On a Key Sampling Formula Relating the Laplace and $\mathcal{Z}$ Transforms

Julio Braslavsky *
julio@seidel.ece.ucsb.edu

Gjerrit Meinsma †
G.Meinsma@math.utwente.nl

Rick Middleton
rick@ee.newcastle.edu.au

Jim Freudenberg ‡
jfr@eecs.umich.edu

Department of Electrical and Computer Engineering
The University of Newcastle, NSW 2308 Australia

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Abstract

This note provides a new, rigorous derivation of a key sampling formula for discretizing an analogue system. The required conditions are formulated in time-domain, and give a clear characterization of the classes of signals and systems to which the formula applies.

Keywords: $\mathcal{Z}$ transform, sampled-data systems, frequency response.

1 Introduction

A formula that is crucial to the understanding of the frequency-domain properties of a sampled-data system is the following,

$$G_d(e^{sT}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s + jk\omega_s), \tag{1}$$

where G is the Laplace transform of a continuous-time signal g, G_d is the $\mathcal{Z}$ transform of the sequence of its samples, $\{g(kT)\}_{k=0}^{\infty}$, and T and $\omega_s = 2\pi/T$ denote the *sampling period* and *sampling frequency*, respectively. This formula displays the fundamental fact that the frequency response of a sampled signal is built upon the superposition of infinitely many copies of its continuous-time frequency response.

The formula has been known for some time in the literature of digital control systems (e.g., Jury, 1958; Ragazzini and Franklin, 1958), and it has recently been at the basis of a considerable number of works on sampled-data systems (Leung et al., 1991; Araki et al., 1993; Goodwin and Salgado, 1994; Yamamoto and Araki, 1994; Freudenberg et al., 1994; Rosenvasser, 1995; Hagiwara et al., 1995; Braslavsky et al., 1995a,b).

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*Presently with the ECE Department, University of California, Santa Barbara, CA 93106-9560, USA

†Presently with the Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

‡Jim Freudenberg is with the Department of Electrical Engineering and Computer Science, University of Michigan, 1301 Beal Ave., Ann Arbor, MI 48109-2122, USA.

Unfortunately, despite the fact that the result appears in many textbooks (e.g., Åström and Wittenmark, 1990; Franklin et al., 1990; Ogata, 1987; Kuo, 1992; Chen and Francis, 1995), it is difficult to find in the literature a proof that is rigorous and self-contained, and which clearly delineates the classes of signals to which it is applicable. Indeed, this fact has stimulated discussion in the past (cf. Pierre and Kolb, 1964; Carroll and W.L. McDaniel, 1966; Phillips et al., 1966, 1968; Doetsch, 1971).

Many of the available proofs for (1) rely on the use of "impulse trains" (e.g., Jury, 1958; Åström and Wittenmark, 1990; Chen and Francis, 1995). An impulse, however, is not well-defined as a function, which brings in technical difficulties in making the proofs rigorous. On the other hand, the proof by Doetsch (1971, p. 183) avoids the impulse trains, but states a frequency-domain condition, and it is not obvious when a given time function satisfies this condition.

In this note we provide a rigorous proof of an extended formulation of (1) for the case in which the signal g has discontinuities. We avoid impulse trains and their associated technical difficulties, and state precise time-domain conditions under which (1) is well-defined. Interestingly, as we show with a counterexample, the existence of the Laplace transform G and the $\mathcal{Z}$ transform of its sampled version G_d does *not* guarantee the validity of (1), even if g is smooth. The bulk of the note is two appendices containing technical details of our main results.

The extended formulation of (1) that we present has implications in characterizing two important classes of signals and systems to which the result applies. These classes are concerned with: (i) sampling the output of a strictly proper finite dimensional linear time-invariant (FDLTI) system, and (ii) computing the discrete equivalent of an analogue system.

Notation. We denote by $\mathbb{C}$ the complex plane, by $\mathbb{C}^+$ the open right half plane, by $\mathbb{R}$ the set of real numbers, and by $\mathbb{R}_0^+$ the segment $[0, \infty)$. The set of natural numbers is denoted by $\mathbb{N}$. The Euclidean norm in $\mathbb{C}^n$ is denoted by $|\cdot|$ and, for functions, $\|f\|_\infty$ denotes the infinity-norm $\sup_t |f(t)|$.

Continuous-time, or analogue, signals are functions mapping $\mathbb{R}_0^+$ into $\mathbb{R}^n$, and are represented with lowercase letters, e.g., u, y, etc. Discrete-time signals are sequences valued in $\mathbb{R}^n$, and are represented with a lowercase letter with the subscript "d", for *discrete*, e.g., $y_d = \{y_k\}_{k=0}^{\infty}$.

By $\mathcal{L}$ we represent the one-sided Laplace transform, defined for a continuous-time signal g as

$$(\mathcal{L}g)(s) = \int_0^\infty e^{-st} g(t)\, dt.$$

The one-sided $\mathcal{Z}$ transform $\mathcal{Z}\, y_d$ of a discrete-time signal y_d is defined as

$$(\mathcal{Z}\, y_d)(z) = \sum_{k=0}^{\infty} y_k z^{-k}.$$

We denote transformed signals with uppercase letters, keeping the subscript "d" to distinguish continuous and discrete domains, e.g., $G = \mathcal{L}\, g$, $Y_d = \mathcal{Z}\, y_d$. A rational function G is said to be *proper* if $|G(\infty)| < \infty$, *biproper* if G and G^{-1} are proper, and *strictly proper* if $|G(\infty)| = 0$. Whenever we write a double series as on the RHS of (1), we mean the limit

$$\sum_{k=-\infty}^{\infty} G(s + jk\omega_s) = \lim_{n\to\infty} \sum_{k=-n}^{n} G(s + jk\omega_s)\ .$$

