Tradeoffs in Multivariable Filter Design with Applications to Fault Detection*

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Abstract

This paper investigates sensitivity limitations in multivariable linear filtering. Integral constraints on appropriate complementary sensitivities are developed that give new insights into fundamental design tradeoffs that apply to all linear filters. The results can be used to obtain performance limitations in a variety of filtering applications where multivariable structural features are of interest. We illustrate our results by exploring design tradeoffs arising in fault detection and isolation.

1 Introduction

All design is ultimately constrained by fundamental limits. These limits have the inescapable consequence that conflicting objectives must be traded against each other. Among these objectives, achieving low sensitivity to disturbances is a primordial design goal, which necessarily leads to tradeoffs when different disturbances affect the problem in a complementary fashion. Feedback control theory has a long history of recognizing design limitations and tradeoffs, which can be traced back to the work of Bode (1945). His analysis demonstrated, for example, that reduction of sensitivity over one frequency range results in an increase of sensitivity in another frequency range.

A well known fact about sensitivity and complementary sensitivity functions of a feedback control system is that they must satisfy certain interpolation constraints at the CRHP poles and zeros of the plant and compensator. Specifically, since the sum of sensitivity and complementary sensitivity is equal to one at all frequencies, the sensitivity function must equal one at the CRHP zeros, and the complementary sensitivity function must equal one at the CRHP poles. As seen in Freudenberg & Looze (1985), these constraints may be used in conjunction with Poisson integrals to describe frequency dependent tradeoffs in control systems design. These results have been recently extended to the multivariable case by Chen (1995) and Gómez & Goodwin (1995). In particular, the work of Gómez & Goodwin (1995) exploits properties of zero directions to study the cost of achieving the design objective of a diagonally decoupled sensitivity function.

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Analogous fundamental constraints and design tradeoffs apply to filtering problems. Goodwin, Mayne & Shim (1995) have defined complementary filtering sensitivity functions that, in the scalar case, were shown to satisfy interpolation conditions imposed by CRHP poles and zeros of the plant and filter. Under the assumption of bounded error filtering (see §2), these constraints were then translated into integral relations that the filtering sensitivity functions must meet independently of the filter chosen. An important consequence of the results was that poor filtering performance was inevitable in some cases irrespective of how the design was carried out. The work by Goodwin et al. applied to filtering problems involving unbiased filters where the only source of errors related to the measured input and output. These results were extended to more general filters and error sources in Seron & Goodwin (1995), and extensions to the problems of prediction and smoothing were explored in Seron (1995).

In this paper we study sensitivity limitations in multivariable linear filtering. We show that integral constraints related to those obtained by Gómez & Goodwin (1995) can be derived on the filtering sensitivities based on the use of directions corresponding to the unstable zeros and poles of the plant to be estimated. These integrals give new insights into fundamental limits that apply to all linear filter designs, with particular relevance to those problems exploiting structural features of multivariable systems. The results have potential application in a vast number of areas, including fault detection, and channel equalization in communication networks. As an example, we investigate the cost of achieving perfect fault detection and isolation, including sensor and actuator faults as particular cases.

Notation. As usual, $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, respectively. We will denote by $\mathbb{C}^+$ and $\mathbb{C}^-$ the open right and left halves of the complex plane, and by $\overline{\mathbb{C}^+}$ and $\overline{\mathbb{C}^-}$ their corresponding closed versions. Sometimes, we will also use the obvious acronyms ORHP, OLHP, CRHP, and CLHP.

Since we will deal with frequency domain descriptions of multivariable systems, we need to recall some standard definitions in this setting. We say that a transfer function, say $H : \mathbb{C} \to \mathbb{C}^{m\times p}$, is stable, if all the eigenvalues of the evolution matrix of any minimal realization of H are in the OLHP. We call these eigenvalues the *poles* of H.

We say that H is proper if all its entries are proper scalar transfer functions. We say that H is biproper (bistable) if both H and H^{-1} are biproper (bistable).

If s_0 is not a pole of H, we say that it is a *zero* of H if there exists a vector $a \in \mathbb{C}^p$ such that $H(s_0)a = 0$, or there exists a vector $b \in \mathbb{C}^m$ such that $b^*H(s_0) = 0$, where b^* denotes the conjugate transpose of b. We say that a and b are the *input* and *output zero directions* associated with s_0. A zero direction is said to be *canonical* if it has only one nonzero component. It is known (e.g., Kailath 1980) that if H admits a left or right inverse, then a pole of H will be a zero of H^{-1}. In this case we will refer to the input and output directions of the pole as those of the corresponding zero of H^{-1} (Chen 1995). A pole in the CRHP is said to be *unstable*, and a zero in the CRHP is called *non-minimum phase* (NMP).

