Performance Limitations in a Class of Single-Input Two-Output Nonlinear Systems*

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Abstract

This paper shows that there exists a structural limitation in reducing the integral of the squared regulation error in strict-feedback nonlinear systems with more outputs (that is, performance variables) than inputs. This limitation arises from the fact that, in these systems, some outputs are controlled only indirectly through other outputs with lower relative degree. It is then impossible to uniformly reduce the overall regulation cost to zero, the infimum value of which is characterised by a reduced-order optimal regulation problem. Although essentially nonlinear, the results are similar in nature and have direct connections with their linear counterparts.

1 Introduction

This paper studies the best achievable performance in optimal quadratic output regulation for a class of single-input two-output (SITO) nonlinear systems. For linear systems, performance limitations — in the quadratic sense and otherwise — arising from the fact that the plant has more outputs than inputs have been known for some time (Kwakernaak & Sivan 1972, Francis 1979, Freudenberg & Looze 1986, Saberi & Sannuti 1987) and are the subject of current research activity (Woodyatt, Freudenberg & Middleton 1997, Freudenberg & Middleton 1998, Woodyatt, Seron, Freudenberg & Middleton 1998).

An effective tool for the analysis of feedback limitations, in particular with quadratic performance, is the optimal cheap control approach, used in many of the aforementioned works. As shown in Seron, Braslavsky, Kokotović, & Mayne (1997), a valuable property of the cheap control approach lies in its applicability to nonlinear systems. Seron et al. (1997) have used a near-optimal cheap control framework to study feedback limitations in a class of nonlinear systems with relative degree one. The same framework has been recently extended in Braslavsky, Seron & Kokotović (1998) to strict-feedback nonlinear systems with arbitrary relative degree.

Using the near-optimal cheap control framework, we prove that, under certain conditions, there exists a positive lower bound for the achievable L_2-norm of the output, which therefore cannot be

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reduced to zero by any controller. We argue that this is a fundamental limitation analogous to that posed by non-right-invertibility in a linear system with more independent outputs than inputs. Moreover, we show that this limitation worsens as the separation between the time scales of the outputs widens, a phenomenon that appears in the linear case as an increase in the variation with frequency of the plant direction (Woodyatt et al. 1998).

As a preliminary step, we consider nonlinear systems in strict-feedback form in which one output has relative degree one, and the other relative degree two with respect to the input. The case with relative degrees n and $n+1$ may be dealt with as in Braslavsky et al. (1998), and has been omitted here for brevity of exposition. The present setting holds for systems in which the two outputs and the input are vectors, but where each set of outputs share the same (uniform) relative degree with respect to the set of inputs. With a slight abuse in the terminology, we will still call these systems SITO, which is the essential underlying structure.

2 Optimal Cheap Control

For the nonlinear system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)\,x_2\,, & x_1 &\in \mathbb{R}^p\,, \\ \dot{x}_2 &= f_2(x_1,x_2) + g_2(x_1)\,u\,, & x_2, u &\in \mathbb{R}^m\,, \end{aligned} \tag{1}$$

where f_1, g_1, f_2, g_2 are continuously differentiable (C^1) vector fields, with $f_1(0) = 0 = f_2(0,0)$, we consider the problem of finding a feedback control that renders its origin $(x_1, x_2) = (0,0)$ asymptotically stable[1] and minimises the functional

$$J_\varepsilon(x_1(0), x_2(0), u) = \frac{1}{2}\int_0^\infty \left(\|x_1(t)\|^2 + \|x_2(t)\|^2 + \varepsilon^2\|u(t)\|^2\right)\,dt\,. \tag{2}$$

The parameter $\varepsilon > 0$ scales the penalty on the control effort. As $\varepsilon \to 0$, the control effort becomes free and the optimal control approaches its ideal "cheap" control performance[2]

$$J_0(x_1(0), x_2(0)) = \lim_{\varepsilon\to 0}\min_u J_\varepsilon(x_1(0), x_2(0), u)\,. \tag{3}$$

Because the cheap control is unconstrained, it yields the lowest achievable L_2-norm of the output (equal to $\sqrt{2J_0}$) among all the controls that asymptotically stabilise (1).

For linear systems, it is well-known that *perfect regulation* — that is, $J_0 = 0$ — can be achieved if and only if the system is minimum phase and right-invertible (Kwakernaak & Sivan 1972, Theorem 3.14). A positive J_0 quantifies a feedback obstruction that can be associated with the lack of a stable right-inverse of the plant — i.e., non-minimum phase (unstable inverse) and/or "tall" (no right-inverse). Explicit expressions for J_0 have been obtained for MIMO square linear plants (Shaked 1980, Qiu & Davison 1993) and also for non-right-invertible linear plants (Woodyatt et al. 1998).

