Frequency Response of Generalized Sampled-data Hold Functions[*]

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Technical Report EE9509
February 1995

Abstract

Alternatives to the traditional Zero Order Hold known as Generalized Sampled-data Hold Functions have been proposed by several authors. More recently, it has been noticed that these new holds incur intersample behavior penalties in many cases. In studying such behavior, one of the issues of relevance is the "frequency response" of the hold. For example, it is known that "non-minimum phase" zeros of the hold impose design tradeoffs in the continuous-time response to disturbances of the sampled-data system. In this paper, we go deeper into the analysis of the frequency response of this devices, and provide some preliminary results on zero location and integral constraints satisfied by the hold. One of the key results of this paper is to show a number of circumstances in which non-minimum phase hold responses may occur. We also show that asymmetric holds have zero locations off the $j\omega$-axis, and therefore, tend to have poor frequency responses, which will deteriorate sensitivity and robustness properties of the sampled-data system.

Keywords: Sampled data systems, Digital-analog conversion, Frequency response, Generalized sampled-data hold functions.

1 Introduction

Generalized Sampled-data Hold Functions (GSHFs) have been proposed to approach several control problems that do not have answers in the traditional sampled-data framework based on a Zero Order Hold (ZOH) (Kabamba 1987, Bai and Dasgupta 1990, Yan *et al.* 1994, Er *et al.* 1994). For example, Kabamba (1987) suggests a GSHF to simultaneously stabilize two systems that violate

[*]Revised March 9, 1995
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the interlacing parity property; Er and Anderson (1994) use the discrete zero-placement capability of GSHFs to achieve perfect discrete-time loop transfer recovery for non-minimum phase systems.

However, serious robustness and sensitivity problems associated to the use of GSHFs, have been pointed out by several authors (Feuer and Goodwin 1994, Freudenberg *et al.* 1994, Zhang and Zhang 1994), showing that many of the most impressive features of GSHFs come at an expensive cost. Moreover, it has been shown (Freudenberg *et al.* 1993) that essential continuous-time design limitations are inherited by the sampled-data system, irrespective of the particular discretization method employed.

On the other hand, it has been recently shown that the general solution to the H_∞-control problem for sampled-data systems, involves a GSHF (*e.g.* Sun *et al.* 1993). Hence, the issue of whether the increased complexity of GSHF design is justified by an overall performance improvement, seems to be still a matter of debate.

Frequency-domain methods have proven to be of great importance in understanding sensitivity and robustness properties of sampled-data systems (*e.g.* Dullerud and Glover 1993, Hagiwara *et al.* 1994). Particularly crucial to these issues, is the frequency response and the zeros of the hold device. For example, open right half plane (ORHP) zeros of the hold may lead to poorly conditioned discretized systems (Middleton and Freudenberg 1995, Middleton and Xie 1994) with serious sensitivity difficulties in the continuous-time response (Freudenberg *et al.* 1993).

In this paper, we study the frequency response, zero location and integral constraints of GSHFs. This allows us to go deeper into the understanding of the tradeoffs associated with the use of these devices. One of the key results of this paper is that holds with zeros off the $j\omega$-axis tend to have poor frequency response, which in turn worsens the overall sensitivity and robustness properties of the sampled-data system (Freudenberg *et al.* 1993).

The organization of the paper is as follows. In §2, we collect several properties of the frequency response of a GSHF. Among these properties are some interesting relations between the frequency response of a generalized hold and the response of a ZOH. In §3, we study the distribution of zeros of the response of GSHFs. In particular, we show a number of circumstances where "non-minimum phase" hold responses, which are known to bring in design limitations to the sampled-data system, may occur. In §4, we present integral constraints satisfied by the hold response, which lead to the previously mentioned result about the additional penalties introduced by zeros off the $j\omega$-axis. Finally, we provide some concluding remarks in §5.

For simplicity, we shall consider only single-input single-output (SISO) devices, although most of our results may be generalized to the multiple-input multiple-output case.

2 Generalized sampled-data holds

The most standard and simplest digital-to-analog (D/A) converter in digital control implementations is the ZOH. A ZOH converts a sequence of discrete values $\{u_k\}_{k=0}^{\infty}$ with sampling period T to a piecewise-constant signal $u(t)$, defined as

$$u(t) = u_k \quad \text{for } kT \le t < (k+1)T.$$

It is well known that the ZOH can be seen as a "filter" with a constant finite impulse response,

$$h(t) = \begin{cases} 1 & t \in [0, T) \\ 0 & \text{otherwise} \end{cases},$$

to which a familiar frequency response function is associated,

$$H(s) = \frac{1 - e^{-sT}}{s}.$$

On the other hand, a GSHF uses a suitable function, $h(t)$, to hold the discrete values through the sampling interval, rather than keeping them constant as in the case of the ZOH (Figure 1). In this way, the GSHF D/A conversion is defined as

$$u(t) = h(t - kT)u_k \quad \text{for } kT \leq t < (k+1)T. \tag{2.1}$$

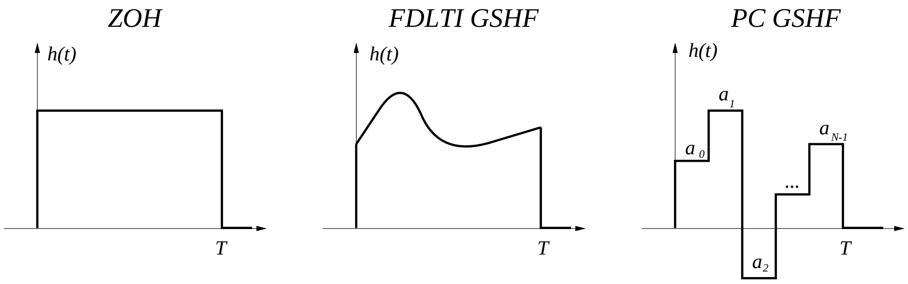

Figure 1: Impulse response of sampled-data hold functions.

We shall assume throughout the paper that the hold function satisfies the following mild technical conditions.

Assumption 1
Assume that

(i) $|h(t)|$ *is bounded and has compact support on* $[0, T)$

(ii) $h(t)$ *has finitely many points of discontinuity and bounded derivative,* $h'(t) = dh(t)/dt$, *where it is defined.*

(iii) $\int_0^T h(t)\, dt \neq 0$.

Under this assumptions, we define the associated GSHF frequency response function as a Laplace transform,

$$H(s) = \int_0^T e^{-st} h(t)\, dt. \tag{2.2}$$

Note that condition (iii) is equivalent to $H(0) \neq 0$, so, in other words, we consider holds with non-zero DC-gain, which is a quite reasonable assumption from a practical point of view.

Since the integration in (2.2) is of a bounded function with compact support in $[0, T)$, the frequency response of a GSHF is an *entire function*. In this section, we collect some interesting properties of $H(s)$ arising from this fact. In Subsections 2.1 and 2.2, we deal with general properties of the frequency response of a GSHF: norms and reconstruction from boundary values. In Subsection 2.3, we compute the frequency response of two particular varieties of GSHFs, common in the related literature.

