $\mathcal{F}^\perp \cap \mathcal{G}$ and $\mathcal{F}^\perp \cap \mathcal{G}^\perp$, where $\mathcal{F}^\perp$ denotes the orthogonal complement of $\mathcal{F}$. Notice that ℓ_2 can be decomposed as $\mathcal{F} \oplus \mathcal{F}^\perp \cap \mathcal{G} \oplus \mathcal{F}^\perp \cap \mathcal{G}^\perp$.

Introduce the operators $\boldsymbol{P}_\mathcal{F}$, $\boldsymbol{P}_{\mathcal{F}^\perp \cap \mathcal{G}}$, and $\boldsymbol{P}_{\mathcal{F}^\perp \cap \mathcal{G}^\perp}$, the orthogonal projections onto the spaces $\mathcal{F}$, $\mathcal{F}^\perp \cap \mathcal{G}$, and $\mathcal{F}^\perp \cap \mathcal{G}^\perp$, respectively. Then, we define the unitary operator

$$\boldsymbol{U} \triangleq \begin{bmatrix} \boldsymbol{P}_\mathcal{F} & \boldsymbol{P}_{\mathcal{F}^\perp \cap \mathcal{G}} & \boldsymbol{P}_{\mathcal{F}^\perp \cap \mathcal{G}^\perp} \end{bmatrix}.$$

Under the decomposition induced by $\boldsymbol{U}$, the operator $\boldsymbol{S}_\omega$ transforms into

$$\boldsymbol{U}^* \boldsymbol{S}_\omega \boldsymbol{U} = \begin{bmatrix} 1-\alpha & 0 & 0 & 0 & \dots \\ -\beta & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $\alpha = \langle \boldsymbol{g}, \bar{\boldsymbol{f}} \rangle_{\ell_2}$, and $\beta = \sqrt{\|\boldsymbol{g}\|_{\ell_2}^2 \|\bar{\boldsymbol{f}}\|_{\ell_2}^2 - |\alpha|^2}$. Since the transformation is unitary, $\|\boldsymbol{S}_\omega\| = \|\boldsymbol{U}^* \boldsymbol{S}_\omega \boldsymbol{U}\|$, which is the maximum of 1 and the norm of

$$M \triangleq \begin{bmatrix} 1-\alpha & 0 \\ -\beta & 1 \end{bmatrix}.$$

Since $\|M\| \geq 1$, then $\|\boldsymbol{S}_\omega\| = \|M\|$.

Getting the square root of the maximum eigenvalue of M^*M, yields

$$\|M\| = \sqrt{1/2 \left(\gamma^2 + 1 - |\alpha|^2 + |1-\alpha|^2 + \sqrt{(\gamma^2 + 1 - |\alpha|^2 + |1-\alpha|^2)^2 - 4|1-\alpha|^2} \right)}, \tag{3.19}$$

where $\gamma \triangleq \|\boldsymbol{g}\|_{\ell_2}^2 \|\bar{\boldsymbol{f}}\|_{\ell_2}^2$. From (3.19) we get

$$\|M\| = 1/2 \left(\sqrt{\gamma^2 - |\alpha|^2 + (|1-\alpha| + 1)^2} + \sqrt{\gamma^2 - |\alpha|^2 + (|1-\alpha| - 1)^2} \right). \tag{3.20}$$

To finish, note that $\gamma = \|\boldsymbol{g}\|_{\ell_2} \|\bar{\boldsymbol{f}}\|_{\ell_2} = \|\boldsymbol{T}_\omega\|$, and that

$$\begin{aligned} \alpha &= \sum_{k=-\infty}^{\infty} G_k(j\omega) F_k(j\omega) \\ &= C_d(e^{j\omega T}) S_d(e^{j\omega T}) \frac{1}{T} \sum_{k=-\infty}^{\infty} F_k(j\omega) P_k(j\omega) H_k(j\omega). \end{aligned} \tag{3.21}$$

Recalling expression (2.3) for the discretized plant, we obtain from (3.21) that $\alpha = T_d(e^{j\omega T})$. Hence, replacing γ and α in (3.20) leads to (3.17), completing the proof. □

Remark 3.2 Due to the finite dimensionality of $\|T_d\|_\infty$ also in the multivariable case, the computation of the norm of these operators is always reduced to $\sup_{\omega \in \Omega_N} \|M(\omega)\|$, where $M(\omega)$ is a finite dimensional transfer matrix (*cf.* Bamieh and Pearson 1992).

3.3 Numerical implementation.

We describe now a method to compute numerically the induced norms of the scalar Sensitivity Operators. The method has been implemented in MATLAB functions, which have been used in an illustrative example included at the end of the section.

In order to compute the induced norms (3.12) and (3.17), we need to evaluate the functions

$$\left(|F|^2\right)_d(e^{j\omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} |F_k(j\omega)|^2 \ . \tag{3.22}$$

and

$$\left(|PH|^2\right)_d(e^{j\omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} |P_k(j\omega)H_k(j\omega)|^2, \tag{3.23}$$

One way to proceed is suggested by Lemma 2.1, *i.e.,* compute (3.22) and (3.23) by discretizing the functions $F(-s)F(s)$ and $P(-s)H(-s)H(s)P(s)$ and evaluate the respective $(|F|^2)_d(z)$ and $(|PH|^2)_d(z)$ at $z = e^{j\omega T}$. Similar techniques have been used by Leung *et al.* (1991) to compute norms of sampled-data systems with bandlimited signals.

Once translated the expressions in terms of rational transfer functions in $z = e^{j\omega T}$, the induced norms are computed by searching maxima over the finite interval Ω_N.

3.3.1 Computation of $(|F|^2)_d(e^{j\omega T})$

We consider the problem of computing the discrete equivalent of the function $F(s)F(-s)$, as illustrated in Figure 2. Since $F(s)$ is a strictly proper rational function, the sampling of the output of $F(s)F(-s)$ is well-defined.

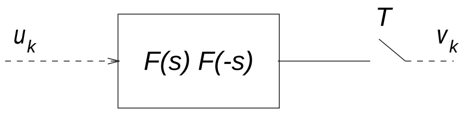

Figure 2: Scheme for computing (3.22).

Let $\{A, B, C, 0\}$ be a minimal state-space realization of it. Then, a minimal realization for $F(s)F(-s)$ is given by

$$\tilde{A} \triangleq \begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix}, \qquad \tilde{B} \triangleq \begin{bmatrix} 0 \\ -C' \end{bmatrix}, \qquad \tilde{C} \triangleq \begin{bmatrix} C & 0 \end{bmatrix} . \tag{3.24}$$

Here, we denoted by A' the transpose of the matrix A.

At the sampling instants the state response of the system is given by

$$\begin{aligned} x_{k+1} &= e^{\tilde{A}T} x_k + \int_0^T e^{\tilde{A}\tau} \tilde{B} u(\tau) d\tau \\ &= e^{\tilde{A}T} x_k + \int_0^T e^{\tilde{A}\tau} \tilde{B}\, \delta(\tau - T) d\tau u_k \ , \end{aligned} \tag{3.25}$$

where $\delta(t)$ is Dirac's delta, since there is no hold device at the input of the system. From (3.25) we get the discrete system

$$\begin{aligned} x_{k+1} &= A_d x_k + B_d u_k \\ v_k &= \tilde{C} x_k \ , \end{aligned}$$

where $A_d \triangleq e^{\tilde{A}T}$ and $B_d \triangleq e^{\tilde{A}T}\tilde{B}$. The discrete transfer function is then

$$(|F|^2)_d(z) = \tilde{C}(zI - A_d)^{-1}B_d. \tag{3.26}$$

Hypotheses of Lemma 2.1 are satisfied for $F(s)F(-s)$, hence evaluating (3.26) at $z = e^{j\omega T}$ gives the desired result.

3.3.2 Computation of $(|PH|^2)_d(e^{j\omega T})$

The case of $(|PH|^2)_d$ is more complicated than the previous one. We shall dicretize the system depicted in Figure 3, *i.e.,* the function $P(s)P(-s)$ cascaded with a GSHF given by $H(s)H(-s)$. Since the hold function results proper by definition, so is the cascade, and therefore the sampling operation is again well-defined.