2 Sensitivity Functions in Bounded Error Filtering

Consider the general filtering configuration of Figure 1, where the signals are as follows,

v : process input, in $\mathbb{R}^m$,	y : measured output, in $\mathbb{R}^q$,
w : measurement noise, in $\mathbb{R}^q$,	$\hat{z}$: estimate, in $\mathbb{R}^p$,
z : signal to be estimated, in $\mathbb{R}^p$,	$\tilde{z}$: estimation error, in $\mathbb{R}^p$.

The plant, G, is a LTI system given by the following state-space representation:

$$G \stackrel{s}{=} \left[\begin{array}{c|cc} A & B & 0 \\ \hline E & D_1 & 0 \\ C & D_2 & I \end{array}\right] ,$$

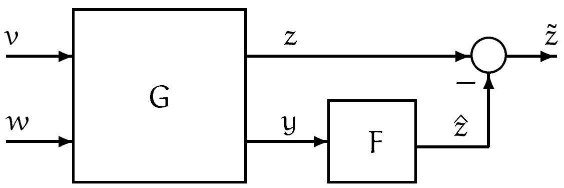

Figure 1: General filtering configuration.

where the pair (A, C) is detectable, and the pair (A, B) is stabilizable[1]. The filter is represented by a proper transfer function F. The basic filtering problem then consists in designing a suitable F such that the target signal z can be estimated from the measured output y. This setting encompasses a large class of filtering applications; for example, it has immediate relevance to fault detection and isolation problems, as we will see in §6.

For convenience, we partition the plant as shown in Figure 2, with the obvious definitions for the transfer functions G_z and G_y. Note that, by the assumption of detectability from y, the function G_y contains all the unstable modes of G, some or all of which may not appear in G_z.

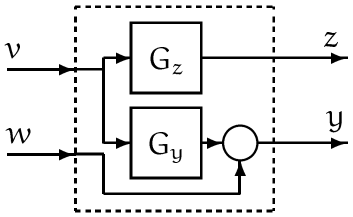

Figure 2: Structure of the plant.

We will further assume that the functions F, G_z, and G_y satisfy the following conditions.

Assumption 1.

(i) F *is stable.*

(ii) G_z *is right invertible and minimum phase.*

(iii) $G_z - FG_y$ *is stable.*

$\circ$

In condition (ii), notice that for G_z to be right invertible it is necessary that there be at least as many process inputs as signals to be estimated, i.e. $m \geq p$. The assumption that G_z has no zeros in the CRHP is made to simplify the analysis.

Since the filter F is stable and the plant G is stabilizable, it follows that condition (iii) is necessary and sufficient for the filter to be a *Bounded Error Estimator* (BEE), as defined below (Seron & Goodwin 1995, Seron 1995).

Definition 2.1 (BEE). We say that a stable filter is a BEE if, for all finite initial states of the plant and the filter, the estimation error $\tilde{z}$ is bounded whenever the system inputs v and w are bounded. $\circ$

It can be inferred from condition (iii) that unstable modes shared by G_z and G_y are canceled[2] by a zero of the difference $G_z - FG_y$, while those unstable poles of G_y that do

[1] Stabilizability of (A, B) is assumed only for simplicity of the presentation.

[2] In frequency and direction.

not appear in G_z must be necessarily canceled by NMP zeros of F. The BEE assumption is the minimum that can be required from a filter and, therefore, the natural starting point of numerous state estimation problems in the literature.

Following Goodwin et al. (1995) and Seron & Goodwin (1995), we now define the *filtering sensitivity* and *complementary sensitivity functions*, denoted by P and M, respectively. Introduce the following input-output operators:

$T_{z\nu}$: mapping process input ν to signal z,

$T_{\hat{z}\nu}$: mapping process input ν to estimate $\hat{z}$, and

$T_{\tilde{z}\nu}$: mapping process input ν to estimation error $\tilde{z}$.

We then define

$$P \triangleq T_{\tilde{z}\nu}T_{z\nu}^{-1}, \qquad \text{and} \qquad M \triangleq T_{\hat{z}\nu}T_{z\nu}^{-1}.$$

Hence, the filtering sensitivity P represents the relative effect of the process input on the estimation error, while the filtering complementary sensitivity M represents the relative effect of the process input on the estimation. In terms of the transfer functions defined above, we can express P and M as follows:

$$P = (G_z - FG_y)G_z^{-1}, \qquad \text{and} \qquad M = FG_yG_z^{-1}. \tag{1}$$

As is the case for their control systems counterparts, P and M are complementary operators, in the sense that they add to the identity, $P + M = I$. There are other interesting properties of these functions that also correspond to the control case. For example, considering the SISO case, it would be desirable that $|P(j\omega)|$ be small over the portion of the imaginary axis where the spectrum of the process input ν is concentrated, which is typically at low frequencies. On the other hand, note that at the same range of frequencies $|M(j\omega)|$ will be close to 1, which is also a sensible goal if one expects the estimation to be close to the target signal (Seron & Goodwin 1995, Seron 1995).