Finally, for a continuous-time signal g, we denote by $g(t^+)$, whenever it exists, the *right limit* of g at point t, i.e.,

$$g(t^+) \triangleq \lim_{\varepsilon \downarrow 0} g(t + \varepsilon), \quad \text{for } \varepsilon > 0.$$

Accordingly,

$$g(t^-) \triangleq \lim_{\varepsilon \downarrow 0} g(t - \varepsilon), \quad \text{for } \varepsilon > 0,$$

denotes the *left limit* of g at point t. ∘

2 A Key Sampling Formula

In this section we present an extended version of (1) that holds for functions that have discontinuities. As noted in Ragazzini and Franklin (1958, p. 25), equation (1) is closely related to an old identity in Fourier analysis known as the *Poisson Summation Formula* (e.g., Dym and McKean, 1972). Following this, we shall refer to our generalized formulation as the *Poisson Sampling Formula*.[1]

In order to state the Poisson Sampling Formula, we need to introduce a class of functions for which the relation will hold. We start by defining the sampling operation.

Definition 2.1 (Sampling Operator). If g is an analogue signal, we define the — ideal — *sampling operator with sampling period* T, denoted by $\mathcal{S}_T$, as

$$\mathcal{S}_T\, g = \{g(kT^+)\}_{k=0}^{\infty}\,. \tag{2}$$

◇

Notice that we have used the right limit to define a sample. This is well-defined even for signals with "jump" discontinuities, which are not uncommon in engineering practice.[2] The sampling operator is *not* well-defined for signals such as $\sin(1/t)$ that vary arbitrarily rapidly at sampling points. Moreover, as we shall see, the velocity of variation is also decisive to the convergence of the series on the RHS of (1). For example, if the frequency content of g is band-limited to the *Nyquist range* $[-\omega_s/2, \omega_s/2]$, then the series is trivially bounded.

Formula (1) is not mathematically meaningful for just *any* function $g : \mathbb{R}_0^+ \to \mathbb{R}^n$. A basic necessary condition is that g be such that both its Laplace transform, G, and the $\mathcal{Z}$ transform of its sampled version, G_d, be well-defined. However, the existence alone of G and G_d does not guarantee the validity of (1), as the following lemma shows.

Lemma 2.1 (A Counterexample). Let $n_p \triangleq 2^{2^{2^p}}$, and let g be the continuous function on $\mathbb{R}_0^+$, depicted in Figure 1, which is defined per interval as

$$g(t) = \sin((2n_p + 1)t) \quad \text{for } t \in [p\pi, (p+1)\pi],\ p \in \mathbb{N}.$$

The Laplace transform $G(s)$ of g and the $\mathcal{Z}$ transform $G_d(e^{sT})$ of its sampled version, with sample period $T = \pi$, are both well-defined in the open right-half plane. Nevertheless,

$$\lim_{n\to\infty} \sum_{k=-n}^{n} G(s + jk\omega_s)$$

does not converge for any $s \in \mathbb{R}_0^+$, and hence, (1) is ill-defined for this g.

Proof. See Appendix B. □

The problem with g is not that it is only continuous — indeed, any smooth function g "close enough" to the above function will render (1) meaningless — the problem with $g(t)$ is that it oscillates arbitrarily rapidly as $t \to \infty$. This we need to exclude.

Definition 2.2 (Bounded and Uniform Bounded Variation). A function g defined on the closed real interval $[a, b]$ is of *bounded variation* (BV) if the *total variation* of g on $[a, b]$,

$$V_g(a, b) \triangleq \sup_{a=t_0<t_1<\cdots<t_{n-1}<t_n=b} \sum_{k=1}^{n} |g(t_k) - g(t_{k-1})| \tag{3}$$

[1] Some authors refer to (1) as the *Impulse Modulation Formula* (Araki et al., 1993; Hagiwara et al., 1995).

[2] The choice of the right limit — rather than the left one — is natural for a "causal" sampler.

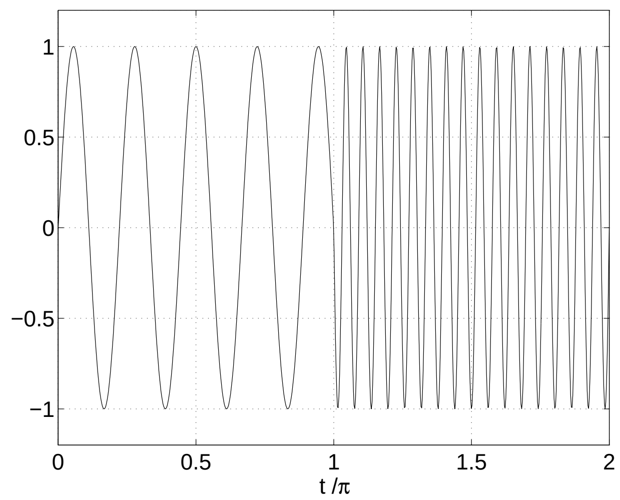

Figure 1: The function g of Lemma 2.1.

is finite. The supremum here is taken over every $n \in \mathbb{N}$ and every partition of the interval $[a, b]$ into subintervals $[t_k, t_{k+1}]$, where $k = 0, 1, \ldots, n-1$, and $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$.