2.1 Norms and the frequency response of a GSHF.

We recall that an entire function, $F(s)$, is a function defined and analytic for all finite values of the complex variable s. Let $H(s)$ be the frequency response of a GSHF defined by (2.2), and suppose that $h(t)$ is in $L_2(0,T)$, that is

$$||h||_2^2 = \int_0^T |h(t)|^2\, dt < \infty.$$

A well known result of Paley and Wiener (1934) says that the Laplace transform of a square integrable function vanishing outside the interval $(-T,T)$ is an entire function of order 1 and type T (*e.g.* Markushevich 1965). Moreover, since $h(t)$ actually vanishes outside $[0,T)$, $H(s)$ is uniformly bounded on the closed right half plane (CRHP). This can be easily verified from the following inequalities,

$$\begin{aligned} |H(re^{j\theta})|^2 &\le \left(\int_0^T |h(t)|^2\, dt\right)\left(\int_0^T e^{-2rt\cos\theta}\, dt\right) \\ &= ||h||_2^2 \left(\frac{1-e^{-2rT\cos\theta}}{2r\cos\theta}\right). \end{aligned}$$

Hence, we see that if θ belongs to the interval $[-\pi/2, \pi/2]$, then

$$|H(re^{j\theta})| \le \sqrt{T}\,||h||_2. \tag{2.3}$$

In particular, (2.3) tells us that if the infinity norm, $||H||_\infty = \sup_\omega |H(j\omega)|$, is large, then the 2-norm,

$$||H||_2 = \left(\int_{-\infty}^{\infty} |H(j\omega)|^2\, d\omega\right)^{1/2},$$

will also be large, since by Parseval's formula $||H||_2 = \sqrt{2\pi}\,||h||_2$.

Denote by $\omega_s = 2\pi/T$ the *sampling frequency*. Another interesting connection between frequency and time domain values is given by the following lemma.

Lemma 2.1 (Parseval's equality for holds.)
For any real number ω and any $H(s)$ defined by (2.2),

$$\frac{1}{T}\sum_{k=-\infty}^{\infty} |H(j\omega + jk\omega_s)|^2 = ||h||_2^2 \tag{2.4}$$

Proof: Consider the function $f_\omega(t) = h(t)\, e^{-j\omega t}$, with compact support in the interval $[0,T)$. Its Fourier series representation is

$$f_\omega(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{jk\omega_s t}, \qquad \text{for } t \text{ in } [0,T)$$

where the Fourier coefficients are

$$\begin{aligned} c_k &= \frac{1}{T}\int_0^T f_\omega(t)\, e^{-jk\omega_s t}\, dt \\ &= \frac{1}{T} H(j\omega + jk\omega_s) \end{aligned} \tag{2.5}$$

Hence, by Parseval's formula we have that

$$\frac{1}{T}\int_0^T |f_\omega(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

The result is then obtained by noting that $|f_\omega(t)| = |h(t)|$, and replacing c_k from (2.5). □

An interpretation in terms of frequency aliasing can be given to the above result. If $\|H\|_\infty$ is large, then evaluation of (2.4) at small values of w still gives a large sum. Assuming (WLOG) that $H(0) = 1$, then there must be a significant number of other terms adding to give a large 2-norm. Thus, a peak of $|H(j\omega)|$ necessarily implies a lot of frequency "folding" going on. In the case of a ZOH, $H(s)$ has zeros at integer multiples of the sampling frequency, ω_s, so evaluating (2.4) at $\omega = 0$ we see that the ZOH has the minimum L_2-norm over all the holds that satisfy $H(0) = 1$ (See also Feuer and Goodwin 1995, § 3.4).

Yet a last property of GSHFs gives us the "gain" of the hold viewed as an input output operator. Let $\mathcal{H}$ denote the operator mapping ℓ_p to L_p, $1 \le p \le \infty$, and defined by (2.1). The lemma below is a generalization to GSHFs of a result for the ZOH in Francis (1991).

Lemma 2.2 (Input-output norm of a hold operator.)
The hold operator $\mathcal{H} : \ell_p \to L_p$ is bounded and of norm $\|h\|_p$.

Proof: We prove this for $p < \infty$; the case $p = \infty$ follows similar steps. Let $u(t)$ be a function in L_p, and $v = \{v_k\}_{k=-\infty}^{\infty}$ a sequence in ℓ_p, such that $u = \mathcal{H}v$. Then,

$$\begin{aligned}
\|u\|_p^p &= \int_{-\infty}^{\infty} |u(t)|^p\,dt \\
&= \sum_{k=-\infty}^{\infty} \int_{kT}^{(k+1)T} |h(t-kT)v_k|^p\,dt \\
&= \left(\int_0^T |h(t)|^p\,dt\right)\left(\sum_{k=-\infty}^{\infty} |v_k|^p\right) \\
&= \|h\|_p^p\,\|v\|_p^p.
\end{aligned}$$

□

For $p = \infty$, it follows from Lemma 2.2, that a large value of $\|h\|_\infty$ implies a "high gain" hold, viewed as a bounded-input bounded-output (BIBO) device. Combining (2.3) with the fact that $\|h\|_2 \le \sqrt{T}\,\|h\|_\infty$, we obtain

$$\|H\|_\infty \le T\,\|h\|_\infty.$$

So, we see that, for a given sampling rate, a large peak in $|H(j\omega)|$ also implies a large BIBO gain. Since the output of the hold is typically the input to the plant, such large gain may introduce serious difficulties due to actuator saturations, present in most real systems (Gilbert 1992).

2.2 GSHF frequency responses from boundary values.

Analytic functions can be reconstructed from their boundary values by means of integral formulas like Poisson's or Cauchy's (*e.g.* Hoffman 1962). Not surprisingly, GSHF frequency responses can be recovered from similar relations, although it is interesting that the frequency response of a ZOH is involved in the reconstruction. Denote by H_{ZOH} the response of a ZOH,

$$H_{\mathrm{ZOH}}(s) = \frac{1 - e^{-sT}}{s}.$$

The following result is a straightforward consequence of the Fourier representation of $h(t)$ (See also Feuer and Goodwin 1995).

Lemma 2.3 (Hold frequency response from boundary values: "discrete" version.) *For any complex number s and any $H(s)$ defined by (2.2),*

$$H(s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H(jk\omega_s)\, H_{ZOH}(s - jk\omega_s) \tag{2.6}$$

Proof: Expand $h(t)$ into Fourier series,

$$h(t) = \sum_{K=-\infty}^{\infty} c_k\, e^{jk\omega_s t}, \quad \text{where } c_k = \frac{1}{T}\int_0^T h(t)\, e^{-jk\omega_s t}\, dt = H(jk\omega_s). \tag{2.7}$$

Then, the Laplace transform of (2.7) gives

$$H(s) = \frac{1}{T} \sum_{K=-\infty}^{\infty} H(jk\omega_s)\, \frac{1 - e^{-sT}}{s - jk\omega_s},$$

completing the proof. □

It is interesting that there exists a — not so obvious — "continuous" version of the above formula, arising from properties of Paley-Wiener spaces of entire functions (De Branges 1968).