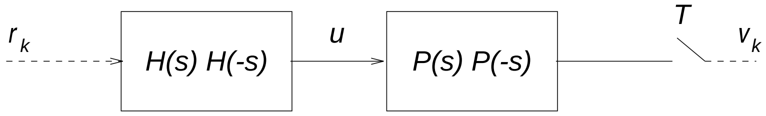

Figure 3: Scheme for computing (3.23).

Suppose the plant $P(s)$ has a minimal relization $\{A, B, C, D\}$. Then, a minimal realization for $P(s)P(-s)$ is given by

$$\tilde{A} \triangleq \begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix}, \quad \tilde{B} \triangleq \begin{bmatrix} BD' \\ -C' \end{bmatrix}, \quad \tilde{C} \triangleq \begin{bmatrix} C & DB' \end{bmatrix}, \quad \tilde{D} \triangleq \begin{bmatrix} DD' \end{bmatrix}$$

A — zero-order — generalized sampled-data hold can be modeled as the response of a linear time-invariant system (LTI GSHF), or by a staircase, piecewise-constant function (PC GSHF) (Middleton and Freudenberg 1993). We analize here the case where $H(s)$ corresponds to a LTI GSHF, the PC GSHF case is deferred to Appendix B.

The LTI GSHF is defined by $u(t) = h(t - kT)u_k$, for $kT \le t < (k+1)T$ and k integer, where

$$h(t) = \begin{cases} Ke^{L(T-t)}M & \text{if } t \in [0, T) \\ 0 & \text{otherwise} \end{cases}, \tag{3.27}$$

for matrices K, L, and M of appropriate dimensions (see Figure 4). Then, the frequency response results $H(s) = K(sI + L)^{-1}(e^{LT} - e^{-sT})M$.

Let $g(t)$ denote the impulse response of the "conjugated" hold whose frequency response is $H(-s) = M'(sI - L')^{-1}(e^{-L'T} - e^{-sT})e^{L'T}e^{sT}K'$. It follows that

$$g(t) = \begin{cases} M'e^{L'(t+T)}K' & \text{if } t \in [-T, 0) \\ 0 & \text{otherwise} \end{cases}. \tag{3.28}$$

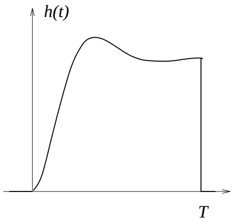

Figure 4: Impulse response of a LTI GSHF.

The impulse response of the "compound" hold corresponding to $H(s)H(-s)$, $h_c(t)$ say, is obtained as the convolution of $h(t)$ and $g(t)$ (Figure 5).

$$\begin{aligned} h_c(t) &= \mathcal{L}^{-1}\{H(s)H(-s)\} \\ &= \int_{-T}^{t} h(t-\tau)g(\tau)d\tau \quad \forall t \in [-T,T] \\ &= \int_{-T}^{t} Ke^{L(T+\tau-t)}MM'e^{L'(\tau+T)}K'd\tau \,. \end{aligned} \tag{3.29}$$

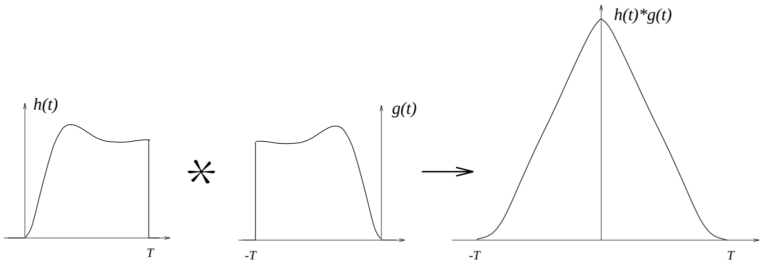

Figure 5: Impulse response of $H(s)H(-s)$.

An apparent difficulty in the computation is that $H(s)H(-s)$, results to be one sampling period non-causal. However, this can be tackled by noting that the signal $u(t)$ produced by the compound hold is a combination of samples at kT and $(k+1)T$. In other words, at each sampling instant we have a term driving the states of $P(s)P(-s)$, and a feedthrough term directly passed to the output of the whole cascade.

Split $h_c(t)$ by introducing two functions defined on $[0,T)$ as follows,

$$h_{c1}(t) \triangleq h_c(t-T) \quad \text{for } t \in [0,T) \tag{3.30}$$

$$h_{c2}(t) \triangleq h_c(-t) \quad \text{for } t \in [0,T). \tag{3.31}$$

Then, the output of the compound hold can be written as

$$u(t) = h_{c1}(t-kT)\,r_{k+1} + h_{c2}(t-kT)\,r_k \quad \text{for } t \in [kT,(k+1)T). \tag{3.32}$$

Using (3.32), the discrete state equations for the system of Figure 3 are

$$\begin{aligned} x_{k+1} &= e^{\tilde{A}T}x_k + \int_0^T e^{\tilde{A}(T-\tau)}\tilde{B}\,u(\tau)\,d\tau \\ &= e^{\tilde{A}T}x_k + \left(\int_0^T e^{\tilde{A}(T-\tau)}\tilde{B}\,h_{c1}(\tau)\,d\tau\right) r_{k+1} + \left(\int_0^T e^{\tilde{A}(T-\tau)}\tilde{B}\,h_{c2}(\tau)\,d\tau\right) r_k \\ &= A_d\,x_k + E_d\,r_{k+1} + F_d\,r_k \,, \end{aligned} \tag{3.33}$$

and the discrete output

$$
\begin{aligned}
v_k &= \tilde{C}x_k + \tilde{D}u(kT) \\
&= \tilde{C}x_k + \tilde{D}\,(h_{c1}(0)\,u_{k+1} + h_{c2}(0)\,u_k) \\
&= \tilde{C}x_k + \tilde{D}W_h\,,
\end{aligned} \tag{3.34}
$$

where $W_h \triangleq h_c(0) = \int_0^T Ke^{L\tau}MM'e^{L'\tau}K'd\tau$, a finite-time Gramian of the hold function.

The discretized system given by (3.33) and (3.34) is represented by the z-transform block diagram of Figure 6. After some block algebra, we obtain Figure 7, from which we get the discrete transfer function from $\{r_k\}$ to $\{v_k\}$

$$
(|PH|^2)_d(z) = D_d + C_d(zI - A_d)^{-1}B_d\,, \tag{3.35}
$$

with

$$
\begin{aligned}
A_d &= e^{\tilde{A}T} \\
B_d &= F_d + A_dE_d \\
C_d &= \tilde{C} \\
D_d &= \tilde{D}W_h + C_dE_d\,.
\end{aligned}
$$

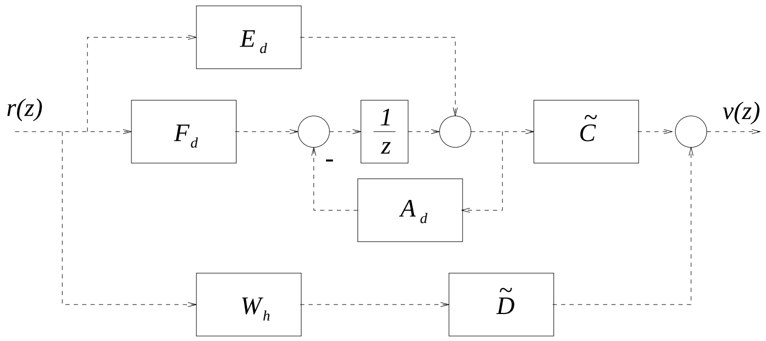

Figure 6: Equivalent block diagram for the discretized system of Figure 3.

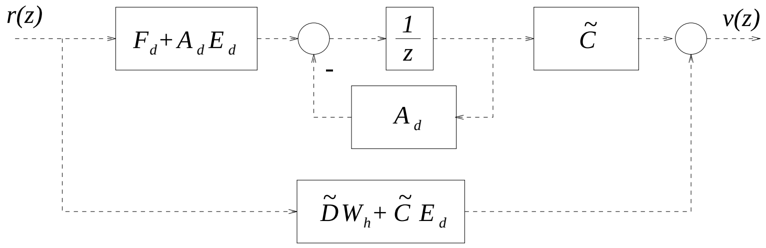

Figure 7: Simplified block diagram.