A function g defined on $\mathbb{R}_0^+$ is of *uniform bounded variation* (UBV) if for some $\Delta > 0$ the total variation $V_g(x, x+\Delta)$ on intervals $[x, x+\Delta]$ of length Δ is uniformly bounded,[3] that is, if

$$\sup_{x \in \mathbb{R}_0^+} V_g(x, x+\Delta) < \infty. \tag{4}$$

◇

A function of BV is not necessarily continuous, but is differentiable almost everywhere and its derivative is a function integrable on $[a, b]$ (Rudin, 1987). Moreover, the limits $g(t^+)$ and $g(t^-)$ are well-defined for every t in (a, b), which means that g can have discontinuities of at most the "finite-jump type". If g is continuously differentiable on $[a, b]$, then the total variation $V_g(a, b)$ equals the L_1-norm of its derivative, i.e.,

$$V_g(a, b) = \int_a^b |\dot{g}(t)| \, dt.$$

Functions $g(t)$ of uniform bounded variation grow at most linearly with t and, as such, the Laplace transform $G(s)$ of g and the $\mathcal{Z}$ transform $G_d(e^{sT})$ of its sampled version are well-defined for any s in the open right-half plane. It is easy to see now that the function of Lemma 2.1, for which (1) fails to converge, is of BV on every finite interval *but not* of UBV.

The following is the version of the sampling formula (1) that holds for functions of UBV.

Theorem 2.2 (Poisson Sampling Formula). If g is a function of UBV on $\mathbb{R}_0^+$, then for every $s \in \mathbb{C}^+$ the following relation holds,

$$G_d(e^{sT}) = \frac{g(0^+)}{2} + \sum_{k=1}^{\infty} \frac{g(kT^+) - g(kT^-)}{2} e^{-skT} + \frac{1}{T} \sum_{k=-\infty}^{\infty} G(s + jk\omega_s) \,. \tag{5}$$

If $e^{-\sigma t} g(t)$ is of UBV on $\mathbb{R}_0^+$ for some $\sigma \in \mathbb{R}$, then (5) holds for every s such that $\operatorname{Re} s > \sigma$.

Proof. See Appendix A. □

[3]Note that the particular value of Δ in (4) over which the total variation is taken is irrelevant to the definition of UBV.

Notice in (5) that if the signal g has discontinuities at the sampling instants, then correction terms of half of the jumps at the corresponding sampling instants have to be included.[4] If g is of UBV and continuous except at most at $t = 0$ — as is the case if the Laplace transform of g is rational and strictly proper — then we obtain the expression

$$G_d(e^{sT}) = \frac{g(0^+)}{2} + \frac{1}{T} \sum_{k=-\infty}^{\infty} G(s + jk\omega_s),$$

which is quoted in some books (e.g., Åström and Wittenmark (1990, p. 104), Jury (1958, p. 9)), although frequently without proof. An exception is Doetsch (1971, p. 183), which derives the result under the assumption that the series $\sum_k G(s + jk\omega_s)$ is uniformly convergent. Theorem 2.2 characterizes the validity of the formula under time-domain conditions.

3 Two Applications of the Poisson Sampling Formula

In this section we indicate how the time-domain conditions needed for Poisson Sampling Formula (5) affect its application.

Theorem 2.2 delineates two important classes of signals and systems to which the formula is applicable, as we shall see in the following two corollaries. The first one is concerned with sampling the response of a strictly proper FDLTI system, as sketched in Figure 2. This represents a common practice in digital control engineering, i.e., to precede the sampler by a low-pass anti-aliasing filter, and also guarantees the boundedness of the sampling operator (e.g., Chen and Francis, 1991).

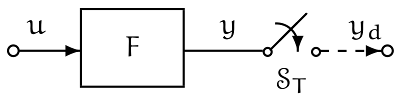

Figure 2: Sampling a filtered signal.

Corollary 3.1 (Sampling of a Filtered Signal). Let u be a signal that is zero for negative time and such that $e^{-\sigma t}u(t)$ is of UBV for some $\sigma \in \mathbb{R}$. Let F be a strictly proper rational function with all its poles to the left of a shifted axis $\{\sigma_F + j\mathbb{R}\}$. Then, for every $s \in \mathbb{C}$ with $\operatorname{Re} s > \max\{\sigma, \sigma_F\}$,

$$(FU)_d(e^{sT}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F(s + jk\omega_s)U(s + jk\omega_s).$$

Proof. For simplicity assume $\max(\sigma, \sigma_F) = 0$. The result follows essentially from two observations: The response y of a stable FDLTI strictly proper system F to input u of UBV is (i) of UBV, and (ii) continuous. The latter implies that $y(0^+) = y(0^-) = 0$ and $y(kT^+) = y(kT^-)$. See Appendix C for more details. ☐

Signals that are steps, ramps, sinusoids or exponentials are of uniform bounded variation when multiplied by some exponential decaying term $e^{-\sigma t}$. Yet, signals like $\sin(e^{t^2})$ and signals that contain impulses are excluded, and thus, Corollary 3.1 establishes the validity of formula (1) for most standard signals in engineering analysis passed through a strictly proper FDLTI filter.

The second corollary deals with sampling the pulse response of a hold device followed by a FDLTI system, as pictured in Figure 3, and displays the relation between the discrete equivalent of this cascade and the corresponding continuous-time Laplace transforms.

[4]This is precisely the same property of the Laplace and Fourier inverse transforms, which converge to the average of the limits of the function from left and right at a jump discontinuity.

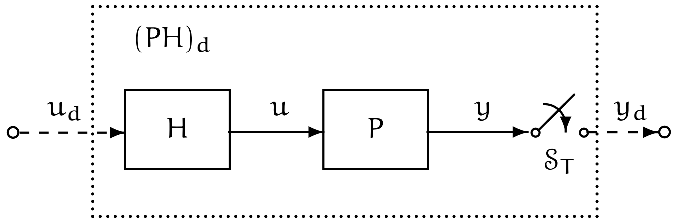

Figure 3: Discrete equivalent of the cascade of a hold and a FDLTI system.