Lemma 2.4 (Hold frequency response from boundary values: "continuous" version.) *For any complex number s and any $H(s)$ defined by (2.2),*

$$H(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)\, H_{ZOH}(s - j\omega) d\omega \tag{2.8}$$

Proof: If $f(t)$ is a function that vanishes outside the interval $[-T/2, T/2]$, and its Laplace transform, $F(s)$, is such that $\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega < \infty$, then $F(s)$ is an entire function of type $T/2$ (De Branges 1968, p. 45). Moreover, for any complex number s,

$$F(s) = \int_{-\infty}^{\infty} F(j\omega)\, \frac{\sinh(j\omega T/2 - sT/2)}{\pi(j\omega - s)}\, d\omega. \tag{2.9}$$

Applying (2.9) to the function $H(s)\, e^{-sT/2}$ gives the result. □

2.3 Two simple types of GSHFs.

One way to build a GSHF is through the response of a finite dimensional linear time-invariant system. This covers, for example, the type of GSHFs suggested by Kabamba (1987).

Definition 2.1 (Finite Dimensional Linear Time-invariant GSHF.) Given suitable dimensioned matrices K, L and M, we define a finite dimensional linear time-invariant GSHF (FDLTI GSHF) by the impulse response

$$h(t) = Ke^{L(T-t)}M \quad \text{for } t \in [0, T). \tag{2.10}$$

The frequency response function of a FDLTI GSHF is easily computed from its definition.

Lemma 2.5 (Frequency response function of a FDLTI GSHF.)
Given a FDLTI GSHF defined by matrices K, L and M, the response function is

$$H(s) = K(sI + L)^{-1}(e^{LT} - e^{-sT}I)M. \tag{2.11}$$

Proof: Straightforward from Definition 2.1. □

Yet another interesting class of GSHFs are the piecewise-constant GSHFs, where the impulse response function $h(t)$ is a "staircase", typically with a regular partition in time of the sampling interval $[0, T)$.

Definition 2.2 (Piecewise-constant GSHF.) A piecewise-constant GSHF (PC GSHF) is given by the following impulse response:

$$h(t) = \begin{cases} a_0 & \text{if } t \in [0, T/N), \\ a_1 & \text{if } t \in [T/N, 2T/N), \\ \dots & \dots \\ a_{N-1} & \text{if } t \in [(N-1)T/N, T). \end{cases} \tag{2.12}$$

Clearly (*e.g.* Yan *et al.* 1994), this type of hold can arbitrarily approximate any GSHF of the form (2.10) by taking N large, and are reasonably simple to implement. The frequency response function for a PC hold is given by the following lemma from Middleton and Freudenberg (1995).

Lemma 2.6 (Frequency response function of a PC GSHF.)
The frequency response function of a PC GSHF defined by (2.12) is:

$$H(s) = \frac{1 - e^{-sT/N}}{s}\mathcal{A}(e^{-sT/N}), \tag{2.13}$$

where $\mathcal{A}(z)$ is the polynomial

$$\mathcal{A}(z) \triangleq \sum_{k=0}^{N-1} a_k z^k. \tag{2.14}$$

Proof: By definition,

$$\begin{aligned} H(s) &= \int_0^T e^{-st}\, h(t)\, dt \\ &= \sum_{k=0}^{N-1} \int_{kT/N}^{(k+1)T/N} a_k e^{-st}\, dt, \end{aligned}$$

from which the result follows. □

3 Distribution of zeros of GSHFs

The location of zeros of an entire function has close connections with its growth properties on the complex plane. As we shall see in §4, some of these connections can be quantified in the form of integral constraints on the GSHF frequency response.

Zeros of the hold are also important in sampled-data control systems. For example, it is known that zeros of $H(s)$ in the ORHP impose a design tradeoff upon the fundamental component of the response of the system to disturbances (Freudenberg *et al.* 1993).

This section focuses on the distribution of zeros of $H(s)$. In Subsection 3.1, we derive a necessary condition for a GSHF to have a frequency response with all its zeros on the $j\omega$-axis. In Subsection 3.2 we precisely describe the location and asymptotic distribution of the zeros of PC and FDLTI holds. To finish, we discuss in Subsection 3.3 some implications arising from discrete zero placement in GSHF systems.

3.1 GSHFs with all zeros on the $j\omega$-axis.

An appealing property of a hybrid control system with a ZOH and an integrator is its ability to reject step disturbances. This arises from the fact that the ZOH frequency response has zeros at multiples of the sampling frequency $\omega_s = 2\pi/T$ on the $j\omega$-axis,

$$H_{\mathrm{ZOH}}(jk\omega_s) = 0, \quad \text{for } k = \pm1, \pm2, \ldots.$$

In addition, these zeros contribute to diminish high frequency components of the plant response that are aliased down to low frequencies. This fact is particularly important in sampled-data control applications, where the low frequency range is typically of great interest.

The response of a GSHF, on the other hand, need not have zeros at these frequencies, and thus high frequency plant behavior (and uncertainty) may have significant effect on the low frequency range of the hybrid control system. Indeed, it is known (Freudenberg *et al.* 1993) that ORHP zeros of the hold impose additional design limitations in the hybrid system, and may originate an ill-conditioned discrete-time system (Middleton and Freudenberg 1995, Middleton and Xie 1994). Moreover, as it will become clear in §4, holds with zeros off the $j\omega$-axis — even OLHP zeros — tend to have poor frequency responses.

For these reasons, it is relevant to characterize the class of GSHFs with all its zeros on the $j\omega$-axis. The following proposition gives a necessary condition for this.

Proposition 3.1
If $H(s)$ has all its zeros on the $j\omega$-axis, then $H(s) = H(-s)\,e^{-sT}$.

Proof: Suppose that $H(s)$ has no zeros in the ORHP. Then, $\log H(s)$ is analytic on the ORHP and, moreover, it admits a Poisson Integral representation (Levinson and Redheffer 1970), so for any complex number $s = x + jy$, with $x > 0$,

$$\log H(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} \log|H(j\omega)|\, W(s,w)\, d\omega, \tag{3.1}$$

where $W(s,w)$ is the Poisson kernel for the half plane,

$$W(s,\omega) = \begin{cases} \dfrac{2x}{x^2+\omega^2} & \text{if } y = 0, \\[2ex] \dfrac{x}{x^2+(\omega-y)^2} + \dfrac{x}{x^2+(\omega+y)^2} & \text{if } y \neq 0. \end{cases} \tag{3.2}$$

Now suppose that $H(s)$ has no zeros in the OLHP. Then, $\log H(-s)e^{-sT}$ is analytic in the ORHP, and also admits a Poisson representation,

$$\log H(-s)e^{-sT} = \frac{1}{\pi}\int_{-\infty}^{\infty} \log |H(-j\omega)|\, W(s,w)\, d\omega, \tag{3.3}$$

Since $|H(j\omega)| = |H(-j\omega)|$, from (3.1) and (3.3) we conclude that $H(s) = H(-s)e^{-sT}$. □

This condition on the frequency response can be translated into a condition in terms of the impulse response, $h(t)$. We say that $h(t)$ has "even" symmetry if $h(t) = h(T-t)$ for each t in the interval $[0,T)$. Then, the following corollary to Proposition 3.1 establish that holds without zeros off the $j\omega$-axis are necessarily even symmetric.