In the MIMO case, these frequency restrictions apply to linear combinations of the scalar entries of P and M, thus, the penalizations imposed by ORHP zeros and poles are somehow "shared" by several elements. However, if in addition to frequency specifications, a particular structure is required for these functions (e.g., diagonal, diagonal-dominant, triangular, etc.), then extra costs may arise (see Gómez & Goodwin 1995).

The following section presents a set of interpolation constraints that P and M must satisfy for BEEs. In §4 we use these interpolation conditions to derive Poisson integral constraints on the MIMO filtering sensitivities. We show an application of these results in §5 and §6, where we analyze the problems of diagonalization and its implications in fault detection and isolation.

3 Interpolation Constraints

It was shown in Seron & Goodwin that, in the scalar case, ORHP poles shared by the system transfer functions $T_{z\nu}$ and $T_{y\nu}$ (mapping process input ν to measured output y) are zeros of P whilst ORHP zeros of the system transfer function $T_{y\nu}$ that are not zeros of $T_{z\nu}$, are zeros of M. The following lemma generalizes these results to the MIMO case.

Lemma 3.1 (Interpolation Constraints). *Suppose that* G *and* F *satisfy the conditions stated in Assumption 1. Then the following conditions are satisfied by* P *and* M*:*

(i) P *and* M *are stable.*

(ii) Let $\alpha \in \overline{\mathbb{C}^+}$ *be a pole of* G_z*. Then, there exists a nonzero vector* $a \in \mathbb{C}^p$ *such that*

$$P(\alpha)a = 0, \qquad \textit{and} \qquad M(\alpha)a = a. \tag{2}$$

(iii) Let $\beta \in \overline{\mathbb{C}^+}$ *be a zero of* G_y. *Then, there exist a nonzero vector* $b \in \mathbb{C}^p$ *such that*

$$P(\beta)b = b, \qquad \textit{and} \qquad M(\beta)b = 0. \tag{3}$$

(iv) Let $\gamma \in \overline{\mathbb{C}^+}$ *be a zero of* $G_z - FG_y$. *Then there exists a nonzero vector* $c \in \mathbb{C}^p$ *such that*

$$c^*P(\gamma) = 0, \qquad \textit{and} \qquad c^*M(\gamma) = c^*. \tag{4}$$

(v) Let $\delta \in \overline{\mathbb{C}^+}$ *be a zero of* F, *and suppose further that the product* $G_yG_z^{-1}$ *does not have a pole at* δ. *Then there exists a nonzero vector* $d \in \mathbb{C}^p$ *such that*

$$d^*P(\delta) = d^*, \qquad \textit{and} \qquad d^*M(\delta) = 0. \tag{5}$$

Proof. Stability of P and M follows from the fact that F is a BEE and G_z is minimum phase. Case (ii) follows in a straightforward manner from (1) by noting that the term $G_z - FG_y$ is stable by Assumption 1. For case (iii), we have that there exists a nonzero vector b_1 such that $G_y(\beta)b_1 = 0$. Since G_z is minimum phase, then there exists a nonzero vector b such that $G_z^{-1}(\beta)b = b_1$. Hence, since F is assumed stable, (3) follows. Case (iv) is immediate on noting that G_z is minimum phase. Finally, for case (v) notice that if δ is not a pole of $G_yG_z^{-1}$, then necessarily there exists a vector d such that $d^*F(\delta)G_y(\delta)G_z^{-1}(\delta) = 0$, and (5) follows. □

Remark 3.1. Conditions (iv) and (v) depend on the particular choice of filter and they are stated for completeness. However, our main focus here is on results which hold irrespective of the particular filter — this is true of the other cases. Yet, as we will see in §6, there are filtering applications where zeros of F are directly determined by the unstable poles of the plant. ⋄

Remark 3.2. Notice that since the plant is detectable from y, the transfer function G_y has all the unstable poles of G_z. In consequence, $G_yG_z^{-1}$ — and hence M — has no zeros at any unstable pole of G. ⋄

4 Poisson Integral Constraints

The interpolation constraints derived in the previous section fix the values of the filtering sensitivity functions at points in the CRHP determined by poles and zeros of the plant and filter. In the SISO case (Seron & Goodwin 1995) the Poisson integral was used to translate these constraints into equivalent integral relations that must be satisfied by P and M along the $j\omega$-axis.

This section extends the results of Seron & Goodwin to MIMO filtering problems using a technique suggested by Gómez & Goodwin (1995).