A hold device performs the inverse operation of a sampler, i.e., it converts a sequence of numbers into an analogue signal. We consider a generalized hold function *à la* Kabamba (1987), defined by the operation

$$u(t) = h(t - kT)\, u_k, \quad \text{for } kT \leq t < (k+1)T, \tag{6}$$

where $\{u_k\}_{k=-\infty}^{\infty}$ is the discrete input to the hold, and h is a function of BV with support on the interval $[0, T)$. As discussed in Middleton and Freudenberg (1995), we can associate a *frequency response function* to this hold device, defined by

$$H(s) = \int_0^T e^{-st}\, h(t)\, dt. \tag{7}$$

Since h is supported on a finite interval, it follows that H is an entire function, i.e., analytic at every finite s in $\mathbb{C}$. For example, in the case of the zero order hold (ZOH), $h = 1$ on $[0, T)$, and we have the well-known response $H(s) = (1 - e^{-sT})/s$. Frequency responses of other holds have been studied in Braslavsky et al. (1995a).

We denote by $(PH)_d$ the discrete equivalent of the cascade connection PH, defined as

$$(PH)_d = \mathcal{Z}\, \mathcal{S}_T\, \mathcal{L}^{-1}\,(PH)\,. \tag{8}$$

Corollary 3.2 (Discretization of an Analogue System). Let H be a hold frequency-response function as described in (7) and suppose that P is a proper rational function. Then, for every $s \in \mathbb{C}$ such that $\operatorname{Re} s$ is larger than the real part of any pole of P,

$$(PH)_d(e^{sT}) = \frac{1}{2} P(\infty)\big(h(0^+) - h(T^-)e^{-sT}\big) + \frac{1}{T}\sum_{k=-\infty}^{\infty} P(s + jk\omega_s)\, H(s + jk\omega_s)\,, \tag{9}$$

where $P(\infty) = \lim_{s\to\infty} P(s)$.

Proof. Write P as $P = P(\infty) + P_0$, where P_0 is strictly proper. Since the pulse response of H is of UBV by definition, applying Theorem 2.2 with $G = P(\infty)H$ yields

$$(P(\infty)H)_d(e^{sT}) = \frac{1}{2} P(\infty)\big(h(0^+) - h(T^-)e^{-sT}\big) + \frac{1}{T} P(\infty) \sum_{k=-\infty}^{\infty} H(s + jk\omega_s)\,, \tag{10}$$

and with $G = P_0 H$,

$$(P_0 H)_d(e^{sT}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} P_0(s + jk\omega_s)\, H(s + jk\omega_s)\,. \tag{11}$$

Notice in (11) that we used the fact that the response of a FDLTI strictly proper system to a bounded input is continuous (Desoer and Vidyasagar, 1975, p. 59). The result then follows after superposition of (10) and (11). □

If the plant P is strictly proper, then $P(\infty) = 0$, and the classic result

$$(PH)_d(e^{sT}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} P(s + jk\omega_s)\, H(s + jk\omega_s)$$

is recovered. Corollary 3.2 establishes that the same relation is not valid for a biproper P unless the hold is such that $h(0^+) = 0 = h(T^-)$. In particular, this condition is *not* satisfied by the ZOH.

4 Conclusions

This paper has presented a generalized formulation of a well-known frequency-domain relation between the Laplace transform of a continuous-time signal and the $\mathcal{Z}$ transform of its sampled version. We have provided a rigorous proof of this result, characterizing in the time-domain an important class of signals to which the formula applies. A key property of these signals is that they are of UBV when multiplied by an exponentially decaying term. This property is sufficient to guarantee the validity of the expression, and moreover, it is almost necessary, since, as we have shown via a counterexample, continuity or BV on their own may render the expression mathematically meaningless.

A Proof of the Poisson Sampling Formula

In this appendix we prove the Poisson Sampling Formula (5), which generalizes (1).

Many of the proofs for (1) available in the literature rely on the use of the "infinite comb"

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT),$$

defined as an infinite sum of impulses, or Dirac's deltas (Pierre and Kolb, 1964; Carroll and W.L. McDaniel, 1966; Phillips et al., 1966; Åström and Wittenmark, 1990; Chen and Francis, 1995). A Dirac's delta is not well-defined as a function, and so special care must be taken regarding the sense in which certain mathematical manipulations are performed (cf. Zemanian, 1965).

Our approach dispenses with the use of δ_T, and instead resources to the *Dirichlet Kernel*, a classical tool in proving convergence of Fourier series. The Dirichlet Kernel is defined by

$$\begin{aligned} D_n(t) &= \frac{\sin((2n+1)t)}{\sin(t)} \\ &= \sum_{k=-n}^{n} e^{-j2kt}\,. \end{aligned} \tag{12}$$

where n is a positive integer. D_n is periodic and its integral on $[0, \pi/2]$ has a fixed value independent of n, i.e.,

$$\int_0^{\pi/2} D_n(t)\, dt = \frac{\pi}{2}\,.$$

A key property of the Dirichlet Kernel is related to the following *Dirichlet Integral*.