Corollary 3.2
If $H(s)$ has all its zeros on the $j\omega$-axis, then $h(t)$ has "even" symmetry.

Proof: From definition,

$$H(s) = \int_0^T e^{-st}\, h(t)\, dt.$$

On the other hand,

$$\begin{aligned} e^{-sT}\, H(-s) &= \int_0^T e^{-s(T-t)}\, h(t)\, dt \\ &= \int_0^T e^{-s(t)}\, h(T-t)\, dt. \end{aligned}$$

Hence, $H(s) = H(-s)e^{-sT}$ if and only if $h(t) = h(T-t)$ for each t (a.e.) in the interval $[0,T)$. Proposition 3.1 completes the proof. □

3.2 Zeros of PC and FDLTI GSHFs.

It is difficult to make general statements about the distribution of the zeros of a GSHF. However, for important special cases, the locations and asymptotic distribution of these zeros can be described precisely. The following lemma characterizes exhaustively the zeros of PC holds, which are the GSHFs of greatest practical significance.

Lemma 3.3 (Zeros of a piecewise constant GSHF.)
Consider a GSHF given by (2.12) with associated frequency response function $H(s)$ given by (2.13) and (2.14). Assume that $a_0 \neq 0$. Then $H(s)$ has zeros at

$$s = j\ell N\omega_s, \quad \text{where } \ell = \pm 1, \pm 2, \ldots, \tag{3.4}$$

and

$$s = -\frac{N}{T}\log \xi_i + jkN\omega_s, \quad \text{with } k = 0, \pm 1, \pm 2, \ldots, \tag{3.5}$$

where ξ_i, with $i = 1, 2, \ldots, N$, is any zero of $\mathcal{A}(z)$.

Proof: From Lemma 2.6, $H(s)$ can be written as (2.13). The zeros of

$$\frac{1 - e^{-sT/N}}{s}$$

are given by (3.4). It remains, therefore, to determine the zeros of $\mathcal{A}(e^{-sT/N})$, which are given precisely by (3.5). The assumption that $a_0 \neq 0$ implies that $\xi_i \neq 0$ for every i, and hence $\log \xi_i$ is defined. $\square$

Zeros of FDLTI holds are harder to determine, but again, we can say something in particular cases. Consider a hold defined by (2.10), and suppose that $h(t)$ is not identically zero. Let m and n denote the smallest nonnegative integers such that $KL^n M \neq 0$ and $KL^m e^{LT} M \neq 0$. Define

$$\eta \triangleq \frac{K(-L)^m e^{LT} M}{K(-L)^n M}. \tag{3.6}$$

Note that

$$\eta = \frac{h^{(m)}(0)}{h^{(n)}(T)},$$

where $h^{(k)}(t)$ denotes the k-derivative of $h(t)$.

Lemma 3.4 (High Frequency Zeros of a FDLTI GSHF.)
If $H(s)$ is the frequency response of a FDLTI GSHF, then it has a sequence of zeros $\{\gamma_\ell\}_{\ell=1}^{\infty}$ converging to infinity. Furthermore, these zeros converge to the roots of the equation $\eta = e^{-\phi T}\phi^{n-m}$. In particular, if $n = m$, the zeros converge to the sequence defined by

$$\phi_\ell = -\frac{1}{T}\log \eta + j\ell\omega_s, \quad \ell = 0, \pm 1, \pm 2, \ldots \tag{3.7}$$

Proof: We first prove, by contradiction, that $n, m < N-1$, where $L \in \mathbb{R}^{N\times N}$. Suppose $n \geq N-1$. Then $KL^i M = 0$ for $i = 1, 2, \ldots, N-1$. By the Cayley-Hamilton Theorem (Chen 1984), $KL^i M = 0$ for *all* i. However, since $d^i h(t)/dt^i\big|_{t=T} = KL^i M$, this implies that $h(t)$ is identically zero (Chen 1984, Appendix B). This argument also shows that the coefficient of e^{-sT} in (2.11) is nonzero. A similar argument shows that $m < N-1$.

Consider next the ratio

$$F(s) = \frac{K(sI+L)^{-1} M e^{-sT}}{K(sI+L)^{-1} e^{LT} M}.$$

Note that $F(s)$ is analytic in a neighborhood of infinity and has an essential singularity at infinity (the latter is due to the presence of e^{-sT} with a nonzero coefficient). It follows from the Great Picard Theorem (Conway 1973, p. 302) that in each neighborhood of infinity $F(s)$ assumes each complex number with one possible exception, infinitely many times. Because of the term e^{-sT}, this exceptional value must equal zero. Hence there exists a sequence $\{\gamma_\ell\}_{\ell=1}^{\infty}$ converging to infinity such that $F(\gamma_\ell) = 1$, and thus $H(\gamma_\ell) = 0$ for all ℓ.

We now show that if $n = m$, then the γ_ℓ's necessarily converge to the values given in (3.7). To do this note that for each integer k,

$$(sI+L)^{-1} = \frac{1}{s}\left(\left(\frac{-L}{s}\right)^{k+1}\left(I + \frac{L}{s}\right)^{-1} + \sum_{i=0}^{k}\left(\frac{-L}{s}\right)^{i}\right)$$

Using this identity and the definitions of m and n yields

$$H(s) = \frac{1}{s^{m+1}}\left(Q(s) - R(s)\,e^{-sT}\,s^{m-n}\right),$$

where

$$Q(s) = K(-L)^m\,e^{LT}M + \frac{1}{s}K(-L)^{m+1}\left(I + \frac{L}{s}\right)^{-1} e^{LT}M,$$

and

$$R(s) = K(-L)^n M + \frac{1}{s} K(-L)^{n+1} \left(I + \frac{L}{s} \right)^{-1} M.$$

For γ_ℓ a zero of $H(s)$, we have $Q(\gamma_\ell) = \gamma_\ell^{m-n}\, R(\gamma_\ell)\, e^{\gamma_\ell T}$. Note that for ℓ sufficiently large, $Q(\gamma_\ell)$ and $R(\gamma_\ell)$ are both nonzero and constant. Taking logarithms and rearranging shows that there exists k such that

$$\gamma_\ell = -\frac{1}{T} \log \frac{Q(\gamma_\ell)}{R(\gamma_\ell)} + \frac{m-n}{T} \log \gamma_\ell + jk\omega_s.$$

If $n = m$, then taking limits yields

$$\gamma_\ell \to -\frac{1}{T} \log \eta + jk\omega_s.$$

Noting that zeros must occur in conjugate pairs yields (3.7). □

A precise description of the zeros of a FDLTI hold is possible in a particular case, as we see in the following lemma.