Conditions for Lemma 2.1 are valid since $P(s)$ was assumed proper and $H(s)$ results strictly proper by definition. Again, evaluating (3.35) at $z = e^{j\omega T}$ gives the desired result.

Remark 3.3 Matrices F_d, E_d and W_h in the above formulae can be readily computed using the following fact (*e.g.* Van Loan 1978)

Fact 1
If A, B, C are real matrices of compatible dimensions, then

$$\int_0^t e^{A(t-\tau)} B e^{C\tau} d\tau = \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} t} \begin{bmatrix} 0 \\ I \end{bmatrix},$$

where I and 0 are identity and null matrices of appropriated dimensions.

For instance, $h_{c1}(t)$ and $h_{c2}(t)$ in (3.30) and (3.31) are given by

$$\begin{aligned} h_{c1}(t) &= \begin{bmatrix} K e^{LT} & 0 \end{bmatrix} e^{\tilde{L} t} \begin{bmatrix} 0 \\ K' \end{bmatrix} \\ h_{c2}(t) &= \begin{bmatrix} -K & 0 \end{bmatrix} e^{\tilde{L}(t-T)} \begin{bmatrix} 0 \\ e^{L^T T} K' \end{bmatrix}, \end{aligned}$$

where

$$\tilde{L} \triangleq \begin{bmatrix} -L & MM' \\ 0 & L' \end{bmatrix}.$$

Hence, F_d and E_d are given respectively by

$$\begin{aligned} F_d &= \begin{bmatrix} -I & 0 \end{bmatrix} e^{\begin{bmatrix} \tilde{A} & \begin{bmatrix} \tilde{B} K e^{-\tilde{L} T} & 0 \end{bmatrix} \\ 0 & \tilde{L} \end{bmatrix} T} \begin{bmatrix} 0 \\ e^{L' T} K' \end{bmatrix} \\ E_d &= \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} \tilde{A} & \begin{bmatrix} \tilde{B} K e^{LT} & 0 \end{bmatrix} \\ 0 & \tilde{L} \end{bmatrix} T} \begin{bmatrix} 0 \\ K' \end{bmatrix}. \end{aligned}$$

3.4 Example

We now consider an example of application of the formulae derived in this section. For a given analog plant and filter, we compare the norms obtained for two differents control designs. The example is a variation of one taken from Goodwin and Salgado (1994).

We refer to Figure 1. Here the plant $P(s)$ is given by

$$P(s) = \frac{1}{(s+3)(s+4)}, \tag{3.36}$$

and we use an antialiasing filter $F(s)$ given by

$$F(s) = \frac{1}{s+10}. \tag{3.37}$$

The hold used is a ZOH with sampling period $T = 0.4s$. The discrete equivalent of the plant and filter with this hold is

$$P_d(z) = \frac{0.0024z^2 + 0.0021z + 0.0001}{z^3 - 0.5214z^2 + 0.07z - 0.0011} \tag{3.38}$$

We consider two different discrete controllers:

Gain	$\max_{\omega\in[0,\omega_N]}$ *Gain*	
	One-step-ahead design	*Second design*
$\|\|\mathcal{S}\|\|$	8.6375	1.8379
$\|S_d\|$	2.0000	1.5559
$\|\|\mathcal{T}\|\|$	8.4059	1.4403
$\|T_d\|$	1.0000	1.0000

Table 1: Norms of hybrid and discrete sensitivity functions.

(i). One-step-ahead design: The discrete controller is designed to obtain a discrete Complementary Sensitivity Function

$$T_{d_1}(z) = \frac{1}{z}. \tag{3.39}$$

The controller obtained is

$$\begin{aligned} C_{d_1}(z) &= \frac{P_d(z)^{-1}}{z-1} && (3.40)\\ &= 423.8178\frac{z^3-0.5214z^2+0.07z-0.0011}{z^3-0.1010z^2-0.8644z-0.0346}, && (3.41) \end{aligned}$$

which contains a discrete pole at $z = 1$.

Note that in this design the zeros of the complementary sensitivity induced by those of the discrete plant were canceled by the controller selected.

(ii). Second design: Similar to the precedent but with a less ambitious T_d, we leave the zeros of $P_d(z)$ in $T_{d_2}(z)$ this time. The discrete complementary sensitivity is then

$$T_{d_2}(z) = 219.1759\frac{0.0024z^2+0.0021z+0.0001}{z^3}. \tag{3.42}$$

The controller to achieve this is

$$\begin{aligned} C_{d_2}(z) &= \frac{T_{d_2}(z)}{(1-T_{d_2}(z))\,P_d(z)} && (3.43)\\ &= 219.1759\frac{z^3-0.5214z^2+0.07z-0.0011}{z^3-0.5171z^2-0.4649z-0.0179}, && (3.44) \end{aligned}$$

also including an integration.

For these two designs we calculate the *hybrid L_2-gains* $\|\boldsymbol{S}_\omega\|$ and $\|\boldsymbol{T}_\omega\|$ and the responses of the discrete Sensitivity Functions, $S_d(e^{j\omega T})$ and $T_d(e^{j\omega T})$.

The plots are shown in Figure 8 for the one-step-ahead design, and in Figure 9 for the second design. The induced L_2 norms of the Sensitivity Operators and the H_∞ norms of the discrete Sensitivity Functions are summarized in Table 1.

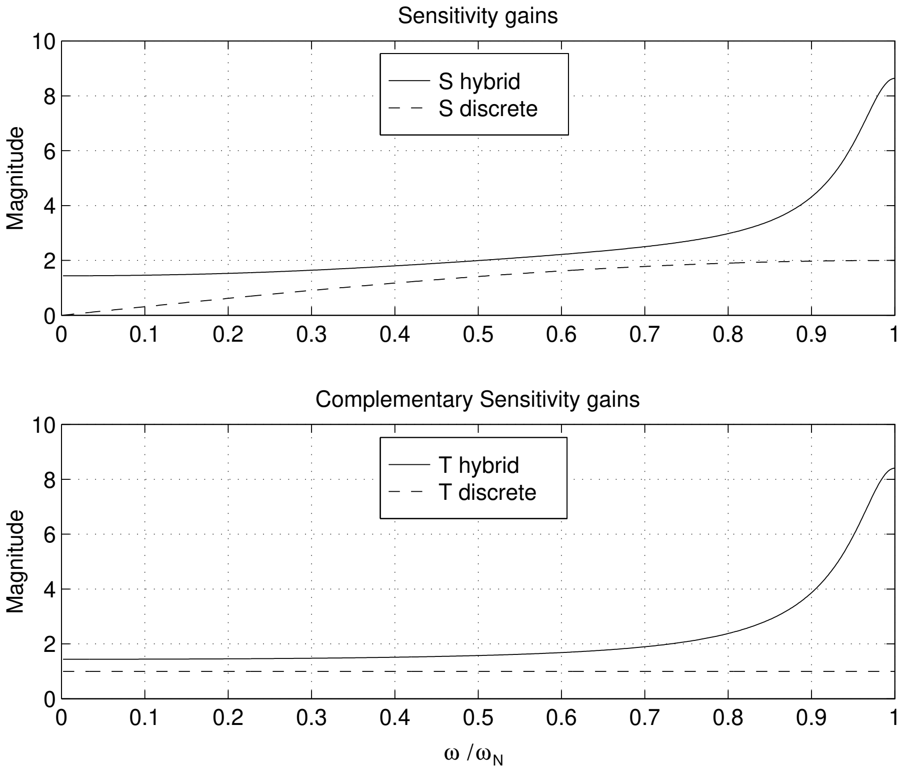

Figure 8: One-step-ahead design: sensitivity and complementary sensitivity gains.