For a class of SISO strict-feedback nonlinear systems, it has been recently shown that J_0 must be positive if the system is non-minimum phase (i.e., it has unstable zero-dynamics) (Seron et al. 1997, Braslavsky et al. 1998). The main result in the present paper shows that J_0 must be positive for the class of non-right-invertible nonlinear systems (1).

For $\varepsilon > 0$ the optimal control problem for system (1) minimising (2) has a solution if there exists a C^2 function $V(x_1, x_2; \varepsilon) \geq 0$, $V(0,0;\varepsilon) = 0$, which satisfies the Hamilton-Jacobi-Bellman (HJB)

[1] We will hereafter shorten to "renders the system asymptotically stable".

[2] The existence of which will be guaranteed by our assumptions.

equation

$$\frac{\partial V}{\partial x_1}[f_1(x_1)+g_1(x_1)x_2]+\frac{\partial V}{\partial x_2}f_2(x_1,x_2)+\frac{1}{2}[x_1^Tx_1+x_2^Tx_2]-\frac{1}{2\varepsilon^2}\frac{\partial V}{\partial x_2}g_2(x_1)\,g_2^T(x_1)\frac{\partial^T V}{\partial x_2}=0, \tag{4}$$

and is such that the feedback control

$$u=-\frac{1}{\varepsilon^2}\,g_2^T(x_1)\frac{\partial^T V}{\partial x_2} \tag{5}$$

asymptotically stabilises (1). Then (5) is the optimal control and V is the optimal value function.

We will not attempt to solve this problem; instead, following Seron et al. (1997), we will construct a feedback control law that is $O(\varepsilon)$-*near-optimal*, in that it incurs a cost that approximates the optimal value with an $O(\varepsilon)$ error. We will then evaluate J_0 as the limit of this approximated value as $\varepsilon \to 0$.

3 Near-Optimality and Cheap Performance

The construction of our near-optimal control law is based on a candidate value function $\hat{V}$ that satisfies the HJB equation (4) with an $O(\varepsilon)$ error. This candidate function has the form

$$\hat{V}(x_1,x_2;\varepsilon)=V_0(x_1)+\varepsilon V_1(x_1,x_2)\,, \tag{6}$$

where V_0 and V_1 are functions to be specified. The motivation for such a choice for $\hat{V}$ is that it removes the singularity of the HJB equation (4) at $\varepsilon=0$; the substitution of V in (4) by $\hat{V}$ from (6) yields the ε-regular expression

$$\frac{\partial V_0}{\partial x_1}[f_1(x_1)+g_1(x_1)x_2]+\frac{1}{2}[x_1^Tx_1+x_2^Tx_2]-\frac{1}{2}\frac{\partial V_1}{\partial x_2}g_2(x_1)g_2^T(x_1)\frac{\partial^T V_1}{\partial x_2}+O(\varepsilon)=0\,. \tag{7}$$

Our control candidate for near-optimality,

$$\hat{u}=-\frac{1}{\varepsilon}\,g_2^T(x_1)\frac{\partial^T V_1}{\partial x_2}\,, \tag{8}$$

is obtained from the substitution of V by $\hat{V}$ in (5) and, like the optimal control law (5), still becomes unbounded when $\varepsilon=0$. We will now make precise the sense in which $\hat{u}$ is required to be near-optimal. Denote by B_R the ball of radius $R>0$ centred at the origin; $B_R=\{(x_1,x_2)\in\mathbb{R}^{p+m}:\|(x_1,x_2)\|\leq R\}$.

Definition 3.1 ($O(\varepsilon)$-near-optimal control). The control $\hat{u}$ from (8) is $O(\varepsilon)$-*near-optimal* with respect to the cost (2) if for each $R>0$ there exists some $\varepsilon_R>0$ such that for all $\varepsilon\in(0,\varepsilon_R]$

(i) the function $\hat{V}$ from (6) satisfies the HJB equation (4) with an $O(\varepsilon)$ error for all (x_1,x_2) in B_R, and

(ii) the trajectories of the closed-loop system (1), (8) starting at any given initial state $(x_1(0),x_2(0))$ in B_R asymptotically converge to the origin incurring a bounded L_2-cost, $J_0(x_1(0),x_2(0))$.