Lemma 3.5 (Zeros of a FDLTI GSHF (Special Case).)
Consider a FDLTI GSHF, and suppose that $KM \neq 0$. Assume that $L = \lambda I$, where I is the identity matrix and λ is a scalar. Then the zeros of $H(s)$ are located precisely at

$$\gamma_\ell = -\lambda + j\ell\omega_s, \quad \ell = \pm 1, \pm 2, \ldots \tag{3.8}$$

Proof: Since $KM \neq 0$, $H(s)$ is not identically zero. The special structure of L implies that

$$H(s) = KM \frac{e^{\lambda T} - e^{-sT}}{s + \lambda},$$

and the result follows. □

Remark 3.1 Note that since a FDLTI hold will undoubtedly be implemented as a PC hold, the additional difficulty in characterizing zeros of FDLTI holds over PC holds is somehow deprived of practical significance.

3.3 Some implications of discrete zero placement.

Even in the case of a ZOH, a simple classification of the zeros of a discretized system is not possible, except in special cases. Åström *et al.* (1984) have given asymptotic formulae for slow and fast sampling rates. Hara *et al.* (1989) examined parity interlacing properties of real zeros and poles. The key result of their work is that the parity[1] of the number of real zeros between any two real poles is preserved under discretization — except possibly for cases of *pathological sampling* (Kalman *et al.* 1963).

At first glance, it might be expected that since GSHFs are capable of zero assignment in the discretized plant (*e.g.* Bai and Dasgupta 1990, Er and Anderson 1994), that no similar property holds. The proposition below, however, shows that there is a generalization of the result of Hara *et al.* (1989) to the case of GSHFs.

[1]Parity is even (odd) if there is an even (odd) number of zeros between the two poles.

We consider a strictly proper, single-input single-output system, $P(s)$, and a GSHF, $H(s)$, in the discrete plant configuration of Figure 2. Denote by $(PH)_d(z)$ the discretized plant with sampling period T,

$$(PH)_d(z) = \mathcal{Z}\{\mathcal{S}_T\{\mathcal{L}^{-1}\{P(s)H(s)\}\}\}, \tag{3.9}$$

where $\mathcal{L}\{\cdot\}$ and $\mathcal{Z}\{\cdot\}$ denote the Laplace and Zed transforms, respectively, and $\mathcal{S}_T\{\cdot\}$ the sampling operation. We assume that $P(s)$ and $H(s)$ satisfy conditions for non-pathological sampling (Middleton and Freudenberg 1995).

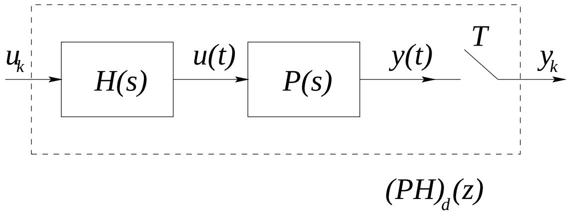

Figure 2: Discrete plant configuration.

Proposition 3.6 (Real pole, zero parity preservation in GSHF systems.)
Suppose that the plant $P(s)$ has simple real poles at $s = p_1$ and $s = p_2$. Then the parity of the number of real zeros of $P(s)H(s)$ in the interval (p_1, p_2) is the same as the parity of the number of real zeros of the discretized plant $(PH)_d(z)$ in the interval $(e^{p_1 T}, e^{p_2 T})$.

Before proving Proposition 3.6, we require the following lemma from Middleton and Freudenberg (1995).

Lemma 3.7 (Discretized plant by Partial Fractions Expansion.)
Suppose $P(s)$ has only simple poles, and thus let

$$P(s) = \sum_{i=1}^{n} \frac{r_i}{s - p_i}.$$

Then,

$$(PH)_d(z) = \sum_{i=1}^{n} r_i \, H(p_i) \frac{e^{p_i T}}{z - e^{p_i T}}.$$

□

Proof of Proposition 3.6: From Lemma 3.7,

$$\begin{aligned}
\lim_{z \to (e^{p_1 T})^+} \operatorname{sgn} (PH)_d(z) &= \operatorname{sgn} r_1 \, H(p_1) \\
&= \lim_{s \to p_1^+} \operatorname{sgn} P(s)H(s).
\end{aligned}$$

Similarly,

$$\lim_{z \to (e^{p_2 T})^-} \operatorname{sgn} (PH)_d(z) = \lim_{s \to p_2^-} \operatorname{sgn} P(s)H(s),$$

from which the result follows. □

Note that for the ZOH case, $H(s)$ has no real zeros, and so we recover the result of Hara *et al.* (1989). Also note that alteration of the zero parity of the discretized plant — *e.g.* as required for strong stabilization of some systems — can only be achieved at the cost of introducing non-minimum phase zeros in the hold response function, $H(s)$, as illustrated in the following example, taken from Kabamba (1987).

Example 3.1 (Simultaneous stabilization by GSHF.) In his Example 2, Kabamba (1987) designs a FDLTI hold that simultaneously stabilizes the two systems given by the following transfer functions:

$$P_1(s) = \frac{1}{s+1} \quad \text{and} \quad P_2(s) = \frac{s-1.5}{s(s-2)}, \tag{3.10}$$

in the parallel configuration of Figure 3. Note that the combined system,

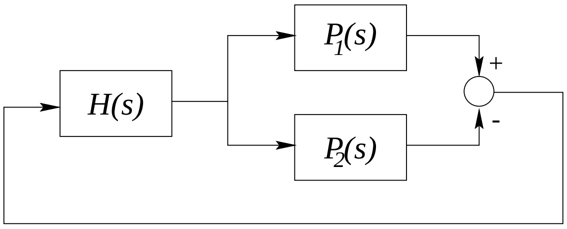

Figure 3: GSHF for simultaneous stabilization.

$$P_1(s) - P_2(s) = \frac{-1.5(s-1)}{s(s+1)(s-2)}, \tag{3.11}$$

does not satisfy the conditions for strong stabilizability, and, since one of them is already asymptotically stable, the two systems cannot be stabilized simultaneously with a single continuous-time dynamic compensator (Vidyasagar 1985). Following the procedures described in Kabamba (1987), we design a FDLTI hold with sampling time $T = 1s$ to simultaneously stabilize the parallel. The hold obtained $H(s)$ is defined by the matrices

$$K = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 87.5619 \\ -616.4937 \\ 1322.6 \end{bmatrix}.$$

The discretized combined plant with this hold is

$$((P_1 - P_2)H)_d(z) = \frac{(z-0.9433)(z-0.3641)}{(z-1)(z-e^{-1})(z-e^2)}, \tag{3.12}$$

and note that it is minimum phase and has one real zero between real poles $z = e^{-1}$ and $z = 1$. Hence, by Lemma 3.6, $(P_1(s) - P_2(s))H(s)$ will have an even number of zeros in the interval $(0, 2)$, and an odd number of zeros in the interval $(-1, 0)$. By inspecting $(P_1(s) - P_2(s))$ in (3.11), we conclude that $H(s)$ has an odd number of zeros in those intervals. Indeed, as we can see in Figure 4, $H(s)$ has a zeros at $s = -0.9967$ and $s = 0.0518$.