Lemma A.1 (Dirichlet Integral). If g is a function of BV on the interval $[0, \pi]$, then

$$\lim_{n\to\infty} \int_0^{\pi} g(t)\, D_n(t)\, dt = \frac{\pi}{2}[g(0^+) + g(\pi^-)]$$

Proof. See for example Carslaw (1950, § 94). □

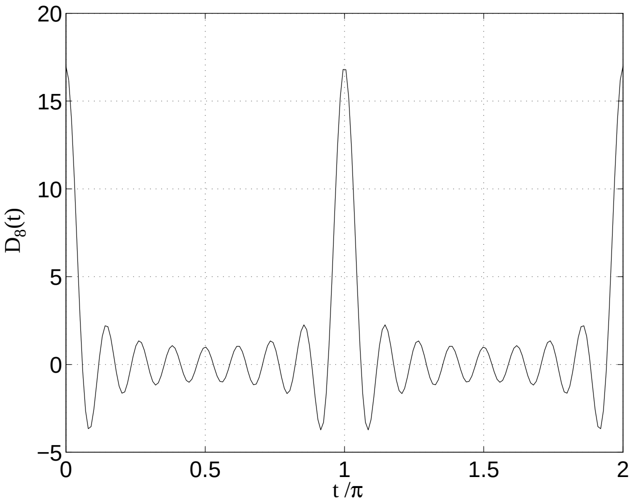

Figure 4: Dirichlet Kernel for $n = 8$.

Figure 4 plots the Dirichlet Kernel for $n = 8$. Note that D_n is very much like an approximation to the infinite comb δ_T, with several common properties, but is well-defined as a function. Convergence of the Dirichlet Integral is slow but is bounded in the following sense.

Definition A.1. $\|g\|_{V[a,b]} \triangleq |g(a)| + V_g(a, b)$. ◇

Lemma A.2. There is an $M > 0$ such that

$$\left| \int_0^{\pi} g(t) D_n(t)\, dt \right| \leq M \|g\|_{V[0,\pi]},$$

for all $n \in \mathbb{N}$ and all functions g of BV on $[0, \pi]$. ◇

As the notation suggests, the $\|\cdot\|_{V[a,b]}$ is indeed a norm (p. 24 Luenberger, 1969, see), and moreover, $\|g\|_{V[a,b]} \geq \|g\|_\infty$ on $[a, b]$. The proof of this lemma relies on the property that any real-valued function g of BV on $[a, b]$ can be expressed as the difference of two bounded non-decreasing functions,

$$g = g_+ - g_-.$$

This is a useful result and is often used (p. 80 Carslaw, 1950, e.g., see). It follows readily by letting

$$g_+(t) \triangleq \frac{1}{2}(V_g(a, t) + g(t)), \quad g_-(t) \triangleq \frac{1}{2}(V_g(a, t) - g(t)). \tag{13}$$

The proof of Lemma A.2 makes use of a Mean Value Theorem.

Lemma A.3 (Second Mean Value Theorem). If a function f is continuous on $[a, b]$ and if g is a bounded, non-decreasing function on $[a, b]$, then for some $c \in [a, b]$,

$$\int_a^b g(t) f(t)\, dt = g(a^+) \int_a^c f(t)\, dt + g(b^-) \int_c^b f(t)\, dt.$$

Proof. See Carslaw (1950, p. 109), Goldberg (1961, p. 5). □

Proof of Lemma A.2. Suppose g is non-decreasing. By the Second Mean Value Theorem we have that

$$\int_0^\pi g(t)D_n(t)\,dt = g(0^+)\int_0^{c_n} D_n(t)\,dt + g(\pi^-)\int_{c_n}^\pi D_n(t)\,dt,$$

for some $c_n \in [0,\pi]$. For any such c_n the integrals $|\int_0^{c_n} D_n(t)\,dt|$ and $|\int_{c_n}^\pi D_n(t)\,dt|$ are bounded by some $L > 0$. (This follows from the alternating property of D_n.) If g is not monotonic we can still write g as the difference of two non-decreasing functions $g = g_+ - g_-$ with $g_\pm$ as in (13). Using the triangle inequality we then get

$$\left|\int_0^\pi g(t)D_n(t)\,dt\right| \le L\left(|g_+(0^+)| + |g_+(\pi^-)| + |g_-(0^+)| + |g_-(\pi^-)|\right).$$

All four $g_\pm(0^-)$ and $g_\pm(\pi^-)$ are over bounded by $\|g\|_{V[0,\pi]}$, which then proves the result with $M = 4L$. □

A final technicality that we need before we can prove the Poisson Sampling Formula is a sub-multiplicative property of the norm $\|\cdot\|_{V[a,b]}$.

Lemma A.4. If f and g are real-valued functions of BV on $[a,b]$, and f is scalar, then $\|fg\|_{V[a,b]} \le \|f\|_{V[a,b]}\|g\|_{V[a,b]}$. If f and g are complex-valued there holds $\|fg\|_{V[a,b]} \le 4\|f\|_{V[a,b]}\|g\|_{V[a,b]}$.

Proof. Write f and g as the sum of a constant term and a term which is zero at a, like

$$f = f(a) + f_0, \quad g = g(a) + g_0$$

Note that f_0 and f have the same total variation and so do g_0 and g. (To avoid clutter we write V_g instead of $V_g(a,b)$.) Because $f_0(a) = g_0(a) = 0$ it may be seen that $V_{f_0 g_0} \le V_{f_0}V_{g_0}$:

$$\begin{aligned}
V_{f_0 g_0} &= V_{(f_+-f_-)(g_+-g_-)} \\
&= V_{f_+g_+-f_-g_+-f_+g_-+f_-g_-} \\
&\le V_{f_+g_+} + V_{f_-g_+} + V_{f_+g_-} + V_{f_-g_-} \\
&= f_+(b)g_+(b) + f_-(b)g_+(b) + f_+(b)g_-(b) + f_-(b)g_-(b) \\
&= (f_+(b) + f_-(b))\,(g_+(b) + g_-(b)) \\
&= V_{f_0}V_{g_0} = V_fV_g\,.
\end{aligned}$$