$\circ$

When we apply the control law (8) to (1), the resulting feedback system is singularly perturbed,

$$\begin{aligned}\dot{x}_1&=f_1(x_1)+g_1(x_1)\,x_2\,,\\ \varepsilon\dot{x}_2&=-g_2(x_1)g_2^T(x_1)\frac{\partial^T V_1}{\partial x_2}+\varepsilon f_2(x_1,x_2)\,.\end{aligned} \tag{9}$$

As shown in Seron et al. (1997), a singular perturbation analysis of (9) reveals the conditions that guarantee the near-optimality of the control law (8). The first condition is basic; it guarantees that the system (9) is in *standard singular perturbation form* (Kokotović, Khalil & O'Reilly 1986). This condition will be satisfied if the system (1) has a well-defined (global) relative degree.

Assumption 1. At each x_1 the matrix $g_2(x_1)$ is nonsingular. ◦

The second condition contains the essential requirement for near-optimality: the existence of the solution to an ε-independent, reduced-order optimal control problem, of which V_0 in (6) is the optimal value.

Assumption 2. There exists a C^2 positive definite and proper function $V_0(x_1)$, $V_0(0) = 0$, which satisfies the HJB equation

$$\frac{\partial V_0}{\partial x_1} f_1(x_1) + \frac{1}{2} x_1^T x_1 - \frac{1}{2}\frac{\partial V_0}{\partial x_1} g_1(x_1) g_1^T(x_1) \frac{\partial^T V_0}{\partial x_1} = 0 \,, \tag{10}$$

and is such that, for

$$\alpha(x_1) \triangleq -g_1^T(x_1)\frac{\partial^T V_0}{\partial x_1} \,, \tag{11}$$

(i) the system $\dot{x}_1 = f_1(x_1) + g_1(x_1)\alpha(x_1)$ is globally asymptotically stable (GAS),

(ii) there exist positive constants δ, k_1 and k_2, such that for each $(x_1, x_2) \in B_\delta$,

$$\left\| f_2(x_1, x_2) - \frac{\partial \alpha}{\partial x_1}\left[f_1(x_1) + g_1(x_1)x_2\right]\right\| \le k_1\|x_2\| + k_2\|\alpha(x_1)\| \,. \tag{12}$$

◦

Assumption 2 (i) implies the existence of a solution to the reduced-order optimal problem: find x_2 to globally asymptotically stabilise the system

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\, x_2 \tag{13}$$

minimising the cost

$$J(x_1(0), x_2) = \frac{1}{2}\int_0^\infty \left(\|x_1(t)\|^2 + \|x_2(t)\|^2\right) dt \,. \tag{14}$$

The minimum of the cost J is then achieved by the optimal "feedback" $x_2 = \alpha(x_1)$.

Assumption 2 (ii) is a local restriction in the interconnection between the x_1- and x_2-subsystems in (9). As discussed in Christofides & Teel (1996, Section VI), some kind of interconnection condition is required in order to guarantee the asymptotic convergence to the origin of the trajectories of a singularly perturbed system. Conditions like Assumption 2 (ii) have been proposed by Saberi & Khalil (1984), and are always satisfied by systems with a stabilisable linearisation at the origin (Saberi & Khalil 1985).

Proposition 3.1 (Near-Optimality). Under Assumptions 1 and 2, there exists a control law of the form (8) that is $O(\varepsilon)$-near-optimal with respect to the cost (2). Moreover, for each initial state (x_1, x_2), there exists some $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, the control (8) incurs the cost

$$J_\varepsilon(x_1, x_2, \hat{u}) = V_0(x_1) + O(\varepsilon) \,, \tag{15}$$

where V_0 is given by Assumption 2.

Proof. See the Appendix. □

The existence of the $O(\varepsilon)$-near-optimal cheap control guarantees the existence of the limiting "cheap" performance (3) as $\varepsilon \to 0$. Because of (15), the cheap performance is the positive definite function V_0; the value of the reduced-order optimal regulator problem (13), (14) (cf. Saberi & Sannuti 1985). This reduced-order regulator problem is a nonlinear counterpart of the *slow regulator problem* known from linear cheap control (Saberi & Sannuti 1987, Section IV A).