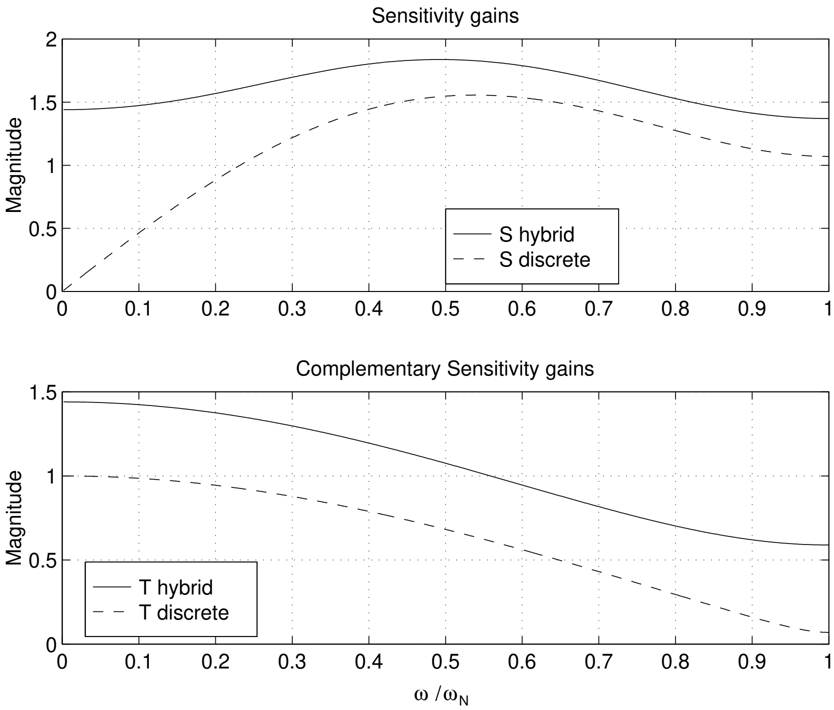

Figure 9: Second design: sensitivity and complementary sensitivity gains.

4 Stability Robustness.

We address the problem of achieving robust stability under linear time-invariant uncertainty in the *analog* plant. Necessary and sufficient conditions for robust stability of sampled-data systems have been derived by Dullerud and Glover (1993) using a frequency domain representation based on time-domain lifting techniques (Bamieh *et al.* 1991, Yamamoto 1994). In the time-domain lifting setup, the system is specified in a state-space representation, from which "transfer matrices" are derived using a generalized z-transform (*e.g.* Dullerud and Glover 1993, Yamamoto and Khargonekar 1994). However, the detour through an internal representation to describe input output operators may complicate the analysis (Yamamoto and Khargonekar 1994).

In this section, we show that similar results can be obtained in a simpler way by approaching the problem directly in frequency domain using the frequency-lifting presented in this paper. We consider *multiplicative* and *divisive* uncertainty models for the perturbed analog plant. We show that, in both cases, the problem of finding necessary and sufficient conditions for robust stability reduces to a structured singular value (μ) test (Doyle 1982). In particular, in the multiplicative single-input single-output case (*cf.* Dullerud and Glover 1992) the μ-problem can be solved exactly and we obtain a weighted bound condition on the Fundamental Complementary Sensitivity Function $T_{fun}(s) = 1/T\, P(s)\, H(s)\, C_d(e^{sT})\, S_d(e^{sT})$ (Freudenberg *et al.* 1993). T_{fun} governs the fundamental component of the intersample response to noise, and plays a similar role to the Complementary Sensitivity Function for LTI systems, see Freudenberg *et al.* (1993) for details.

4.1 Multiplicative perturbation.

Consider the multivariable sampled-data system depicted in Figure 10. The perturbed plant is represented by the multiplicative uncertainty model

$$\tilde{P}(s) = (I + W(s)\Delta(s))\, P(s) \ , \tag{4.1}$$

where $\Delta(s)$ is a stable transfer function satisfying $\|\Delta\|_\infty < 1$, and $W(s)$ is a fixed stable, proper, minimum-phase weighting function.

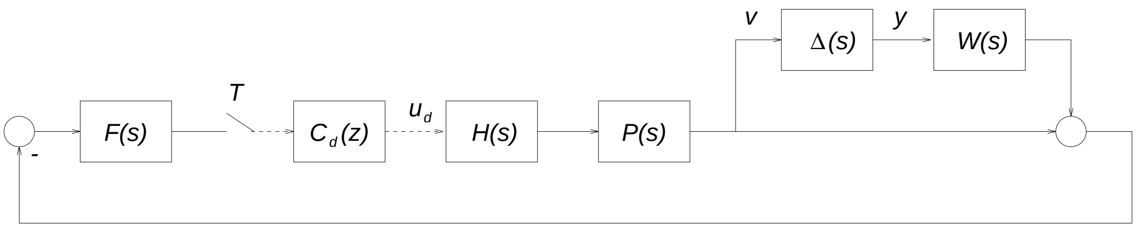

Figure 10: System with multiplicative uncertainty.

Assuming closed loop stability of the nominal hybrid system, $\Delta(s) = 0$, we shall determine necessary and sufficient conditions for the disturbed system to remain stable under the class of admissible uncertainties.

Suppose that we discretize the system of Figure 10 by opening the loop at the input and output of the discrete controller C_d. Then, we obtain the simplified discrete diagram of Figure 11, where $(F\tilde{P}H)_d(z)$ is the discretized series of hold, perturbed plant and antialiasing filter. Applying

Lemma 2.1 to $(F\tilde{P}H)_d(z)$ yields

$$(F\tilde{P}H)_d(e^{sT}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F_k(s)(I + W_k(s)\Delta_k(s))P_k(s)H_k(s)\ . \tag{4.2}$$

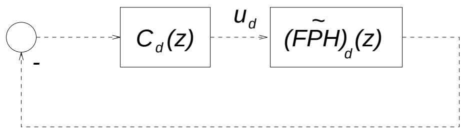

Figure 11: Discretized perturbed system.

Equation (4.2) describes the effect of the sampling operation in frequency domain. This relationship is translated directly into the block diagram of Figure 12. It is important to note that this diagram does not show explicitely the sampler but its action, now structurally embedded in the graphic model. Based on this picture, we can now work out conditions for robustness keeping track of every harmonic component in the hybrid system.

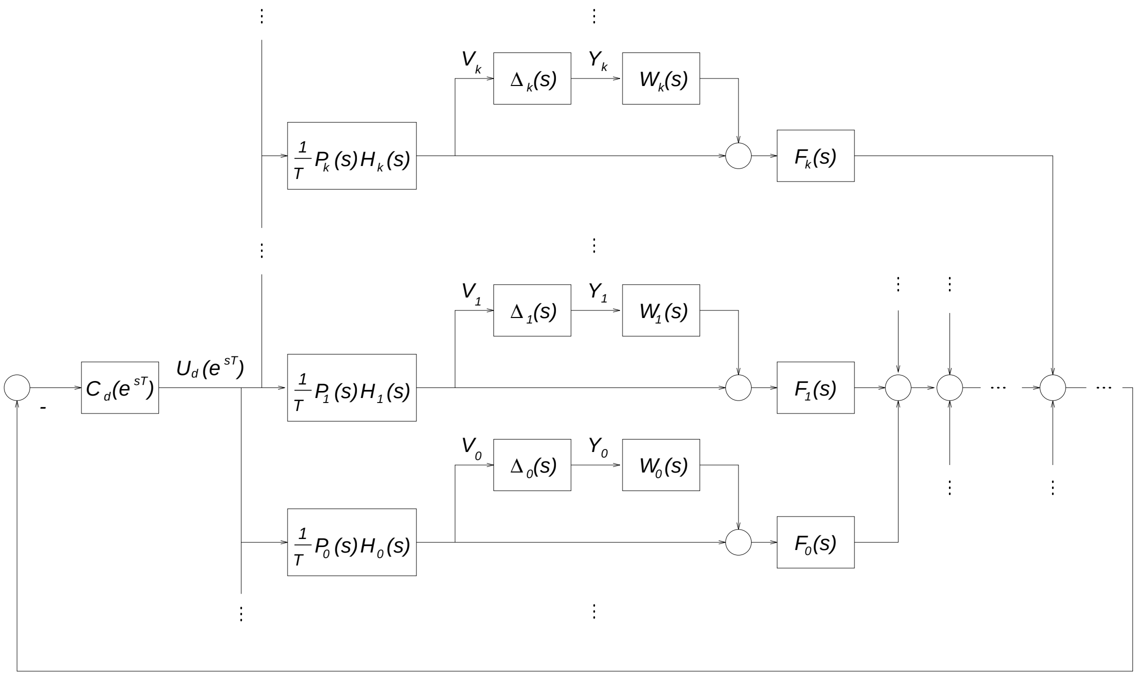

Figure 12: Harmonic structure of the perturbed sampled-data system with multiplicative uncertainty.