Then

$$\begin{aligned}
\|fg\|_{V[a,b]} &= |f(a)g(a)| + V_{(f(a)+f_0)(g(a)+g_0)} \\
&= |f(a)g(a)| + V_{f(a)g(a)+f(a)g_0+f_0g(a)+f_0g_0} \\
&\le |f(a)g(a)| + V_{f(a)g(a)} + V_{f(a)g_0} + V_{f_0g(a)} + V_{f_0g_0} \\
&= |f(a)g(a)| + 0 + |f(a)|V_{g_0} + |g(a)|V_{f_0} + V_{f_0g_0} \\
&\le (|f(a)| + V_{f_0})\,(|g(a)| + V_{g_0}) = \|f\|_{V[a,b]}\,\|g\|_{V[a,b]}\,.
\end{aligned}$$

The complex-valued case follows from a decomposition into real and imaginary parts. □

Proof of Theorem 2.2. Without loss of generality we take $\sigma = 0$. So we have that g is of UBV. First we rewrite $\sum_{k=-n}^n G(s + jk\omega_s)$.

$$\begin{aligned}
\frac{1}{T}\lim_{n\to\infty}\sum_{k=-n}^{n} G(s+jk\omega_s) &= \frac{1}{T}\lim_{n\to\infty}\sum_{k=-n}^{n}\int_0^\infty e^{-(s+jk\omega_s)t}g(t)\,dt \\
&= \frac{1}{T}\lim_{n\to\infty}\int_0^\infty e^{-st}g(t)D_n(\omega_s t/2)\,dt.
\end{aligned}$$

We need to show that the above limit is well-defined and that it equals

$$G_d(e^{sT}) - \frac{g(0^+)}{2} - \sum_{k=1}^{\infty} \frac{g(kT^+) - g(kT^-)}{2} e^{-skT} = \sum_{k=0}^{\infty} \frac{g(kT^+)e^{-skT} + g((k+1)T^-)e^{-s(k+1)T}}{2}.$$

That is, we need to show that for every $\varepsilon > 0$ there is an N such that

$$\left| \frac{1}{T} \int_0^{\infty} e^{-st} g(t) D_n(\omega_s t/2)\, dt \; - \sum_{k=0}^{\infty} \frac{g(kT^+)e^{-skT} + g((k+1)T^-)e^{-s(k+1)T}}{2} \right| \tag{14}$$

is well-defined and less than ε for all $n > N$. Using the rule "$|(x+y) - (a+b)| = |(x-a) + y - b| \leq |x-a| + |y| + |b|$" it may be seen that (14) is indeed less than ε if the following three inequalities hold for some $q \in \mathbb{N}$.

$$\left| \frac{1}{T} \int_0^{qT} e^{-st} g(t) D_n(\omega_s t/2)\, dt - \sum_{k=0}^{q-1} \frac{g(kT^+)e^{-skT} + g((k+1)T^-)e^{-s(k+1)T}}{2} \right| < \frac{\varepsilon}{3}, \tag{15}$$

$$\left| \frac{1}{T} \int_{qT}^{\infty} e^{-st} g(t) D_n(\omega_s t/2)\, dt \right| < \frac{\varepsilon}{3}, \tag{16}$$

$$\left| \sum_{k=q}^{\infty} \frac{g(kT^+)e^{-skT} + g((k+1)T^-)e^{-s(k+1)T}}{2} \right| < \frac{\varepsilon}{3}. \tag{17}$$

We will show that such a q can be found, independent of n. The most difficult bound to establish is (16). First we examine the integral over $[kT, (k+1)T]$.

$$\left| \frac{1}{T} \int_{kT}^{(k+1)T} e^{-st} g(t) D_n(\omega_s t/2)\, dt \right| \leq \frac{M}{T} \left\| e^{-s\cdot} g \right\|_{V[kT,(k+1)T]} \quad \text{(by Lemma A.2)} \tag{18}$$

$$\leq \frac{4M}{T} \left\| e^{-s\cdot} \right\|_{V[kT,(k+1)T]} \left\| g \right\|_{V[kT,(k+1)T]}. \tag{19}$$

The last inequality follows from Lemma A.4. By assumption g is of uniform bounded variation and therefore (19) decays exponentially as $k \to \infty$. But then the integral

$$\left| \frac{1}{T} \int_{qT}^{\infty} e^{-st} g(t) D_n(\omega_s t/2)\, dt \right| \leq \sum_{k=q}^{\infty} \left| \frac{1}{T} \int_{kT}^{(k+1)T} e^{-st} g(t) D_n(\omega_s t/2)\, dt \right| \tag{20}$$

is well-defined and also decays exponentially as $q \to \infty$. For every $\varepsilon > 0$ there is therefore a Q such that (16) holds for all $q > Q$. Among these qs there is a large enough $q \in \mathbb{N}$ such that also (17) holds. Finally, the Dirichlet Integral Theorem guarantees that, given such a q, there will be an $N \in \mathbb{N}$ for which (15) is satisfied for all $n > N$. □

B A Counterexample

In this appendix we prove Lemma 2.1, thus showing that there are functions $g : \mathbb{R}_0^+ \to \mathbb{R}$ that are bounded, are continuous everywhere, are of BV, and whose Laplace transform is well-defined in the open right-half plane, but for which the Poisson Sampling Formula is not well-defined in any half-plane. Needless to say, the function in Lemma 2.1 is not of UBV. This serves to illustrate that the condition of UBV in the Poisson Sampling Formula (5) can not be relaxed to mere BV.