Corollary 3.2 (Cheap Performance). Under the conditions of Proposition 3.1, for every initial state (x_1, x_2), the optimal cheap value for the problem (1), (3) satisfies

$$J_0(x_1, x_2) = V_0(x_1)\,. \tag{16}$$

Proof. Since the cost incurred by the $O(\varepsilon)$-near-optimal control (8) satisfies (15), and the optimal control cannot produce a larger cost, (15) will therefore hold for the optimal value. In the limit when $\varepsilon \to 0$, (15) yields (16). □

It follows from Corollary 3.2 that only the initial conditions in x_1 (the outputs with greater relative degree) are relevant to the limiting cost. This is so because, as $\varepsilon \to 0$, x_2 is driven increasingly rapidly towards the manifold $x_2 = \alpha(x_1)$, where x_2 will henceforth drive x_1. The cost of the fast transition of x_2 from $x_2(0)$ to $\alpha(x_1(0))$ vanishes with ε, and what remains is the cost $V_0(x_1(0))$ of the slow motion required to drive x_1 (and x_2) to the origin subject to the constraint (14). This phenomenon shows, in the class of nonlinear systems (1), a structural constraint to L_2-performance that is analogous to that imposed by non-right-invertibility in linear systems. We further investigate this analogy in the next section.

4 Interpretations: Variation with Frequency of the Plant Direction

The obstacle to perfect regulation imposed by the non-right-invertibility of the plant has been quantified in linear systems as the "variation with frequency of the plant (transfer matrix) direction" (Woodyatt et al. 1998). The *direction* of a $p \times m$ plant $P(s)$ is defined as the (s-dependent) range of the full-rank $p \times m$ polynomial matrix $N(s)$ in the factorisation $P(s) = N(s)K(s)$, where $K(s)$ is an $m \times m$ proper transfer matrix. We assume that the rows of P are linearly independent over the field of real numbers. This assumption implies that the plant has p independent outputs, and implies that plant direction will always vary with frequency.

Perfect regulation may be achieved for a minimum phase plant when its direction is constant with frequency (that is, the range of $N(j\omega)$ is the same as that at DC, $N(0)$, at each $\omega \in \mathbb{R}$) (Woodyatt et al. 1998, Corollary 3.5). When the direction does vary with frequency, the optimal cheap value must be positive, and is quantified by a Bode-type integral.

We will now see that the cost (16) quantifies a nonlinear equivalent of this obstacle for the class (1) of SITO nonlinear systems. Moreover, we show that this obstacle worsens as the dynamics of the output with higher relative degree get slower, a phenomenon that appears in the linear case as an increment in the variation with frequency of the plant direction (Woodyatt et al. 1998).

4.1 Linear Case

From Woodyatt et al. (1998), the variation with frequency of the plant direction can be quantified as follows:

(i) Choose an orthonormal matrix $M \in \mathbb{R}^{p\times p}$ such that $M^T N(0) = \left[\begin{smallmatrix} V \\ 0 \end{smallmatrix}\right]$, where V is a full-rank square matrix.

(ii) Change $N(s)$ to the "DC coordinates" $M^T N(s) = \left[\begin{smallmatrix} N_1(s) \\ N_2(s) \end{smallmatrix}\right]$, and define $R = N_1^{-1} N_2$.

The transfer matrix R represents the ratio between the outputs in "DC coordinates". The cost due to the variation with frequency of the plant direction, denoted $\operatorname{var} P$, is

$$\operatorname{var} P = \frac{1}{\pi} \int_0^\infty \log\det\left(I + R^T(-j\omega)R(j\omega)\right) \frac{d\omega}{\omega^2} . \tag{17}$$

Because the outputs are independent, R cannot be identically zero, and hence, $\operatorname{var} P$ must be positive.

Now suppose we have the linear system

$$\begin{aligned} \mu\dot{x}_1 &= f_1 x_1 + g_1 x_2 , & x_1 &\in \mathbb{R}^p \\ \dot{x}_2 &= f_{21} x_1 + f_{22} x_2 + g_2 u , & x_2, u &\in \mathbb{R}^m \end{aligned} \tag{18}$$

which has the structure (1). Here $\mu > 0$ is a parameter that changes the time scale of the output x_1 *without modifying the DC value of the plant.* Denote by P_μ the transfer matrix corresponding to the μ-perturbed system (18)

$$P_\mu(s) = \begin{bmatrix} \mu(\mu s I - f_1)^{-1} g_1 \\ \mu I \end{bmatrix} \left[\mu s I - f_1 - g_1(sI - f_{22})^{-1} f_{21}\right]^{-1} g_2 . \tag{19}$$

We see from (19) that the direction of the plant depends on the x_1-equation alone and moreover, it turns out that $R_\mu(s) = R(\mu s)$. For the scalar case $p = 1 = m$, for example, we get

$$R_\mu(s) = \frac{\mu s g_1}{g_1^2 + f_1^2 - \mu s f_1} .$$

It then follows from (17) that $\operatorname{var} P_\mu = \mu \operatorname{var} P$, which shows that the parameter μ scales the variation with frequency of the plant direction, as measured by (17). Hence, if μ decreases, the dynamics of x_1 gets faster, the variation with frequency of the plant direction decreases, and so does the limiting cost. On the other hand, if μ increases, the dynamics of x_1 gets slower, and the variation with frequency of the plant direction and the limiting cost increase.