We first introduce some preliminary notation. Recalling that P and M are $p \times p$ square transfer matrices, we denote by P_{ik}, and M_{ik} their respective elements in the i-row and k-column. If h is a vector in $\mathbb{C}^p$, we denote its elements by h_i, $i = 1, \dots, p$, i.e. $h = [h_1^*, h_2^*, \dots, h_p^*]^*$. Also, we introduce the index set $\mathcal{I}_h \triangleq \{i \in \mathbb{N} : h_i \neq 0\}$ as the set of indices of the nonzero elements of h.

Given a set $\Psi = \{\zeta_i, i = 1, \dots, \ell\}$ of complex numbers in the ORHP, we define its *Blaschke product* as the function

$$B_k(s) = \prod_{i=1}^{\ell} \frac{s - \xi_i}{s + \bar{\xi}_i}.$$

If Ψ is empty, we define $B_k(s) = 1$, $\forall s$. We then have the following result.

Theorem 4.1 (Poisson Integral for P). *Let* $\beta = \sigma_\beta + j\omega_\beta$, $\sigma_\beta > 0$, *be a zero of* G_y, *and let* $b \in \mathbb{C}^p$, $b \neq 0$, *be the corresponding zero direction of* M, *as described in Lemma 3.1 (iii). Then, assuming that* P *is proper, for each number* k *in* $\mathcal{I}_b$,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \log\left|\sum_{i=1}^{p} P_{ki}(j\omega)\frac{b_i}{b_k}\right| \frac{\sigma_\beta}{\sigma_\beta^2 + (\omega_\beta - \omega)^2} d\omega \geq 0. \tag{6}$$

Proof. The proof follows the same general line as that in Gómez & Goodwin (1995), and is an application of the Poisson integral formula, as in Freudenberg & Looze (1985), to the elements of the vectorial function $Pb : \mathbb{C} \to \mathbb{C}^p$, which are proper, stable scalar rational functions. Pick one of these elements, say $\rho_k(s) \triangleq \sum_{i=1}^{p} P_k i(s) b_i$, where k is in $\mathcal{I}_b$, and let B_k be the Blaschke product of the zeros of ρ_k in the ORHP. Then, the function $\tilde{\rho}(s) \triangleq B_k^{-1}(s)\rho_k(s)$ is proper, stable, and NMP, which implies that $\log\tilde{\rho}(s)$ is analytic in the ORHP and satisfies the conditions of the Poisson integral formula (Levinson & Redheffer 1970, p. 225). Evaluating this at $s = \beta$, and using the fact that $|\tilde{\rho_k}(j\omega)| = |\rho_k(j\omega)|$ yields

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty} \log\left|\sum_{i=1}^{p} P_{ki}(j\omega)b_i\right| \frac{\sigma_\beta}{\sigma_\beta^2 + (\omega_\beta - \omega)^2} d\omega &= \log|B_k^{-1}(\beta)| + \log\left|\sum_{i=1}^{p} P_{ki}(\beta)b_i\right| \\ &= \log|B_k^{-1}(\beta)| + \log|b_k|,\end{aligned} \tag{7}$$

where the last step follows from the interpolation condition $P(\beta)b = b$ (notice that $b_k \neq 0$ by assumption). Since

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_\beta}{\sigma_\beta^2 + (\omega_\beta - \omega)^2} d\omega = 1,$$

subtracting $\log|b_k|$ from both sides of (7) yields

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \log\left|\sum_{i=1}^{p} P_{ki}(j\omega)\frac{b_i}{b_k}\right| \frac{\sigma_\beta}{\sigma_\beta^2 + (\omega_\beta - \omega)^2} d\omega = \log|B_k^{-1}(\beta)|. \tag{8}$$

The proof is then completed on noting that $|B_k^{-1}(s)| \geq 1$ at any point s in $\mathbb{C}^+$, and so $\log|B_k^{-1}(\beta)| \geq 0$. □

Theorem 4.1 establishes that for each NMP zero of the plant G_y, there is a set of integral constraints that limit the values of P on the $j\omega$-axis in an intrinsically vectorial fashion. More specifically, depending on the characteristics of the input directions associated to the zero, up to p integral constraints may arise on linear combinations of elements of P in the same row. If the zero direction happens to be canonical or P is designed to be diagonal, then the constraints reduce to those of the SISO case.

Recall that, in the SISO case, if the plant is NMP, then requiring that (as often desirable) $|P(j\omega)| < 1$ over a frequency range, say Ω, implies that, necessarily, $|P(j\omega)| > 1$ at other frequencies. The severity of this tradeoff depends on the relative location of the NMP zero and the frequency range Ω. More details and interpretations may be found in Seron & Goodwin (1995).