Moreover, note that since $H(s)$ has a zero very close to the open loop pole of the continuous plant at the origin, sampling is almost pathological, and so the discretized plant is very close to be non-stabilizable/non-detectable. Indeed, the singular values of the Hankel matrix for the discretized system are

$$\Sigma = \begin{bmatrix} 23163 & 0.1178 & 0.0005 \end{bmatrix},$$

which clearly indicate a very ill-conditioned system.

4 Integral Relations

Integral relations quantifying inherent limitations in the achievable performance of feedback control systems have been known for some time. Bode and Poisson integrals on the sensitivity and

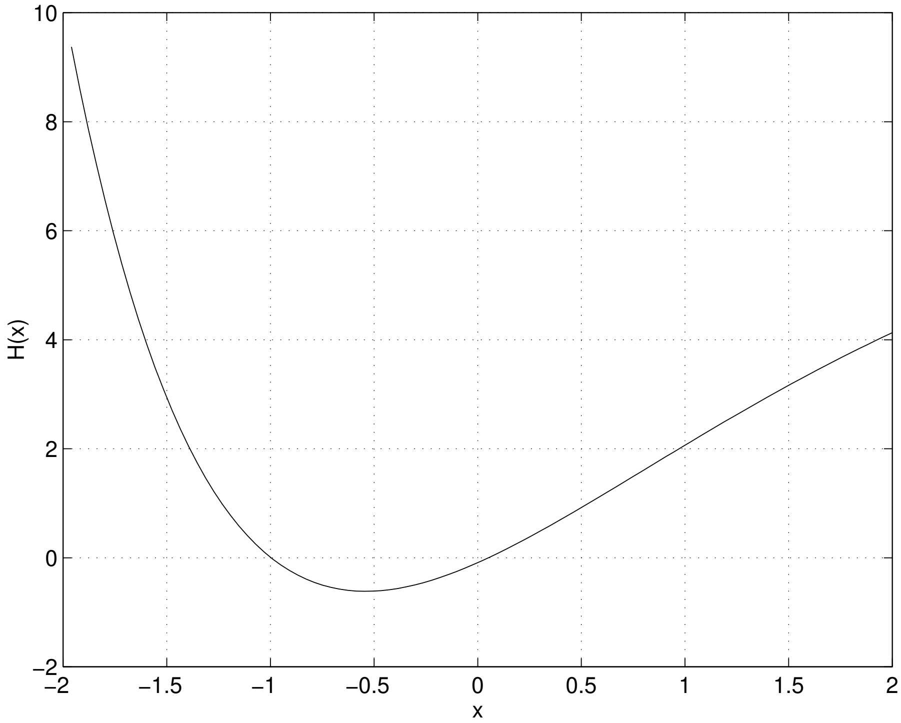

Figure 4: $H(s)$ on the real axis.

complementary sensitivity functions of continuous-time systems have been used to describe design tradeoffs arising from the feedback structure, and the requirement of closed-loop stability (Freudenberg and Looze 1985). Similar results have been also extended to discrete-time systems (Sung and Hara 1988, Middleton and Goodwin 1990), and recently, to sampled-data systems (Freudenberg *et al.* 1993).

In this section, we present two types of integral relations for the frequency response of a GSHF. These integrals translate the connection between growth characteristics and the distribution of zeros, into constraints that the magnitude of the frequency response of the GSHF must satisfy on the $j\omega$-axis.

4.1 A Poisson integral for GSHFs.

Let $H(s)$ be the frequency response function of a GSHF defined by (2.2). Recall the definition of the Poisson kernel for the half plane, $W(s,\omega)$, in (3.2). The following proposition presents a Poisson integral relation for $1 - H(s)$.

Proposition 4.1 (Poisson integral for $1 - H(s)$.)
Let $\xi = x + jy$ be zero of $H(s)$. Then

(i) if $x > 0$,

$$\int_0^\infty \log|1 - H(j\omega)|\, W(\xi,\omega)\, d\omega \geq 0; \tag{4.1}$$

(ii) if $x < 0$,

$$\int_0^\infty \log|1 - H(j\omega)\,e^{j\omega T}|\,W(-\xi,\omega)\,d\omega \geq 0. \tag{4.2}$$

Proof: Since $H(s)$ is entire, so is $F(s) = 1 - H(s)$, and therefore, we may factorize it as

$$F(s) = \tilde{F}(s)\,B(s),$$

where $\tilde{F}(s)$ is an entire function without zeros on the ORHP, and $B(s)$ is the Blaschke product of the sequence of zeros of $F(s)$, $\{\phi_k\}_{k=1}^{N_\phi}$ (with N_ϕ possibly infinite), on the ORHP,

$$B(s) = \prod_{k=1}^{N_\phi} \frac{\phi_k - s}{\overline{\phi}_k + s}.$$

Note that $\log\tilde{F}(s)$ is analytic on the ORHP, and furthermore, it satisfies the conditions for a Poisson Integral representation (*cf.* Proof of Proposition 3.1). Hence, we write,

$$\begin{aligned} \int_0^\infty \log|\tilde{F}(j\omega)|\,W(s,\omega)\,d\omega &= \pi\log|\tilde{F}(s)| \\ &= \pi\log|F(s)| - \pi\log|B(s)|. \end{aligned} \tag{4.3}$$

Since $|\tilde{F}(j\omega)| = |F(j\omega)|$, evaluating (4.3) at an ORHP zero of $H(s)$, and noting that $-\log|B(s)| \geq 0$ for each s in the ORHP, we finally obtain (4.1).

Inequality (4.2) is obtained in a similar way considering the function $F(s) = 1 - H(-s)\,e^{-sT}$. □

It follows from (4.1) and (4.2) that zeros of the hold off the $j\omega$-axis impose design tradeoffs on its frequency response. More specifically, if we require that $|H(jw)|$ be close to 1 over some range of frequencies, it will necessarily show a peak somewhere else. The extent of this difficulty is linked to the relative location of these zeros, as we see now.

Given an interval of frequencies $\Omega = [\omega_1, \omega_2]$ and a point $\xi = x + jy$ in the ORHP, let $\Theta(\xi,\Omega)$ denote the weighted length of Ω by

$$\Theta(\xi,\Omega) = \int_{\omega_1}^{\omega_2} W(\xi,\omega)\,d\omega, \tag{4.4}$$

with $W(\xi,\omega)$ defined by (3.2). See Freudenberg and Looze (1988) for a detailed phase lag interpretation of $\Theta(\xi,\Omega)$.

Suppose that we require $|H(j\omega)|$ to be close to 1 over a low range of frequencies,

$$|1 - H(j\omega)| < \alpha, \qquad \text{for } \omega \text{ in } \Omega = [0,\omega_b], \tag{4.5}$$

with $\omega_b < \omega_s/2$, and α a small number. Then, the following result is a straightforward consequence of Proposition 4.1.