4.2 Nonlinear Case

The same phenomenon occurs in the nonlinear case. For if we perturb the system (1) to

$$\begin{aligned} \mu\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 , \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1)u , \end{aligned} \tag{20}$$

then Proposition 3.2 characterises its lowest achievable cost J_0 as the solution $V_{0\mu}$ of the HJB equation

$$\frac{1}{\mu}\frac{\partial V_{0\mu}}{\partial x_1} f_1(x_1) + \frac{1}{2} x_1^T x_1 - \frac{1}{2\mu^2}\frac{\partial V_{0\mu}}{\partial x_1} g_1(x_1) g_1^T(x_1) \frac{\partial^T V_{0\mu}}{\partial x_1} = 0 . \tag{21}$$

Thus if V_0 solved (10), corresponding to the unperturbed system ($\mu = 1$), then $V_{0\mu} = \mu V_0$ solves (21). Therefore, at each initial state, J_0 for the perturbed system (21) reduces when the dynamics of x_1 is made faster ($\mu < 1$), and increases when it is made slower ($\mu > 1$), as it happens in the linear case.

5 Examples

Example 5.1 (Nonlinear Dynamics "Help"). The system

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_1^3 + x_2\,, \\ \dot{x}_2 &= x_1^2 + u\,, \end{aligned} \tag{22}$$

belongs to the class (1), with $f_1(x_1) = -(x_1 + x_1^3)$, $f_2(x_1, x_2) = x_1^2$, and $g_1(x_1) = 1 = g_2(x_1)$. Assumption 1 is trivially satisfied ($g_2 = 1$). Let us check whether Assumption 2 is satisfied. Equation (10) here is

$$-\frac{\partial V_0}{\partial x_1}(x_1 + x_1^3) + \frac{x_1^2}{2} - \frac{1}{2}\left[\frac{\partial V_0}{\partial x_1}\right]^2 = 0$$

and admits the smooth positive definite solution

$$V_0(x_1) = -\frac{{x_1}^2}{2} - \frac{{x_1}^4}{4} + \frac{({x_1}^2+1)\sqrt{(x_1^2+1)^2+1}}{4} + \frac{\operatorname{arcsinh}({x_1}^2+1)}{4} - \frac{\sqrt{2}+\operatorname{arcsinh}(1)}{4}\,.$$

The manifold $x_2 = \alpha(x_1) = x_1 + x_1^3 - x_1\sqrt{(x_1^2+1)^2+1}$ yields, in the x_1-subsystem in (22), the "closed-loop" $\dot{x}_1 = -x_1\sqrt{(x_1^2+1)^2+1}$, which is GAS. Hence Assumption 2 (i) is satisfied. Assumption 2 (ii) is also satisfied because the linearisation of the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= u\,. \end{aligned} \tag{23}$$

is stabilisable. It then follows from Corollary 3.2 that for a given initial state $(x_1(0),\ x_2(0))$, the value of $\int_0^\infty [x_1^2(t) + x_2^2(t)]\,dt$ cannot be reduced below $V_0(x_1(0))$ by any asymptotically stabilising control.

We also can use Corollary 3.2 to compute the optimal cost (3) for the linearisation (23). The linear $O(\varepsilon)$-near-optimal control for (23) is

$$\hat{u}^\ell = -\frac{1}{\varepsilon}\left(x_2 + (\sqrt{2}-1)x_1\right) \tag{24}$$

with *linear* optimal cheap value

$$V_0^\ell(x_1) = \frac{(\sqrt{2}-1)x_1^2}{2}\,.$$

Note that V_0^ℓ is *not* the same as the limiting cost $J^\ell \triangleq \lim_{\varepsilon\to 0} J_\varepsilon(x_1(0), x_2(0), \hat{u}^\ell)$ incurred by $\hat{u}^\ell$ when applied to the nonlinear system (20). We can see this in Figure 1, which shows plots of V_0, V_0^ℓ, and J^ℓ — the three are functions of x_1 alone. The difference between the plots of V_0 and J^ℓ is indicative of the degree of extra cost reduction that may be achieved by allowing nonlinear control in (22) (not very large in this example). Here V_0^ℓ appears as a conservative estimate of the best achievable regulation, as measured by $\int_0^\infty [x_1^2(t) + x_2^2(t)]\,dt$.