Concentrate on the k-harmonic of the signal $V(s)$, input to the uncertainty block in Figure 12. Then, we can write

$$V_k(s) = \frac{1}{T}P_k(s)\,H_k(s)\,U_d(e^{sT}), \tag{4.3}$$

where $U_d(z)$ is the z-transform of the output of the controller. In the rest of this computation we shall drop the independent variables to ease notation, understanding that all signals and transfer functions are functions of s, save for the discrete ones, like C_d and U_d, which are functions of e^{sT}.

But U_d is given by

$$\begin{aligned} U_d &= -C_d \sum_{k=-\infty}^{\infty} F_k \left(W_k Y_k + 1/T P_k H_k U_d\right) \\ &= -C_d \sum_k F_k W_k Y_k - C_d \left(1/T \sum_k F_k P_k H_k\right) U_d \ . \end{aligned} \tag{4.4}$$

Noting that $1/T \sum_k F_k P_k H_k$ is the nominal discretized plant, from (4.4) we get

$$U_d = -S_d C_d \sum_k F_k Y_k \ , \tag{4.5}$$

where

$$S_d(z) = [I + C_d(z)(FPH)_d(z))]^{-1} \tag{4.6}$$

is the nominal discrete Sensitivity Function. Now, replacing U_d from (4.5) and $Y_k = \Delta_k V_k$ into (4.3) yields

$$V_k = \frac{1}{T} P_k H_k S_d C_d \sum_m F_m W_m \Delta_m V_m \ . \tag{4.7}$$

In the lifted domain, (4.7) is written as

$$(\boldsymbol{I} + \boldsymbol{T}_\omega \boldsymbol{W}_\omega \Delta_\omega)\, \boldsymbol{v} = 0, \tag{4.8}$$

where $\boldsymbol{I}$ is the infinite identity matrix (the identity operator in ℓ_2) and $\boldsymbol{T}_\omega$ is the infinite matrix representation of the *Complementary Sensitivity Operator*, defined in (3.6). $\boldsymbol{W}_\omega$ and Δ_ω are infinite dimensional block diagonal matrices, $\boldsymbol{W}_\omega \triangleq \mathrm{diag}[\ldots, W_k(j\omega), W_{k-1}(j\omega), \ldots]$, and $\Delta_\omega \triangleq \mathrm{diag}[\cdots, \Delta_k(j\omega), \Delta_{k-1}(j\omega), \cdots]$. Finally $\boldsymbol{v}$ is the lifted vector

$$\boldsymbol{v}(\omega) \triangleq \begin{bmatrix} \vdots \\ V_1(\omega) \\ V_0(\omega) \\ V_{-1}(\omega) \\ \vdots \end{bmatrix} . \tag{4.9}$$

Equation (4.8) collects together system knowns and perturbations in two separated blocks, displaying the undelying structure of a μ-problem (Doyle 1982). The standard "feedback" interpretation associated to (4.8) is shown in Figure 13.

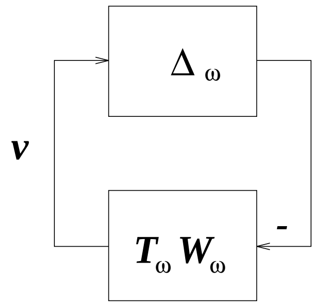

Figure 13: Collapsed block diagram: multiplicative uncertainty.

Recall that the structured singular value of a given complex matrix M, M in $\mathbb{C}^{n\times n}$, is a number defined respect to a set $\boldsymbol{\Delta}$ of perturbation matrices Δ, Δ in $\mathbb{C}^{n\times n}$, of prescribed structure.

Definition 4.1 (Doyle (1982)) For M in $\mathbb{C}^{n\times n}$, $\mu_{\mathbf{\Delta}}(M)$ is defined

$$\mu_{\mathbf{\Delta}}(M) \triangleq \frac{1}{\min\{\|\Delta\|_2 : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0\}}, \tag{4.10}$$

unless no $\Delta \in \mathbf{\Delta}$ makes $I - M\Delta$ singular, in which case $\mu_{\mathbf{\Delta}}(M) \triangleq 0$.

The perturbation set $\mathbf{\Delta}$ is normally defined of the form

$$\begin{aligned}\mathbf{\Delta} \triangleq \{&[\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_S I_{r_S}, \Delta_{S+1}, \ldots, \Delta_{S+F}] \\ &: \delta_i \in \mathbb{C}, \Delta_{S+j} \in \mathbb{C}^{m_j\times m_j}, i = 1, 2, \ldots, S, j = 1, 2, \ldots, F\},\end{aligned} \tag{4.11}$$

where I_{r_i} are $\mathbb{C}^{r_i\times r_i}$ identity matrices. Note that for dimensional consistency it is necessary that $\sum_{i=1}^{S} r_i + \sum_{j=1}^{F} m_j = n$.

With these definitions, we can now state necessary and sufficient conditions for robust stability of the sampled-data system of Figure 10, adapted from the result obtained by Dullerud and Glover (1993). The result reduces to a μ-problem on the infinite dimensional matrices of Figure 13, and it is expressed as a sequence of all the finite dimensional μ-problems obtained by truncating the original matrices. Define $\boldsymbol{T}_\omega^n$, $\boldsymbol{W}_\omega^n$ and Δ_ω^n as the corresponding truncations keeping all harmonics between $-n$ and n, for some positive integer n. For each ω in Ω_N, Δ_ω^n has a block diagonal structure, where each block $\Delta_k(j\omega)$ is in $\mathbf{\Delta}$, introduced in (4.11). Let $\mathbf{\Delta}^n$ denote the set of all those block diagonal matrix perturbations capturing that structure, $\mathbf{\Delta}^n \triangleq \{\mathrm{diag}[\Delta_n, \ldots, \Delta_{-n}] : \Delta_i \in \mathbf{\Delta}\}$.

Proposition 4.1 (Dullerud and Glover (1993))
For all $\Delta(s)$ such that $\|\Delta\|_\infty < 1$ the system of Figure 10 is internally stable if and only if for each integer $n > 0$ the following inequality is satisfied

$$\max_{\omega\in\Omega_N} \mu_{\mathbf{\Delta}^n}(\boldsymbol{T}_\omega^n \boldsymbol{W}_\omega^n) \le 1\ . \tag{4.12}$$

Proof: See Dullerud and Glover (1993). □

In the particular case of single-input single-output systems, the μ-problem can be solved explicitely, yielding the following corollary.

Corollary 4.2
If the system of Figure 10 is single-input single-output, then, for all $\Delta(s)$ satisfying $\|\Delta\|_\infty < 1$, the system is internally stable if and only if

$$\sum_{k=-\infty}^{\infty} |T_{fun}(j(\omega + k\omega_s))\, W(j(\omega + k\omega_s))| \le 1 \qquad \forall\omega \in \Omega_N \tag{4.13}$$

Proof: By Proposition 4.1, the system will be robustly stable if and only if all truncated systems satisfy μ-condition (4.12). Fix an integer $n > 0$ and ω in Ω_N. The standard approach to evaluate $\mu_{\mathbf{\Delta}^n}(\boldsymbol{T}_\omega^n)$ is through the computation of upper and lower bounds. Define the sets

$$\mathcal{Q} \ \triangleq\ \{\boldsymbol{Q} \in \mathbf{\Delta}^n : \boldsymbol{Q}^*\boldsymbol{Q} = I\} \tag{4.14}$$

$$\mathcal{D} \ \triangleq\ \{\boldsymbol{D} \in \mathbf{\Delta}^n : \boldsymbol{D} = \boldsymbol{D}^* > 0 \text{ and } \boldsymbol{D}\Delta_\omega^n = \Delta_\omega^n\boldsymbol{D} \text{ for all } \Delta_\omega^n \in \mathbf{\Delta}^n\}\ . \tag{4.15}$$

Then we have the following inequalities

$$\max_{\boldsymbol{Q}\in\mathcal{Q}} \rho(\boldsymbol{Q}\boldsymbol{T}_\omega^n) \le \mu_{\mathbf{\Delta}^n}(\boldsymbol{T}_\omega^n) \le \inf_{\boldsymbol{D}\in\mathcal{D}} \|\boldsymbol{D}\boldsymbol{T}_\omega^n\boldsymbol{D}^{-1}\|_2\ . \tag{4.16}$$

Note that the structure of the uncertainty in this case is diagonal, $\Delta^n_\omega = \mathrm{diag}[\delta_n(j\omega), \ldots, \delta_{-n}(j\omega)]$, with $\delta_i(j\omega)$ in $\mathbb{C}$. As the truncated $\boldsymbol{T}^n_\omega$ is rank-one, we can work out in closed form the values of $\rho(\boldsymbol{Q}\boldsymbol{T}^n_\omega)$ and $\|\boldsymbol{D}\boldsymbol{T}^n_\omega \boldsymbol{D}^{-1}\|_2$ in (4.16). We show that there exist matrices $\boldsymbol{Q}_0$ and $\boldsymbol{D}_0$ such that upper and lower bounds in (4.16) coincide, and so we get the expression for $\mu_{\Delta^n}(\boldsymbol{T}^n_\omega)$. To lighten notation we eliminate in the remaining the superindex n and the subindex ω in truncated matrices and vectors.