Corollary 4.2
Suppose that $|H(j\omega)|$ *satisfies (4.5). Then, if* $\xi = x + jy$ *is a zero of* $H(s)$*, with* $x \neq 0$,

(i) if $x > 0$,

$$\sup_{\omega > \omega_b} |H(j\omega)| \geq \left(\frac{1}{\alpha}\right)^{M_\alpha(\xi)} - 1, \tag{4.6}$$

(ii) if $x < 0$,

$$\sup_{\omega > \omega_b} |H(j\omega)| \geq \left(\frac{1}{\alpha}\right)^{M_\alpha(-\xi)} - 1, \tag{4.7}$$

where

$$M_\alpha(\xi) = \frac{\Theta(\xi, \Omega)}{\pi - \Theta(\xi, \Omega)}$$

□

Note from (4.6) and (4.7), that the effect of zeros in the OLHP is as detrimental as the effect of zeros in the ORHP. To illustrate this bounds, suppose that we wish to design a hold satisfying specification (4.5), and that due to the type of hold chosen, there will be a real zero $\xi = x$. Assume also that the "bandwidth" ω_b is a 5% of the sampling frequency, and that $\tau_H = 0$. Figure 5 shows plots of the bounds (4.6) and (4.7) relative to the location of the zero in Ω, and for different values of the specification α.

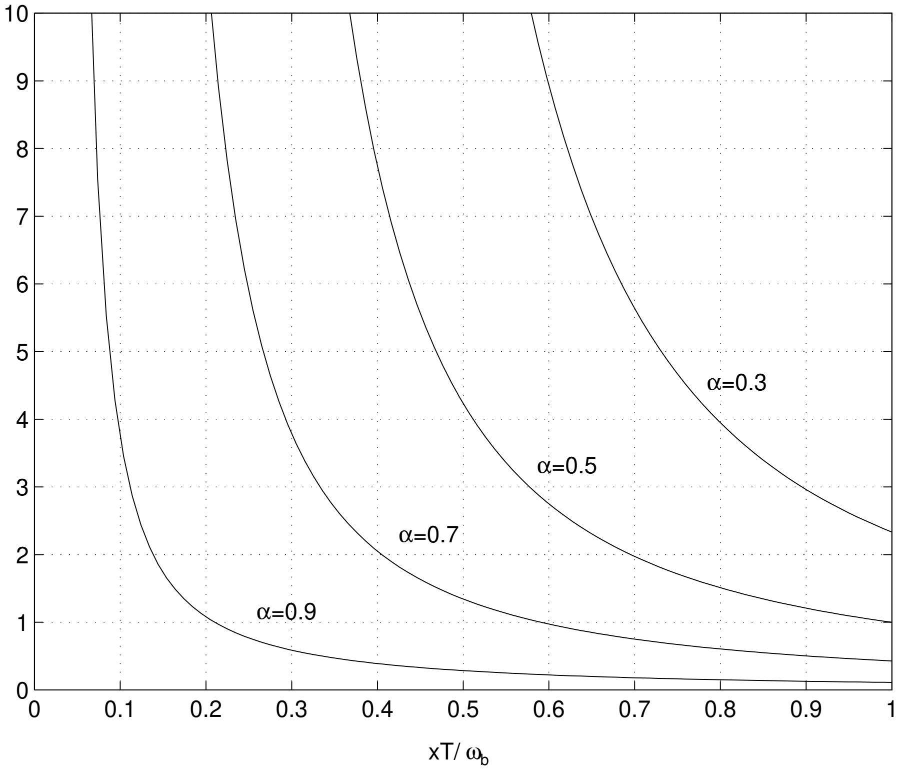

Figure 5: Lower bound (4.6) (or (4.7)) for a real zero.

4.2 Middleton integral for GSHFs.

Another integral relation that evidences the penalties imposed by zeros off the $j\omega$-axis over the frequency response of the hold is the Middleton Integral (*cf.* Freudenberg *et al.* 1993). In this case, we obtain a single integral for both case of zeros in the OLHP and ORHP.

We first present a general expression of this relation, and then, in the particular case of PC holds, a more compact form.

Denote by $\{\lambda_k\}_{k=1}^{N_\lambda}$ and $\{\rho_k\}_{k=1}^{N_\rho}$ the sequence of zeros of $H(s)$, counted with multiplicities, in the OLHP and ORHP, respectively. Typically, $N_\lambda + N_\rho = \infty$. WLOG, we also assume $H(0) = 1$. Then, we have the following result.

Proposition 4.3 (Middleton integral for $H(s)$.)

$$\int_0^\infty \frac{\log|H(j\omega)|}{\omega^2}\,d\omega = \frac{\pi}{4}\left(-T - 2\sum_{k=1}^{N_\lambda}\frac{1}{\lambda_k} + 2\sum_{k=1}^{N_\rho}\frac{1}{\rho_k}\right). \tag{4.8}$$

Proof: Factorize $H(s)$ as $H(s) = \tilde{H}(s)\,B_\rho(s)$, where $B_\rho(s)$ is the Blaschke product of the ORHP zeros of $H(s)$,

$$B_\rho(s) = \prod_{k=1}^{N_\rho}\frac{\rho_k - s}{\overline{\rho}_k + s}.$$

From similar arguments to those in the proof of Proposition 4.1, we obtain the following Poisson Integral relation, which we evaluate at a real $s = x$, $x > 0$,

$$\int_0^\infty \log|H(j\omega)|\,\frac{2x}{x^2+\omega^2}\,d\omega = \pi\log|H(x)| - \pi\log|B_\rho(x)|. \tag{4.9}$$

Dividing both sides of (4.9) by x, and taking the limit when $x \to 0$ yields[2]

$$\int_0^\infty \frac{\log|H(j\omega)|}{\omega^2}\,d\omega = \lim_{x\to 0}\pi\frac{\log|H(x)|}{x} - \lim_{x\to 0}\pi\log|B_\rho(x)|. \tag{4.10}$$

The application of L'Hopital's rule to the limits on the RHS of (4.10), our assumption $H(0) = 1$, and the fact that zeros of $H(s)$ must occur in complex conjugate pairs, yield

$$2\int_0^\infty \frac{\log|H(j\omega)|}{\omega^2}\,d\omega = \left.\frac{dH(s)}{ds}\right|_{s=0} + 2\sum_{k=1}^{N_\rho}\frac{1}{\rho_k}. \tag{4.11}$$

From analogous arguments applied to $H(-s)e^{-sT}$, and noting that $|H(j\omega)| = |H(-j\omega)|$, we obtain an integral relation for the OLHP zeros of $H(s)$,

$$2\int_0^\infty \frac{\log|H(j\omega)|}{\omega^2}\,d\omega = -T - \left.\frac{dH(s)}{ds}\right|_{s=0} - 2\sum_{k=1}^{N_\lambda}\frac{1}{\lambda_k}. \tag{4.12}$$

Finally, adding term-to-term (4.11) and (4.12) yields (4.8), completing the proof. □

In the particular case of a PC hold, the location of zeros is well determined, and therefore, we obtain a more specific result. As it follows from Lemma 3.3, zeros off the $j\omega$-axis for a PC hold are determined by the zeros of the discrete polynomial $\mathcal{A}(z)$. Denote by $\{\phi_k\}_{k=1}^{N_\phi}$ and $\{\psi_k\}_{k=1}^{N_\psi}$ the set of zeros of $\mathcal{A}(z)$ inside and outside the unit circle, respectively (note that there is a finite number of them, $N_\psi + N_\phi \le N$). The following corollary states the analog to Proposition 4.3 for PC holds.