Perfect regulation may be achieved for this system if only one output appears in the cost. Indeed, if only the output x_2 is in the cost, its zero dynamics are the x_1-subsystem in (22). It then follows from the SISO results in Seron et al. (1997) that J_0 is given as the value of the minimum energy problem: find x_2 to asymptotically stabilise the zero dynamics subsystem $\dot{x}_1 = -x_1 - x_1^3 + x_2$ while minimising $\int_0^\infty x_2^2\,dt$. Because this zero dynamics subsystem is GAS when $x_2 = 0$, the minimum energy value is zero, and so is J_0. On the other hand, if only x_1 is in the cost, the system has no zero dynamics at all, and hence, by the same token, J_0 is zero. ∘

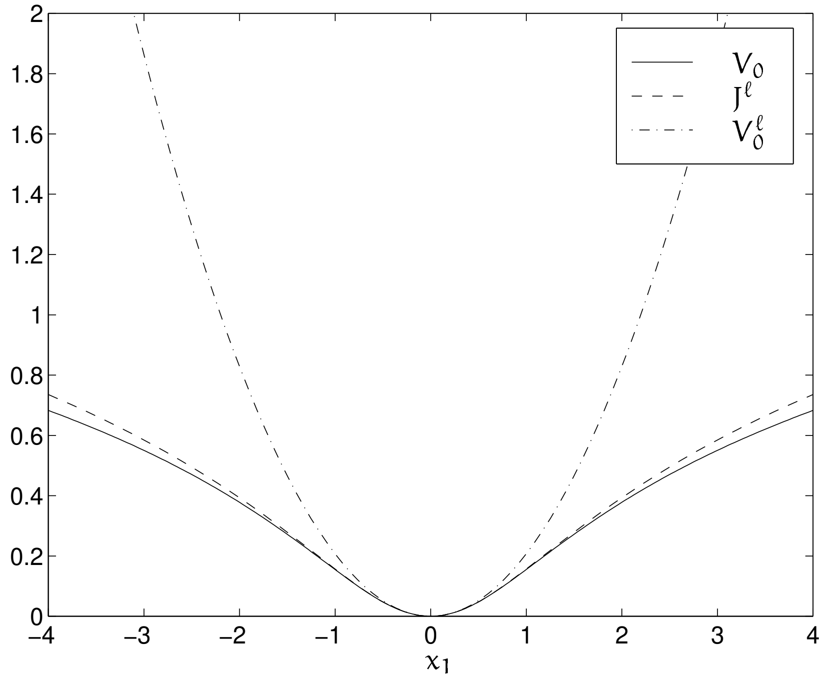

Figure 1: Optimal cheap values for systems (22) and (23), and cost incurred by the linear optimal cheap control on system (22).

Example 5.2 (Nonlinear Dynamics "Hurt"). We modify the system (22) to

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^3 + x_2\,, \\ \dot{x}_2 &= x_1^2 + u\,, \end{aligned} \tag{25}$$

by changing the sign of x_1^3 in the first equation. Assumption 2 is still satisfied, now with

$$V_0(x_1) = -\frac{{x_1}^2}{2} + \frac{{x_1}^4}{4} + \frac{({x_1}^2-1)\sqrt{(x_1^2-1)^2+1}}{4} + \frac{\operatorname{arcsinh}({x_1}^2-1)}{4} + \frac{\sqrt{2}+\operatorname{arcsinh}(1)}{4}\,.$$

Because the linearisation of (25) is again (23), the linear optimal cheap value V_0^ℓ here is the same as in Example 5.1. Differently in this case, the linear control can achieve asymptotic stability *only for* $|x_1(0)| < \sqrt[4]{2}$, so the cost J^ℓ is infinite outside that interval, as seen in Figure 2.

We also observe in Figure 2 that the optimal value V_0 is considerably larger than that obtained in Example 5.1. This additional cost may be interpreted as an extra amount of energy required to compensate the instability introduced by x_1^3 in the x_1-subsystem in (25). The linear optimal value V_0^ℓ appears in this case as an underestimate of the best achievable regulation.

Can perfect regulation be achieved here if only one output is weighted in the cost? The answer is yes if x_1 alone is weighted, because then, as in the previous example, there are no zero dynamics. However, the situation is quite different when only x_2 is in the cost, because the zero dynamics subsystem $\dot{x}_1 = -x_1 + x_1^3$ is now (globally) unstable when $x_2 = 0$. It follows from the results in Seron et al. (1997) that a nonzero amount of energy in x_2 will be required for stability, and therefore a positive value of J_0 must be expected.