We compute the spectral radius $\rho(\boldsymbol{Q}\boldsymbol{T})$. As in Section 3, we have that the Complementary Sensitivity Matrix can be written as an outer product, $\boldsymbol{T} = \boldsymbol{g}\,\boldsymbol{f}^T$, where the vectors

$$\boldsymbol{g} = \frac{1}{T} S_d\, C_d \begin{bmatrix} P_n\, H_n \\ P_{n-1}\, H_{n-1} \\ \vdots \\ P_{-n}\, H_{-n} \end{bmatrix} \quad \text{and} \quad \boldsymbol{f} = \begin{bmatrix} F_n \\ F_{n-1} \\ \vdots \\ F_{-n} \end{bmatrix} .$$

Then, $\boldsymbol{Q}\boldsymbol{T}$ is also a rank-one matrix, and its only eigenvalue is $\lambda = \boldsymbol{f}^T \boldsymbol{Q}\boldsymbol{g}$, so $\rho(\boldsymbol{Q}\boldsymbol{T}) = |\boldsymbol{f}^T\boldsymbol{Q}\boldsymbol{g}|$.

Consider the particular matrix $\boldsymbol{Q}_0 = \mathrm{diag}[Q_n, Q_{n-1}, \ldots, Q_{-n}]$, with

$$Q_i \triangleq \begin{cases} \frac{P_i^*\, H_i^*\, C_d^*\, S_d^*\, F_i^*}{|P_i\, H_i\, C_d\, S_d\, F_i|} & \text{if } P_i\, H_i\, C_d\, S_d\, F_i \neq 0 \\ 1 & \text{otherwise} \end{cases} . \tag{4.17}$$

Then

$$\rho(\boldsymbol{Q}_0\boldsymbol{T}) = \sum_{i=-n}^{n} |P_i\, H_i\, C_d\, S_d\, F_i| \,, \tag{4.18}$$

and $\boldsymbol{Q}_0$ is certainly in $\mathcal{Q}$.

The 2-norm of a rank-one matrix $\boldsymbol{T} = \boldsymbol{g}\,\boldsymbol{f}^T$ is given by $\|\boldsymbol{T}\|_2 = \rho(\boldsymbol{T}^*\boldsymbol{T})^{1/2} = \|\boldsymbol{g}\|_2\,\|\boldsymbol{f}\|_2$. Consider $\|\boldsymbol{D}_0\boldsymbol{T}\boldsymbol{D}_0^{-1}\|_2$ with $\boldsymbol{D}_0 = \mathrm{diag}[D_n, D_{n-1}, \ldots, D_{-n}]$ and

$$D_i \triangleq \begin{cases} \left|\frac{T F_i}{P_i H_i C_d S_d}\right|^{1/2} & \text{if } P_i H_i C_d S_d F_i \neq 0, \\ 1 & \text{otherwise.} \end{cases} \tag{4.19}$$

Therefore,

$$\|\boldsymbol{D}_0\boldsymbol{T}\boldsymbol{D}_0^{-1}\|_2 = \sum_{i=-n}^{n} |P_i\, H_i\, C_d\, S_d\, F_i| \,, \tag{4.20}$$

with $\boldsymbol{D}_0$ in $\mathcal{D}$.

From (4.16), (4.18) and (4.20), we conclude that

$$\rho(\boldsymbol{Q}\boldsymbol{T}) = \sum_{i=-n}^{n} |P_i\, H_i\, C_d\, S_d\, F_i| \,. \tag{4.21}$$

Note that (4.21) is valid for an arbitrary integer $n > 0$ and ω in Ω_N.

The proof is completed by recalling that $T_{fun}(s) = 1/T\, P(s)\, H(s)\, C_d(e^{sT})\, S_d(e^{sT})$ and using Proposition 4.1. □

Remark 4.1 This result can also be obtained without the μ-framework. Note that the perturbed discrete Sensitivity Function can be written as

$$\begin{aligned} \tilde{S}_d &= \frac{S_d}{1 + (F\tilde{P}H)_d S_d C_d} \\ &= \frac{S_d}{1 + \sum_{n=-\infty}^{\infty} T_{fun_k} W_k \Delta_k} \end{aligned} \tag{4.22}$$

We prove both implications by contrapositive arguments.

($\Leftarrow$) Suppose that there exist an admissible $\Delta(s)$ such that $\tilde{S}_d$ is unstable. Then, by continuity arguments there also exist some admissible $\Delta'(s)$ such that $\tilde{S}_d$ is *marginally* stable, *i.e.,* , it has a pole at $s = j\omega_0$, for some ω_0. From (4.22) it follows that

$$\sum_{n=-\infty}^{\infty} T_{fun_k}(\omega_0) W_k(\omega_0) \Delta'_k(\omega_0) = -1 \ .$$

Hence,

$$\sum_{n=-\infty}^{\infty} |T_{fun_k}(\omega_0) W_k(\omega_0) \Delta'_k(\omega_0)| \geq 1 \ ,$$

but as $\|\Delta'\|_\infty < 1$, then

$$\sum_{n=-\infty}^{\infty} |T_{fun_k}(\omega_0) W_k(\omega_0)| > 1 \ .$$

($\Rightarrow$) Suppose that there exists ω_0 in Ω_N such that

$$\sum_{n=-\infty}^{\infty} |T_{fun_k}(\omega_0) W_k(\omega_0)| = \alpha > 1 \ .$$

Then, it is possible to find an admissible perturbation $\Delta(s)$ that interpolates

$$\Delta(j(\omega_0 + k\omega_s)) = -1/\alpha e^{-j\theta_k},$$

where $\theta_k \triangleq \angle T_{fun_k}(\omega_0) W_k(\omega_0)$. Then $\tilde{S}_d$ has a pole at $z = e^{j\omega_0 T}$ and so the perturbed system is not asymptotically stable.

4.2 Divisive perturbation.

We now look at the divisive uncertainty model of Figure 14, where the perturbed plant is represented by

$$\tilde{P}(s) = (I + W(s)\Delta(s))^{-1} P(s) \ . \tag{4.23}$$

Again, we have on $\Delta(s)$ and $W(s)$ the same assumptions made for the multiplicative uncertainty in the last subsection. This time, the discretized perturbed plant is given by

$$(F\tilde{P}H)_d(e^{sT}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F_k(s)(I + W_k(s)\Delta_k(s))^{-1} P_k(s) H_k(s) \ . \tag{4.24}$$

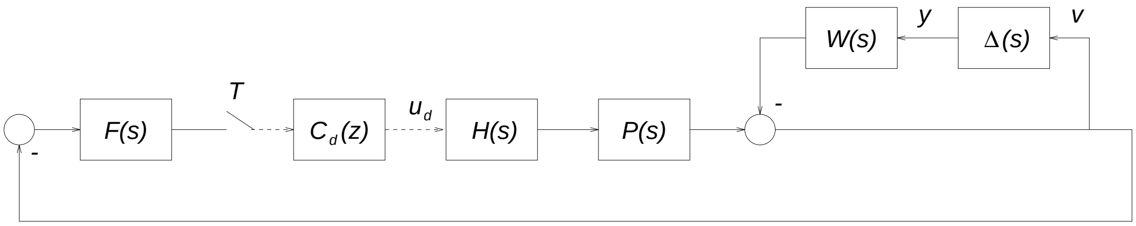

Figure 14: System with divisive uncertainty.