[2] The interchange between limit and integration on the LHS is valid by the Lebesgue Dominated Convergence Theorem (Riesz and Sz.-Nagy 1955). For more details see a similar proof in Freudenberg *et al.* (1993, Lemma B.1).

Corollary 4.4 (Middleton integral for PC GSHFs.)

$$\int_0^\infty \frac{\log|H(j\omega)|}{\omega^2}\,d\omega = \frac{\pi T}{4}\left(\frac{1}{N}\sum_{k=1}^{N_\psi}\frac{\psi_k+1}{1-\psi_k}+\frac{1}{N}\sum_{k=1}^{N_\phi}\frac{\phi_k+1}{\phi_k-1}-1\right). \tag{4.13}$$

Proof: From Lemma 3.3 we have that zeros of $H(s)$ in the ORHP are

$$\rho_{k,i} = -\frac{N}{T}\log\phi_i + jkN\omega_s, \quad \text{with } i=1,\ldots,N_\phi \text{ and } k=0,\pm1,\pm2\ldots \tag{4.14}$$

and zeros in the OLHP are

$$\lambda_{k,i} = -\frac{N}{T}\log\psi_i + jkN\omega_s, \quad \text{with } i=1,\ldots,N_\psi \text{ and } k=0,\pm1,\pm2\ldots, \tag{4.15}$$

From (4.14) and (4.15), and using the identity (Rudin 1966, p. 197)

$$\frac{e^{2\pi x}+1}{e^{2\pi x}-1} = \frac{1}{\pi}\sum_{k=-\infty}^{\infty}\frac{x}{x^2+k^2},$$

in the form

$$\frac{e^{2\pi\alpha/\beta}+1}{e^{2\pi\alpha/\beta}-1} = \frac{\beta}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{\alpha+jk\beta},$$

we obtain the following closed forms for the series on the RHS of (4.8),

$$\sum_{i=1}^{N_\phi}\sum_{k=1}^{\infty}\frac{1}{\lambda_{k,i}} = \frac{T}{2N}\sum_{i=1}^{N_\phi}\frac{\phi_k+1}{\phi_k-1},$$

and

$$\sum_{i=1}^{N_\psi}\sum_{k=1}^{\infty}\frac{1}{\rho_{k,i}} = \frac{T}{2N}\sum_{i=1}^{N_\psi}\frac{\psi_k+1}{\psi_k-1}.$$

Replacing these in (4.8) yields (4.13), and finishes the proof. □

4.3 Example.

To illustrate the above results, we take an example from Er and Anderson (1994), where a PC hold of two steps is used to achieve discrete-time perfect loop transfer recovery of a non-minimum phase continuous plant. The zero-placement capabilities of GSHFs are used in their algorithm. The PC hold obtained for a sampling time $T = 0.04s$, is the following:

$$h(t) = \begin{cases} -1957 & \text{for } 0 \le t < 0.02 \\ 1707 & \text{for } 0.02 \le t < 0.04 \end{cases} \tag{4.16}$$

From Lemma 3.3 we see that there is an infinite sequence of zeros at

$$s = -6.8338 + jk\omega_s, \quad k = 0,\pm1,\pm2,\ldots,$$

with $\omega_s = 157.0796$. Figure 6 shows the normalized magnitude of $H(j\omega)$ (such that $H(0)=1$), compared to that of a ZOH. The effect of the real zero can be further evaluated by considering the bound (4.7). In this case, the bandwidth of the closed loop system is given by $\omega_b = 15.3$rad/s, so the ratio xT/ω_b is

$$xT/\omega_b = 0.0179.$$

This value in Figure 5 gives an indication of the peak expected in $|H(j\omega)|$ if we require that specification (4.5) be satisfied.

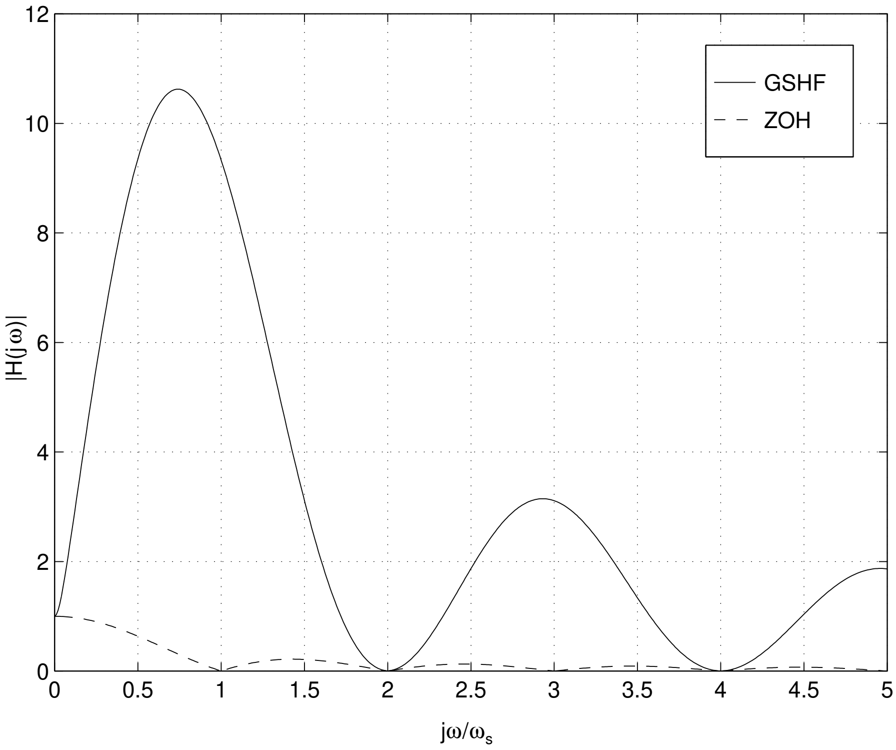

Figure 6: Frequency responses of GSHF and ZOH.

5 Conclusions

In this paper, we have considered the frequency response of non-traditional D/A devices known as GSHFs. We have studied the zero distribution of these holds, and showed several instances where problematic ORHP zeros may arise. Particularly, we have seen that "asymmetric" holds will necessarily have zeros off the $j\omega$-axis.

In addition, we have derived integral relations displaying the connections between zero locations and the frequency response of the hold. Our results indicate that holds with zeros off the $j\omega$-axis tend to have poor frequency response, with deleterious consequences in a sampled-data system.

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