We can see this by considering the (SISO) problem with cost $\frac{1}{2}\int_0^\infty [x_2^2 + \varepsilon^2 u^2]\,dt$ as the limit of the (SITO) problem with cost $\frac{1}{2}\int_0^\infty [\nu^2 x_1^2 + x_2^2 + \varepsilon^2 u^2]\,dt$ when $\nu \to 0$. A slight modification of the arguments in this paper will show that the limiting cost of this later SITO problem when $\varepsilon \to 0$, *for*

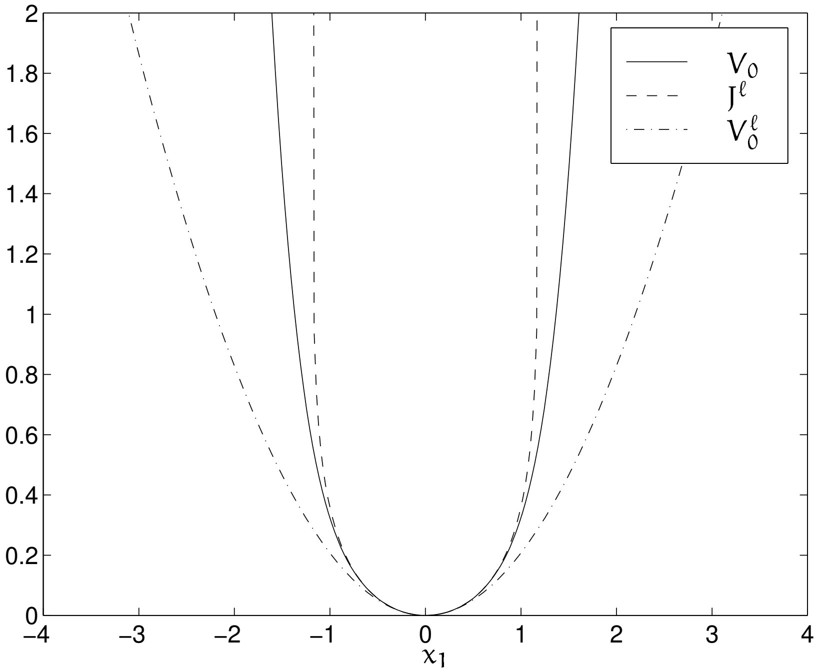

Figure 2: Optimal cheap values for systems (25) and (23), and cost incurred by the linear optimal cheap control on system (25).

a fixed ν, is the positive definite solution $V_{0\nu}$ of the HJB equation

$$\frac{\partial V_{0\nu}}{\partial x_1}(-x_1 + x_1^3) + \frac{\nu^2 x_1^2}{2} - \frac{1}{2}\left[\frac{\partial V_{0\nu}}{\partial x_1}\right]^2 = 0 \tag{26}$$

that renders $\dot{x}_1 = -x_1 + x_1^3 - \partial V_{0\nu}/\partial x_1$ GAS. It can be checked by direct differentiation that for any $\nu > 0$ the smooth positive definite function

$$V_{0\nu}(x_1) = -\frac{{x_1}^2}{2} + \frac{{x_1}^4}{4} + \frac{({x_1}^2 - 1)\sqrt{(x_1^2-1)^2+\nu^2}}{4} + \frac{\nu^2}{4}\operatorname{arcsinh}\left(\frac{{x_1}^2-1}{\nu}\right) + \frac{\sqrt{1+\nu^2}+\nu^2\operatorname{arcsinh}(\nu^{-1})}{4},$$

satisfies (26), $\partial V_{0\nu}/\partial x_1 = -x_1 + x_1^3 + x_1\sqrt{(x_1^2-1)^2+\nu^2}$, and gives the GAS "closed-loop" x_1-subsystem $\dot{x}_1 = -x_1\sqrt{(x_1^2-1)^2+\nu^2}$.

Figure 3 plots $V_{0\nu}$ for $\nu = 0.01$ and $\nu = 1$. The value $V_{0\nu}$ for $\nu = 0.01$ will be very close to J_0 for the SISO problem where only x_2 is weighted in the cost. The value $V_{0\nu}$ for $\nu = 1$ corresponds to the SITO problem where x_1 and x_2 are equally weighted. Although J_0 diminishes with ν, it is clear from the plot that "almost" perfect regulation would only be achieved (as $\nu \to 0$) for initial states $x_1(0) \in (-1, 1)$, for which the zero dynamics $\dot{x}_1 = -x_1 + x_1^3 = -(1 - x_1^2)x_1$ are already asymptotically stable. For initial states outside this interval a significant amount of energy will be required to compensate for the instability introduced by the nonlinear term x_1^3, which thus imposes an unavoidable performance limitation in the regulation problem. There will be no indication of this cost from the linearised analysis, which, for either (22) or (25), would yield $J_0 = 0$ if only x_2 is weighted in the cost. ∘