A similar integral holds for the filtering complementary sensitivity function, as established below.

Theorem 4.2 (Poisson Integral for M). *Let* $\alpha = \sigma_\alpha + j\omega_\alpha$, $\sigma_\alpha > 0$, *be a pole of* G_z, *and let* $a \in \mathbb{C}^p$, $a \neq 0$, *be the corresponding zero direction of* P, *as described in Lemma 3.1 (ii). Then, assuming that* M *is proper, for each number* k *in* $\mathcal{I}_a$,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \log\left|\sum_{i=1}^{p} M_{ki}(j\omega)\frac{a_i}{a_k}\right| \frac{\sigma_\alpha}{\sigma_\alpha^2 + (\omega_\alpha - \omega)^2} d\omega \geq 0. \tag{9}$$

Proof. The proof follows that of Theorem 4.1, this time using the interpolation constraint (ii) in Lemma 3.1. □

A straightforward corollary of these results emphasizes the vectorial nature of the associated tradeoffs. Let $\Omega_1 \triangleq [0, \omega_1]$ denote a given range of frequencies of interest, and assume that the k-row of M satisfies the following design specifications:

$$|M_{ki}(j\omega)| \le \epsilon_{ki} \quad \text{for } \omega \text{ in } [\omega_1, \infty],\ i = 1, \dots, p, \tag{10}$$

where ϵ_{ki}, $i = 1, \dots, p$, are small positive numbers. Define the *weighted length of* Ω_1 as

$$\Theta_\alpha(\omega_1) \triangleq \int_0^{\omega_1} \frac{\sigma_\alpha}{\sigma_\alpha^2 + (\omega_\alpha - \omega)^2} + \frac{\sigma_\alpha}{\sigma_\alpha^2 + (\omega_\alpha + \omega)^2}\, d\omega. \tag{11}$$

Then, we have the following corollary (Gómez & Goodwin 1995).

Corollary 4.3. *Assume that all the conditions of Theorem 4.2 hold. Then, if the* k*-row of* M *achieves specifications* (10)*, the following inequality must be satisfied,*

$$\|M_{kk}\|_\infty + \sum_{\substack{i=1\\ i\neq k}}^{p} \left|\frac{a_i}{a_k}\right| \|M_{ki}\|_\infty \ge \left(\frac{1}{\epsilon_{kk} + \sum_{\substack{i=1\\ i\neq k}}^{p} \left|\frac{a_i}{a_k}\right| \epsilon_{ki}} \right)^{\frac{\pi - \Theta_\alpha(\omega_1)}{\Theta_\alpha(\omega_1)}} \tag{12}$$

Proof. The proof follows immediately from Theorem 4.2 by using specifications (10) on M (see Gómez & Goodwin 1995, Corollary 2.1). □

As in the SISO case, the corollary shows that integral relation (9) implies lower bounds on the infinity norm of elements of M. Notice that the exponent on the RHS of (12) is a positive number and its base is likely to be larger than one if the ϵ_{ki} are small enough; hence, the more demanding the specifications, the larger these lower bounds are.

If the zero direction is not canonical, an important difference in the MIMO case is that the lower bounds apply to a combination of norms of elements, which somehow relaxes the constraint over the SISO case (only one element). Also, the lower bounds are smaller than in the SISO case due to the presence of extra positive terms in the denominator of the RHS of (12). In consequence, if M is required to be diagonal (as often occurs in a number of applications), the constraints on the values of M on the $j\omega$-axis worsen since the off-diagonal elements disappear. We analyze this in more detail in the following section.

5 The Cost of Diagonalization

As was concluded from Corollary 4.3, when the zero direction is not canonical the cost that it induces on each row of P or M can be shared among various elements of the row. On the other hand, as discussed in Gómez & Goodwin (1995), a diagonally decoupled system loses this potential alleviation of the cost, since all the price is paid by the diagonal elements alone.

By combining the ideas of Gómez & Goodwin and a parametrization of all diagonalizing post-compensators given by Kinnaert & Peng (1995), we can make a precise statement about the cost of diagonal decoupling. We first recall the result of Kinnaert & Peng for convenience. Given a $q \times p$ stable, proper, left-invertible transfer function H, it can be factorized as $H = R[\bar{H}^* \ 0]^*$, where R is a $q \times q$ bistable and biproper transfer function and $\bar{H}$ is an upper triangular $q \times q$ matrix (Dion & Commault 1988). The matrix $[\bar{H}^* \ 0]^*$ is a column Hermite form of H over the ring of stable proper rational functions and it is unique up to units of this ring. With the aid of this factorization, Kinnaert & Peng (1995) derived a parametrization of all stable, proper postcompensators F such that FH is a stable, proper diagonal matrix with non-zero diagonal elements. Using this parametrization for F, the product FH has the form

$$FH = LT \triangleq L\,\mathrm{diag}[t_1, t_2, \dots, t_p], \tag{13}$$

where L is an arbitrary $p \times p$ stable, proper diagonal matrix with non-zero diagonal elements, and the t_k, $k = 1, \dots p$, are functions

$$t_k(s) \triangleq \frac{\prod_{i=1}^{m_k}(s - \xi_i)}{(s + \zeta_k)^{m_k + r_k}}. \tag{14}$$