We split the proof in several parts. Since g is continuous it is "samplable" and because $\|g\|_\infty = 1$ it is direct that $G(s)$ and $G_d(e^{sT})$ are well-defined in the open right-half plane. Denote by Γ_n the function $\Gamma_n(s) = \sum_{k=-n}^{n} G(s + jk\omega_s)$. From Appendix A (first steps in the proof of Theorem 2.2), we know that

$$\Gamma_{n_l}(s) = \int_0^\infty e^{-st} g(t) D_{n_l}(t)\, dt.$$

We prove Lemma 2.1 by showing that $|\Gamma_{n_l}(s)|$ is unbounded, (which, obviously precludes convergence). The idea, roughly, is that both g and D_{n_l} change sign so often that the value of the integral $\int_0^\infty e^{-st} g(t) D_{n_l}(t)\, dt$ will be dominated by the integral over $[l\pi, (l+1)\pi]$ which is the only interval where the integrand $e^{-st} g(t) D_{n_l}(t)$ does not change sign. The contribution over that interval grows without bound as l goes to infinity.

We need one technical lemma. The bounds obtained in this lemma are conservative but are sufficient for our purposes.

Lemma B.1. There are constants $C_1 > 0$ and $C_2 > 0$ such that for any $s \in \mathbb{R}_0^+$ and any $q, n \in \mathbb{N}, q > 1, n > 1$ we have that

$$\left| \int_0^\pi e^{-st} \sin((2q+1)t)\, D_n(t)\, dt \right| \begin{cases} \leq & C_1 \frac{q^2}{n}, \quad \text{if } q < n, \\ \geq & C_2 \log(n) e^{-s\pi}, \quad \text{if } q = n, \\ \leq & C_1 \frac{n^2}{q}, \quad \text{if } q > n. \end{cases}$$

Proof. First consider the case that $q = n$.

$$\begin{aligned} \int_0^\pi e^{-st} \sin((2q+1)t)\, D_n(t)\, dt &= \int_0^\pi e^{-st} \frac{\sin^2((2n+1)t)}{\sin(t)}\, dt \\ &\geq e^{-s\pi} \sum_{k=0}^{2n} \int_{\frac{k}{2n+1}\pi}^{\frac{k+1}{2n+1}\pi} \frac{\sin^2((2n+1)t)}{\sin(t)}\, dt \\ &\geq e^{-s\pi} \sum_{k=0}^{2n} \int_{\frac{k}{2n+1}\pi}^{\frac{k+1}{2n+1}\pi} \frac{\sin^2((2n+1)t)}{\frac{k+1}{2n+1}\pi}\, dt \end{aligned}$$

(this is because $\sin(x) \leq x$)

$$\begin{aligned} &= e^{-s\pi} \sum_{k=0}^{2n} \frac{\frac{1}{2}\frac{1}{2n+1}\pi}{\frac{k+1}{2n+1}\pi} \\ &= e^{-s\pi} \sum_{k=0}^{2n} \frac{1/2}{k+1} \\ &\geq e^{-s\pi} C_2 \log(n)\ , \end{aligned}$$

for some $C_2 > 0$ because $n > 1$.

The case that $q < n$ follows from the case that $q > n$ by interchanging q and n. Suppose $q > n$, and let $h_n(t) = e^{-st} D_n(t)$.

$$\begin{aligned} &\left| \int_0^\pi e^{-st} \sin((2q+1)t)\, D_n(t)\, dt \right| \\ &\quad = \left| -h_n(t) \frac{\cos((2q+1)t)}{2q+1} \Big|_{t=0}^{t=\pi} + \int_0^\pi \frac{\cos((2q+1)t)}{2q+1}\, \dot{h}_n(t)\, dt \right| \qquad (21) \\ &\quad \leq \frac{1}{2q+1} \left(2\|h_n\|_\infty + \int_0^\pi |\dot{h}_n(t)|\, dt \right). \qquad (22) \end{aligned}$$

The term $\int_0^\pi |\dot{h}_n(t)|\, dt$ can be bounded with help of Lemma A.4 as follows.

$$\int_0^\pi |\dot{h}_n(t)|\, dt = V_{h_n}(0,\pi) \le \|h_n\|_{V[0,\pi]} \le \|e^{-s\cdot}\|_{V[0,\pi]}\, \|D_n\|_{V[0,\pi]} = 2\|D_n\|_{V[0,\pi]}.$$

From the plot of the Dirichlet Kernel it is direct that the interval $[0,\pi]$ can be divided in $2n$ subintervals on each of which D_n is monotonic, and that on each of these subintervals D_n varies at most $2(2n+1)$. (This may also be verified formally.) Hence,

$$\begin{aligned} \|D_n\|_{V[0,\pi]} &= |D_n(0)| + V_{D_n}(0,\pi) = (2n+1) + V_{D_n}(0,\pi) \\ &\le (2n+1) + 2n\, 2(2n+1) \le Cn^2 \quad \text{for some } C > 0 \text{ because } n > 0. \end{aligned}$$

So we have that $\int_0^\pi |\dot{h}_n(t)|\, dt \le 2Cn^2$, and because $\|h_n\|_\infty = 2n+1$ the expression in (22) is overbounded by some function of the form $C_1 n^2/q$. □