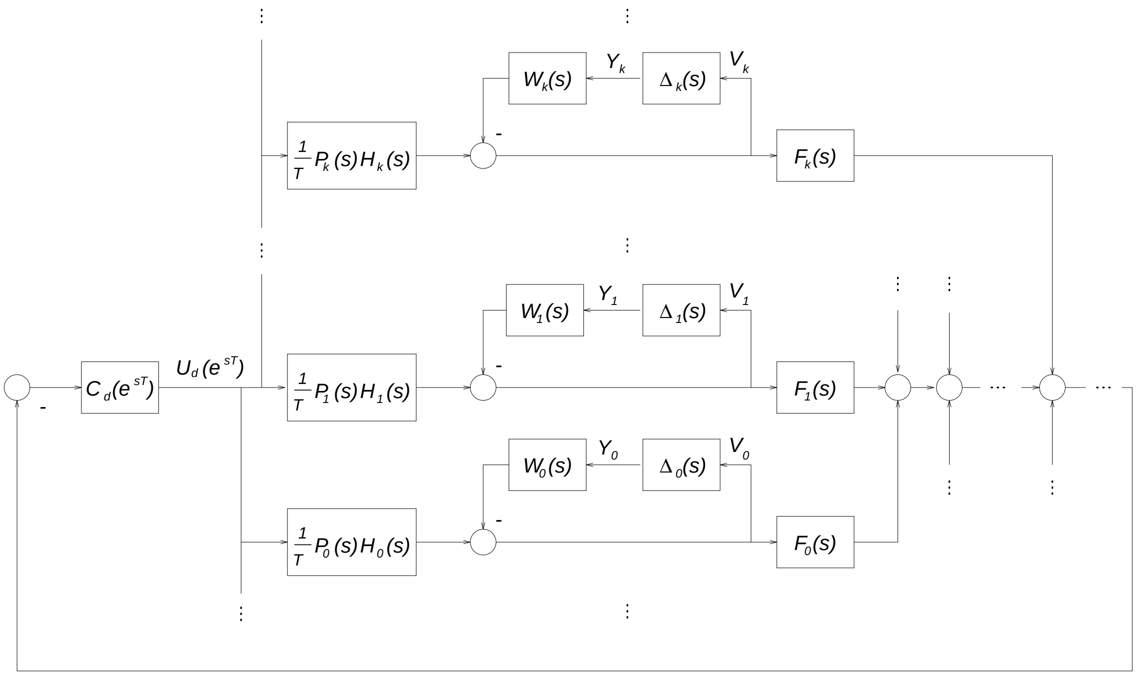

Figure 15: Harmonic structure of the perturbed sampled-data system with divisive uncertainty.

Following similar steps to those for the multiplicative case, we obtain a block diagram of Figure 15 displaying the harmonic structure of the system arising from the sampling process. Based on this picture, we compute an expression analogous to (4.7) for the k-harmonic of the input to the uncertainty block, $V(s)$, getting this time

$$V_k = -W_k \Delta_k V_k + \frac{1}{T} P_k H_k S_d C_d \sum_m F_m W_m \Delta_m V_m \ . \tag{4.25}$$

The lifted version of (4.25) is then

$$(\boldsymbol{I} + \boldsymbol{S}_\omega \boldsymbol{W}_\omega \Delta_\omega)\, \boldsymbol{v} = 0 \ , \tag{4.26}$$

with the same notation used in Subsection 4.25. As could have been intuitively expected, now the *Sensitivity Operator* appears in the formula by its infinite matrix representation $\boldsymbol{S}_\omega$, defined in (3.7). As before, the structure of a μ-problem is evident from (4.26), which is ilustrated in Figure 16.

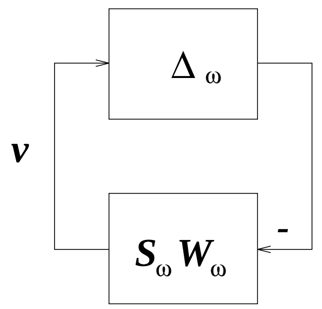

Figure 16: Collapsed block diagram: divisive uncertainty.

We should expect that a result similar to Proposition 4.12 will be valid in this case also, that is, the system will be robust stable under divisive perturbation if and only if all the truncated μ-problems corresponding to Figure 16 satisfy a stability condition. In other words, if and only if for all admissible LTI perturbations and each integer $n > 0$

$$\max_{\omega \in \Omega_N} \mu_{\Delta^n}(\boldsymbol{S}_\omega^n \boldsymbol{W}_\omega^n) \leq 1 \ . \tag{4.27}$$

At the present time, we can only conjecture that condition (4.27) is the appropriated, as the operator given by $\boldsymbol{S}_\omega \boldsymbol{W}_\omega$ does not fulfill some technical assumptions made in Dullerud and Glover (1993). In fact, (4.27) will be valid if we additionally assume that $W(s)$ is *strictly* proper. This issue is the subject of current research.

5 Conclusions

This paper has considered a frequency domain, input-output based approach to the analysis of hybrid sampled-data control systems. The technique of "frequency-domain lifting" has been expounded, and some applications of this technique were illustrated. The particular applications that we have considered are:

(a) Worst-case L_2 disturbance/noise performance (induced L_2 norm) of a hybrid system (and hence, stability robustness for LTV unstructured perturbations),

(b) Stability robustness of a hybrid system to unstructured, linear time-invariant perturbations to the analog plant.

In the SISO case, simplifications of the required calculations can easily be performed. In this paper we have shown that

(a) The induced L_2 norm of the Sensitivity or Complementary Sensitivity Operator can be found as a $\sup_{\omega \in \Omega_N}$ of a readily computable discrete-time system;

(b) Robustness to multiplicative LTI perturbations can be reduced to a rank-one μ-problem, and hence considered analytically.

The frequency-domain lifting approach described above we believe to give, in some cases, simpler, more intuitive methods than the equivalent time-domain lifting approaches to the same problems.

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Appendix A The space $\boldsymbol{L_2(\Omega_N;\ell_2)}$.

Consider the space of time-domain signals $x(t)$, defined for $t > 0$, and with values over $\mathbb{C}^n$. Let $L_2(0,\infty)$ denote the space of those Lebesgue square integrable signals for which

$$\|x\|_{L_2(0,\infty)} \triangleq \left(\int_0^\infty |x(t)|^2\,dt\right)^{1/2} < \infty\,. \tag{A.1}$$

Here, we indicate with $|\cdot|$ the standard Euclidean norm in $\mathbb{C}^n$.

Similarly, consider the space of frequency-domain signals $X(j\omega)$, defined for all real ω, and with values on $\mathbb{C}^n$. Let $L_2(-\infty,\infty)$ denote the corresponding space of signals for which

$$\|X\|_{L_2(-\infty,\infty)} \triangleq \left(\int_{-\infty}^\infty |X(j\omega)|^2\,dt\right)^{1/2} < \infty\,. \tag{A.2}$$

If $x(t)$ and $X(j\omega)$ are a Fourier-transform pair, *i.e.,*

$$X(j\omega) = \int_{-\infty}^\infty e^{-j\omega t}\,x(t)\,dt,$$

then, by Parseval's Theorem, their norms are related by

$$\|x\|_{L_2(0,\infty)} = 2\pi\,\|X\|_{L_2(-\infty,\infty)}\,. \tag{A.3}$$

We also introduce the space ℓ_2, which consists of all sequences $\nu = \{\nu_k\}_{k=-\infty}^\infty$, ν_k in $\mathbb{C}^n$, such that

$$\|\nu\|_{\ell_2} \triangleq \left(\sum_{k=-\infty}^\infty |\nu|^2\right)^{1/2} < \infty\,. \tag{A.4}$$

Now, consider the class of functions defined over a real segment Ω with values in the space ℓ_2. A function $\boldsymbol{f}(\omega)$ in this class assigns to almost every[1] value ω in Ω a vector in ℓ_2.