In (14), $\xi_i, i = 1, \dots m_k$ are the unstable poles of the k-row of $\bar{H}^{-1}$, ζ_k is an arbitrary positive real number and r_k is determined so that

$$\lim_{s \to \infty} t_k(s)\,[k\text{-row of } \bar{H}^{-1}] = \text{a non-zero row vector.}$$

Notice that the functions t_k contain information about the zeros of H, and their numerators depend on H alone.

Assume next that we are given a filtering problem where the design objective is to achieve a diagonal complementary sensitivity M. This is the case, for example, if the filter is to perform detection and isolation of system faults, as we will discuss in §6. We then have the following result.

Theorem 5.1 (Cost of Diagonalization). *Let $\alpha = \sigma_\alpha + j\omega_\alpha$, $\sigma_\alpha > 0$, be a pole of G_z, and let $a \in \mathbb{C}^p$, $a \neq 0$, be the corresponding zero direction of P, as described in Lemma 3.1 (ii). Assume that $G_y G_z^{-1}$ is a stable, proper and left invertible transfer function, with Hermite form*

$$G_y G_z^{-1} = R \begin{bmatrix} \bar{G} \\ 0 \end{bmatrix}. \tag{15}$$

Suppose that the filter F has been selected such that M in (1) is a stable, proper diagonal matrix with non-zero diagonal elements. Then, for each number k in $\mathcal{I}_a$,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \log|M_{kk}(j\omega)|\,\frac{\sigma_\alpha}{\sigma_\alpha^2 + (\omega_\alpha - \omega)^2}\,d\omega = \log|B_k^{-1}(\alpha)|, \tag{16}$$

where B_k is the Blaschke product of the unstable poles of the k-row of $\bar{G}^{-1}$.

Proof. Note that the assumptions imply that Theorem 4.2 holds specialized to the diagonal case, i.e.

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \log|M_{kk}(j\omega)|\,\frac{\sigma_\alpha}{\sigma_\alpha^2 + (\omega_\alpha - \omega)^2}\,d\omega \geq 0.$$

The inequality above is turned into an equality similar to (8) by adding the Blaschke product of ORHP zeros of ρ_k (defined in the proof of Theorem 4.1). Since M has the form (13) with $H = G_y G_z^{-1}$, this Blaschke product is independent of the filter and equal to the unstable poles of the k-row of $\bar{G}^{-1}$, with $\bar{G}$ defined in (15). This completes the proof. □

The following example illustrates the result.

Example 5.1. Consider a plant having the following transfer functions.

$$G_y(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{s-1}{s+1} & 0 \\ 0 & 0 & \frac{4-s}{(s-3)(s+2)} \\ \frac{4}{s^2+3s+5} & 0 & \frac{4}{s^2+3s+5} \end{bmatrix}, \qquad G_z^{-1}(s) = \begin{bmatrix} 0 & \frac{s-3}{s+2} \\ \frac{s+1}{s+3} & \frac{s+1/3}{s+2} \\ 0 & \frac{3-s}{s+2} \end{bmatrix}. \tag{17}$$

Note that G_z^{-1} has a zero at $\alpha = 3$ with input direction

$$a = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

which implies that integrals of the form (9) hold that constrain the frequency response of the rows of the complementary sensitivity M. These constraints are:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \log|M_{k1}(j\omega) - M_{k2}(j\omega)| \frac{3}{\omega^2+9}\, d\omega \geq 0, \quad k = 1, 2. \tag{18}$$

Next we compute the product $G_y G_z^{-1}$, i.e.

$$G_y(s)G_z^{-1}(s) = \begin{bmatrix} \frac{s-1}{s+3} & \frac{s-5/3}{s+1} \\ 0 & \frac{s-4}{(s+2)^2} \\ 0 & 0 \end{bmatrix},$$

which is in the Hermite form (15) with $R = I$. If the filter F is selected to achieve diagonalization of M, then necessarily M has the form (13) with $H = G_y G_z^{-1}$, i.e.

$$M(s) = L(s) \begin{bmatrix} \frac{(s-1)(s-4)}{(s+\zeta_1)^3} & 0 \\ 0 & \frac{s-4}{(s+\zeta_2)^2} \end{bmatrix}.$$