Proof of Lemma 2.1. As argued at the beginning of this appendix, we only need to show that for each $s \in \mathbb{R}_0^+$ the $\Gamma_{n_l}(s)$ diverges as l goes to ∞. We have

$$\begin{aligned} |\Gamma_{n_l}(s)| &= \left| \int_0^\infty e^{-st} g(t) D_{n_l}(t)\, dt \right| \\ &= \left| \sum_{k\in\mathbb{N}} \int_{k\pi}^{(k+1)\pi} e^{-st} \sin((2n_k+1)t)\, D_{n_l}(t)\, dt \right| \\ &\ge \left| \int_{l\pi}^{(l+1)\pi} e^{-st} \sin((2n_l+1)t)\, D_{n_l}(t)\, dt \right| \\ &\quad - \sum_{\substack{k\in\mathbb{N}\\ k\ne l}} \left| \int_{k\pi}^{(k+1)\pi} e^{-st} \sin((2n_k+1)t)\, D_{n_l}(t)\, dt \right| \\ &= e^{-sl\pi} \left| \int_0^\pi e^{-st} \sin((2n_l+1)t)\, D_{n_l}(t)\, dt \right| \\ &\quad - \sum_{\substack{k\in\mathbb{N}\\ k\ne l}} \left| e^{-sk\pi} \int_0^\pi e^{-st} \sin((2n_k+1)t)\, D_{n_l}(t)\, dt \right| \\ &\ge e^{-sl\pi} e^{-s\pi} C_2 \log(n_l) - \sum_{k=1}^{l-1} e^{-sk\pi} C_1 \frac{n_k^2}{n_l} - \sum_{k=l+1}^{\infty} e^{-sk\pi} C_1 \frac{n_l^2}{n_k} \end{aligned}$$

(this is by application of the previous lemma)

$$\begin{aligned} &\ge e^{-s(l+1)\pi} C_2 \log(2)\, 2^{2^l} - C_1 (l-1) \frac{n_{l-1}^2}{n_l} - C_1 \sum_{k=l+1}^{\infty} \frac{n_l^2}{n_k} \\ &\ge e^{-s(l+1)\pi} C_2 \log(2)\, 2^{2^l} - C_1 \left((l-1) + \sum_{k=l+1}^{\infty} \frac{n_l^2}{n_k} \right). \end{aligned}$$

The last inequality follows from the fact that $n_{l-1}^2/n_l < 1$. The last term in the last inequality can be over bounded as follows,

$$\sum_{k=l+1}^{\infty} \frac{n_l^2}{n_k} = \sum_{q=1}^{\infty} \frac{2^{2^{2^l}} \cdot 2^{2^{2^l}}}{2^{2^{2^{l+q}}}} = \sum_{q=1}^{\infty} \frac{2^{2^{(2^l+1)}}}{2^{2^{(2^l+2^q)}}} \le \sum_{q=1}^{\infty} \frac{1}{2^q} = 1.$$

Finally, then, we get

$$|\Gamma_{n_l}(s)| \ge e^{-s(l+1)\pi} C_2 \log(2)\, 2^{2^l} - C_1 l.$$

For any $s \in \mathbb{R}_0^+$ the term $e^{-s(l+1)\pi} C_2 \log(2)\, 2^{2^l}$ grows without bound as l goes to infinity, and hence, also $|\Gamma_{n_l}(s)|$ grows without bound, which is what we set out to prove. □

C A Property of Functions of UBV

Lemma C.1. Let y be the response of a FDLTI system P to input u. If $\sigma \in \mathbb{R}$ is such that $e^{-\sigma t}u(t)$ is of UBV and $P(s+\sigma)$ is exponentially stable, then $e^{-\sigma t}y(t)$ is of UBV.

Proof. Let $P(s) = C(sI - A)^{-1}B + D$ be a minimal realization of P, and suppose for the moment that $\sigma = 0$, i.e., that P is stable and u is of UBV. It suffices to show that x is of UBV. The state x of this system satisfies

$$x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)\, d\tau = \int_0^t e^{Az}Bu(t-z)\, dz.$$

Using this it may be seen that for any pair $t_{k+1} > t_k$ we have

$$x(t_{k+1}) - x(t_k) = \int_0^{t_k} e^{Az}B(u(t_{k+1}-z) - u(t_k - z))\, dz + \int_{t_k}^{t_{k+1}} e^{Az}Bu(t_{k+1}-z)\, dz.$$

It then follows that

$$|x(t_{k+1}) - x(t_k)| \leq V_u(t_k, t_{k+1}) \int_0^{t_k} \|e^{Az}B\|\, dz \; + (t_{k+1} - t_k) \sup_{z \in [t_k, t_{k+1}]} (|u(t_{k+1} - z)|\|e^{Az}B\|).$$

Here $\|\cdot\|$ denotes the spectral norm for matrices. By stability of A there exist $M > 0$ and $N > 0$ such that $|x(t_{k+1}) - x(t_k)| \leq MV_u(t_k, t_{k+1}) + (t_{k+1} - t_k)N$. ($N$ depends on $u(t)$, $t \in [0, t_{k+1} - t_k]$.) This finally shows that x is of UBV, because given any $\Delta > 0$ there holds

$$V_x(t, t+\Delta) = \sup_{t=t_0<\cdots<t_n=t+\Delta} |x(t_{k+1}) - x(t_k)| \leq MV_u(t, t+\Delta) + \Delta N.$$

By uniform boundedness of u the last expression is independent of t, hence, x is of UBV, which completes the proof.

The case that σ is nonzero can be reduced to zero case by a substitution: It follows directly from the convolution integral $x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)\, d\tau$ that

$$e^{-\sigma t}x(t) = \int_0^t e^{(A-\sigma I)(t-\tau)}Be^{-\sigma\tau}u(\tau)\, d\tau.$$

This substitution shows that $e^{-\sigma t}x$ is nothing but the state of system with $A - \sigma I$ as its (stable) "A-matrix" and $e^{-\sigma t}u(t)$ as its input (which is of UBV). □

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