For any sequence ν in ℓ_2, the inner product $\langle\nu,\boldsymbol{f}(\omega)\rangle_{\ell_2}$ is a scalar valued function of ω in Ω. We say that $\boldsymbol{f}(\omega)$ is *weakly measurable* if $\langle\nu,\boldsymbol{f}(\omega)\rangle_{\ell_2}$ is a Lebesgue measurable function on Ω for every ν. The set of weakly measurable functions $\boldsymbol{f}$ for which the integral

$$\int_\Omega \|\boldsymbol{f}(\omega)\|_{\ell_2}^2 d\omega < \infty \tag{A.5}$$

converges is a Hilbert space (see for example (Balakrishnan 1981)), with norm given by the square root of the integral in (2.6), and a well-defined inner product given by

$$\langle\boldsymbol{f},\boldsymbol{g}\rangle \triangleq \int_\Omega \langle\boldsymbol{f}(\omega),\boldsymbol{g}(\omega)\rangle_{\ell_2}\,d\omega. \tag{A.6}$$

The Hilbert space just defined is an L_2 space of functions mapping Ω to ℓ_2, and it is denoted by $L_2(\Omega;\ell_2)$.

[1] Respect to the standard Lebesgue measure.

Proof of Lemma 2.2 Let $Y(j\omega)$ be in L_2. Then we have that

$$\begin{aligned} \|Y\|^2 &= \int_{-\infty}^{\infty} |Y(j\omega)|^2 \, d\omega & (A.7)\\ &= \sum_{k=-\infty}^{\infty} \int_{(2k-1)\omega_N}^{(2k+1)\omega_N} |Y(j\omega)|^2 \, d\omega \\ &= \sum_{-\infty}^{\infty} \int_{-\omega_N}^{\omega_N} |Y(j(\omega + k\omega_s))|^2 d\omega. & (A.8)\end{aligned}$$

As $\|Y\|^2$ is finite by assumption, the series $\sum \int |Y_k(j\omega)|^2 \, d\omega$ is convergent. Then, by Levi's Theorem (Riesz and Sz.-Nagy 1955), we can interchange summation and integration in (A.8), and using (2.4), we have that

$$\begin{aligned} \sum_{-\infty}^{\infty} \int_{-\omega_N}^{\omega_N} |Y_k(j\omega)|^2 d\omega &= \int_{-\omega_N}^{\omega_N} \sum_{-\infty}^{\infty} |Y_k(j\omega)|^2 d\omega & (A.9)\\ &= \|\boldsymbol{y}\|^2 & (A.10)\end{aligned}$$

From (A.7)-(A.10) it follows that there is an isometry between $L_2(-\infty, \infty)$ and $L_2(\Omega_N; \ell_2)$. To see that the isometry is isomorphic, we have to show that it is *onto*, that is, *each* function in $L_2(\Omega_N; \ell_2)$ is the image of a function in $L_2(-\infty, \infty)$. It will suffice to show that this is the case for each element in a basis for $L_2(\Omega_N; \ell_2)$.

Let $\{\phi_k\}_{k=-\infty}^{\infty}$ be an orthonormal basis for ℓ_2, and $\{\psi_k(\omega)\}_{k=-\infty}^{\infty}$ an orthonormal basis for $L_2(\Omega_N)$. It can be proved that the sequence

$$\{\psi_n(\omega) \, \phi_m\}_{n,m=-\infty}^{\infty}$$

is an orthonormal base on $L_2(\Omega_N; \ell_2)$. Now, take for example $\psi_n(\omega) \, \phi_m$, for fixed integers n, m. This element of $L_2(\Omega_N; \ell_2)$ is precisely

$$\begin{array}{cc} \begin{bmatrix} \vdots \\ 0 \\ \psi_n(\omega) \\ 0 \\ \vdots \end{bmatrix} & \begin{array}{c} \vdots \\ {\scriptstyle m+1} \\ {\scriptstyle m} \\ {\scriptstyle m-1} \\ \vdots \end{array} \end{array}, \qquad (A.11)$$

which corresponds to the function

$$\psi(\omega) = \begin{cases} \psi_n(\omega) & \text{if } \omega \in [-\omega_N + m\omega_s, \omega_N + m\omega_s] \\ 0 & \text{otherwise} \end{cases} . \qquad (A.12)$$

But $\psi(\omega)$ is in $L_2(-\infty, \infty)$, as it is a function of finite support and integrable there. Therefore, every element in $L_2(\Omega_N; \ell_2)$ is the image of an element in $L_2(-\infty, \infty)$ and the proof is completed. □

Appendix B Computation of $(|PH|^2)_d(e^{j\omega T})$ for a PC GSHF

We consider Figure 3 where $H(s)$ is the frequency response of a PC GSHF. The PC GSHF is defined by a simple function

$$h(t) = \begin{cases} h_1 & \text{if } t \in [0, T/N) \\ h_2 & \text{if } t \in [T/N, 2T/N) \\ \vdots & \vdots \\ h_N & \text{if } t \in [(N-1)T/N), T \end{cases} , \qquad (B.1)$$

where $h_1, h_2, \ldots, h_N$ is a sequence of real numbers (Figure 17). The frequency response for this hold is given by (Middleton and Freudenberg 1993)

$$H(s) = \frac{1-e^{-sT/N}}{s}\sum_{k=1}^{N} h_k e^{-sT(k-1)/N}.$$

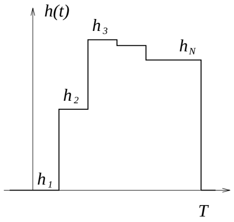

Figure 17: Impulse response of a PC GSHF.

Introduce the polynomial $p_h(z) \triangleq h_1 + h_2\, z + \cdots + h_N z^{N-1}$. Then $H(s)$ compacts to

$$H(s) = \frac{1-e^{-sT/N}}{s} p_h(-sT/N).$$

Then

$$\begin{aligned} H(-s) &= -\frac{1-e^{sT/N}}{s} p_h(sT/N) \\ &= e^{sT}\frac{1-e^{-sT/N}}{s} p_g(-sT/N), \end{aligned}$$

where $p_g(z) \triangleq h_N + h_{N-1}\, z + \cdots + h_1 z^{N-1}$. Notice here how the response of the "conjugated" hold is clearly a causal hold with a time shift of one sampling period. The trick used to handle the non-causality of the resulting $H(s)H(-s)$ is similar to the one used for LTI GSHF. Write

$$\begin{aligned} H(s)H(-s) &= \frac{e^{sT}}{s^2}(1-2e^{-sT/N}+e^{-2sT/N})\, p_h(e^{-sT/N})\, p_g(e^{-sT/N}) \\ &= \frac{e^{sT}}{s^2}\, p_f(e^{-sT/N}), \end{aligned}$$

where $p_f(z) = f_1 + f_2 z + \cdots + f_N z^{N-1} + \cdots + f_{2N+1} z^{2N}$ is a $2N$-degree polynomial whose coefficients, are given by the discrete convolution

$$[f_1, f_2, \ldots, f_{2N+1}] = [1, -2, 1] * [h_1, h_2, \ldots, h_N] * [h_N, h_{N-1}, \ldots, h_1].$$

Following the lines used in §3, the output of the compound hold can be written as

$$u(t) = h_{c1}(t-kT)\, r_{k+1} + h_{c2}(t-kT)\, r_k \quad \text{for } t \in [kT, (k+1)T), \tag{B.2}$$

where

$$h_{c1}(t) \triangleq \begin{cases} f_1 t & t \in [0, T/N) \\ f_1 t + f_2(t-T/N) & t \in [T/N, 2T/N) \\ \vdots & \vdots \\ \displaystyle\sum_{i=1}^{N} f_i t - (i-1) f_i T/N & t \in [(N-1)T/N, T) \end{cases} \tag{B.3}$$

and

$$h_{c2}(t) \triangleq \begin{cases} \sum_{i=1}^{N+1} f_i(t+T) - (i-1)f_iT/N & t \in [0, T/N) \\ \sum_{i=1}^{N+2} f_i(t+T) - (i-1)f_iT/N & t \in [0, T/N) \\ \vdots & \vdots \\ \sum_{i=1}^{2N} f_i(t+T) - (i-1)f_iT/N & t \in [0, T/N) \end{cases} \tag{B.4}$$

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