In this case, Theorem 5.1 holds, giving the following integral constraints on the diagonal elements of M:

$$\begin{aligned} \frac{1}{\pi}\int_{-\infty}^{\infty} \log|M_{11}(j\omega)| \frac{3}{\omega^2+9}\, d\omega &= 2.6391, \\ \frac{1}{\pi}\int_{-\infty}^{\infty} \log|M_{22}(j\omega)| \frac{3}{\omega^2+9}\, d\omega &= 1.9459. \end{aligned} \tag{19}$$

Now we see that *even without imposing specifications on the values of* M *on the* $j\omega$*-axis*, there are non trivial bounds arising from the diagonalization. Indeed, on the one hand it follows from (18) that

$$\begin{aligned} \|M_{11} - M_{12}\|_\infty &\geq 1, \\ \|M_{21} - M_{22}\|_\infty &\geq 1. \end{aligned}$$

On the other hand, it follows from (19) — after diagonalizing, i.e. $M_{12} = 0 = M_{21}$ — that

$$\begin{aligned} \|M_{11}\|_\infty &\geq 14, \\ \|M_{22}\|_\infty &\geq 7. \end{aligned} \tag{20}$$

Therefore, a diagonal M is obtained at the cost of large peaks in its diagonal entries. These peaks might be highly undesirable if they happen to occur at frequencies where M is required to be small. Moreover, it is not difficult to see that if frequency specifications on the entries of M are imposed in addition, the design tradeoffs arising from (20) will be even worse. ⋄

Theorem 5.1 has immediate application in filtering problems where diagonal sensitivities are desirable. In the next section, we illustrate our results by studying the problem of detection and isolation of faults in the context of multivariable filtering.

6 Application to Fault Detection

Fault detection and isolation (FDI) finds application in complex multiple-component systems where, for reasons of safety or economy, tolerance to component failure is required. An approach to achieving fault tolerance[3] consists in exploiting the redundancy inherent in the system model, which is known as *model-based* FDI.

A common method for model-based FDI is to design a filter that generates a *residual*, or fault-sensitive signal, that is distinguishable from zero when a component of the system fails but remains close to zero otherwise. If the residual has different properties for different component faults, then it is also possible to isolate the faulty component. A filter that generates a residual that allows both detection and isolation of faults is called an FDI filter. A number of techniques are available for constructing FDI filters; see for example Massoumnia, Verghese & Willsky (1989) and the references therein.

Two issues are important in FDI filter design. Firstly, the residual should be sensitive to faults at those frequencies where the energy of the fault is likely to be concentrated. Secondly, the residual should be insensitive to other process disturbances and noise. It is well understood that these are conflicting objectives, since both sensitivity to faults and insensitivity to disturbances cannot be achieved at the same frequency. However, the implications of ORHP poles and zeros of G for achieving design objectives over a range of frequencies are less well appreciated.

In this section we show that, by approaching the FDI problem in the context of multivariable filtering of §2, the Poisson integral constraints of §4 are useful to precisely quantify the limitations imposed on FDI filters by ORHP poles and zeros of G. Furthermore, in connection with §5, these constraints are shown to be more severe due to a structural requirement of diagonalization imposed by the condition of isolation.

We recast the FDI problem as a filtering problem within the general framework of Figure 1 by taking z as the fault to be detected, and $\hat{z}$ to be the residual. By choosing the structure of G, it is possible to model a wide variety of faults. For example, the scheme of Figure 3 represents the important cases of *actuator* and *sensor* faults. Then fault of the i-th actuator (or sensor) is modeled by $z_i \neq 0$. Nothing is assumed about the mode of fault, i.e. z is an arbitrary function of time.

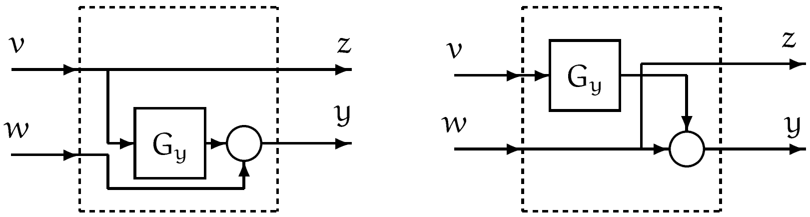

Figure 3: Plant models for actuator (left) and sensor (right) faults.

In this context, M maps the fault z to the residual $\hat{z}$, and P maps the fault to the "detection error" $\tilde{z}$. Thus, M and P quantify the quality of detection.

We will say that F is an FDI filter if

(i) (detection) for all initial conditions in G and F, and $v, w = 0$, $\hat{z}(t) \to 0$; and

(ii) (isolation) if $z_i \neq 0$, then $\hat{z}_i \neq 0$ and $\hat{z}_j = 0, \; j \neq i$.

Condition (i) is equivalent to the requirement that F is a BEE. Condition (ii) is equivalent to the requirement that M is diagonal. This is not the only structure that allows isolation

[3]Avoiding the expense of hardware redundancy.

(Massoumnia et al. 1989), but is often studied because (a) it renders the problem of isolation trivial, and (b) it permits isolation of simultaneous faults.

The filtering sensitivity functions corresponding to the problems of actuator and sensor FDI have the forms given by Table 1.

	P	M
Actuator FDI	$I - FG_y$	FG_y
Sensor FDI	$I - F$	F

Table 1: Filtering sensitivity functions for actuator and sensor FDI.

We illustrate the ideas for the case of sensor faults; similar analyses can be performed for the case of actuator faults. Notice that, at first sight, the problem of sensor FDI may appear to be unconstrained, since P and M only depend on the filter, which is the variable of design. Yet, even for detection alone, due to the fact that the filter is a BEE, F must incorporate information of the unstable dynamics of the plant. To see this, let $G_y = \tilde{D}^{-1}\tilde{N} = ND^{-1}$, where $(\tilde{N}, \tilde{D})$, and (N, D) are left and right coprime factorizations of G_y (Vidyasagar 1985). Then, using a parametrization of all BEEs (see e.g., Seron & Goodwin 1995), it is easy to show that F necessarily has the form $F = Q\tilde{D}$, where Q is any stable, proper transfer function of appropriate dimensions. Therefore, M in the case of sensor FDI has zeros at the unstable poles of the plant. Moreover, if isolation is also required, then the filter must not only be a BEE, but also diagonalize the filtering complementary sensitivity M. As we have seen in §5, this has the associated cost — quantified by (16) — of further worsening the design limitations arising from unstable poles of the plant with non canonical directions.

For sensor FDI, M not only measures the quality of detection as mentioned above, but it also maps measurement noise w and the relative effect of the input disturbance v to both $\hat{z}$ and $\tilde{z}$, the residual and detection error respectively. Hence P and M may be used to specify the properties of the filter to adequately detect faults and reject disturbances. These objectives can sometimes be conflicting, as is the case for input disturbances and low frequency faults, but can successfully be achieved, for example, in the case of low frequency faults and high frequency disturbances. In the latter case, a suitable shape for the diagonal elements of M will require that $|M_{kk}(j\omega)|$ be close to one at low frequencies[4] and close to zero at high frequencies where the power spectrum of the disturbance concentrates its energy. These objectives can be translated into the following design specs:

(i) $|M_{kk}(j\omega)| \approx 1$ on $[0, \omega_1]$.

(ii) $|M_{kk}(j\omega)| < \epsilon_k < 1$ on $[\omega_2, \infty]$,

with $\omega_1 < \omega_2$. Using these requirements and Theorem 5.1 it is easy to see that the following lower bound holds for the peak of M_{kk} between ω_1 and ω_2.

Corollary 6.1. *Assume that the conditions of Theorem 5.1 hold. Then the following inequality is satisfied:*

$$\sup_{\omega\in[\omega_1,\omega_2]} |M_{kk}(j\omega)| \geq \left(\frac{1}{\epsilon_k}\right)^{\frac{\pi-\Theta_\alpha(\omega_2)}{\Theta_\alpha(\omega_2)-\Theta_\alpha(\omega_1)}}. \tag{21}$$

Proof. Immediate from Theorem 5.1 and the given specs. □

We can argue from the above corollary that a large peak may occur if ϵ_k is too small, or if ω_1 is too close to ω_2. As just discussed, this may have a deleterious impact on detection if there are input disturbances with frequency content within $[\omega_1, \omega_2]$.

[4] For the k-residual to be sensitive to the k-fault, we require $\tilde{z}_k$ to be small, and therefore $|M_{kk}(j\omega)|$ to be close to one at frequencies where the fault is assumed to occur.

Corollary 6.1 shows the tradeoffs in sensor FDI due to the constraints imposed by unstable poles of the plant G_y. Similar tradeoffs occur in the case of actuator FDI in connection to NMP zeros of G_y. These results may also be extended to the case of general component FDI, where there are tradeoffs due to both ORHP zeros and poles of the plant.

7 Conclusions

In this paper we have presented new results on integral constraints for multivariable filtering problems. We have used the definitions of filtering sensitivity functions to quantify important properties in filter design, such as quality of estimation and relative effect of disturbances. Differently to the SISO case, a new dimension arises in the analysis of MIMO filtering limitations since the structural properties of these functions have non-neglectable impact. In particular, we have shown that problems in which diagonalization is required may introduce additional limitations. As an example of application, we have considered problems of fault detection and isolation in a multivariable filtering setting.

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