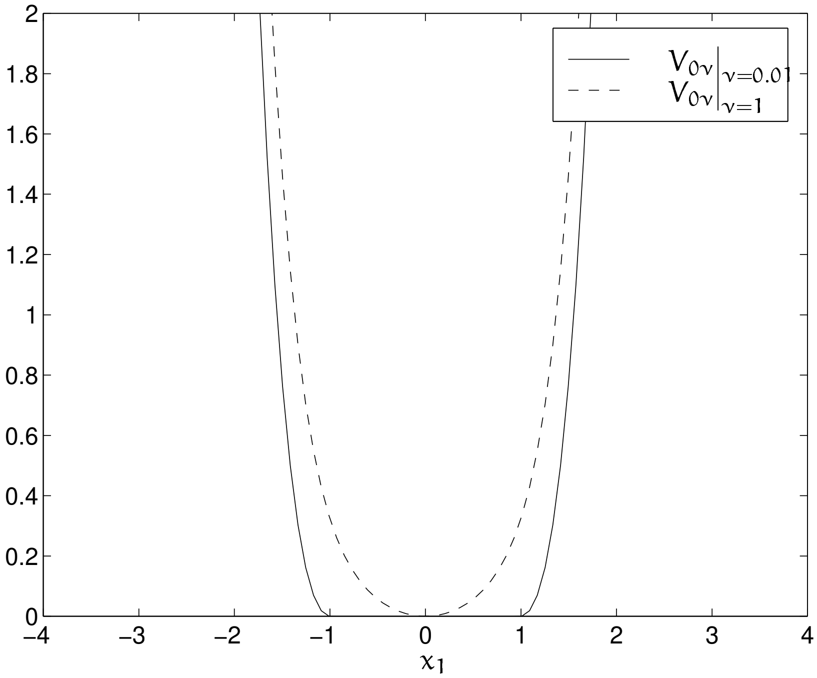

Figure 3: Optimal cheap values for system (25) corresponding to the cost $\frac{1}{2}\int_0^\infty [\nu^2 x_1^2 + x_2^2]\,dt$ for $\nu = 0.01, 1$.

6 Conclusions

The main result in this paper shows that there exists an unavoidable feedback limitation in reducing the output regulation error — as measured by L_2-norm — for a class of SITO strict-feedback nonlinear systems. We have established that the lowest achievable value for this error is the optimal value of a reduced-order regulation problem, in which the outputs with lower relative degree drive those with higher relative degree. Although essentially nonlinear, this result has interpretations that link directly with similar results for linear multivariable systems, as illustrated in the examples.

Notwithstanding that our conclusions hold for a particular class of SITO strict-feedback nonlinear systems, this feedback limitation seems inherent to the non-right-invertible nature of the strict-feedback structure. Its quantification for a wider class of systems is under current development.

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Appendix: Proof of Proposition 3.1

We first determine V_1 such that $\hat{V}$ in (6) satisfies the equation (4) with an $O(\varepsilon)$ error.

Lemma 6.1 ($O(\varepsilon)$ HJB Condition). Let Assumptions 1 and 2 be satisfied. Then $\hat{V}$ from (6) satisfies (4) with an $O(\varepsilon)$ error if V_0 is given by Assumption 2 and

$$V_1(x_1, x_2) = \frac{1}{2}[x_2 - \alpha(x_1)]^T [g_2(x_1) g_2^T(x_1)]^{-1/2} [x_2 - \alpha(x_1)]\,. \tag{27}$$

Proof. The substitution of V by $\hat{V}$ in (4) yields (7), so we only need to prove that V_0 and V_1 satisfy (7) with an $O(\varepsilon)$ error. By adding and subtracting $\frac{1}{2}\|\alpha(x_1)\|^2 = \frac{1}{2}\|g_1^T(x_1)\partial^T V_0/\partial x_1\|^2$ to (7), and using the fact that V_0 satisfies (10), we can rewrite (7) as

$$\frac{1}{2}\left[x_2-\alpha(x_1)\right]^T\left[x_2-\alpha(x_1)\right] - \frac{1}{2}\frac{\partial V_1}{\partial x_2}g_2(x_1)g_2^T(x_1)\frac{\partial^T V_1}{\partial x_2} = O(\varepsilon)\,. \tag{28}$$

Differentiation of V_1 from (27) with respect to x_2 gives

$$\frac{\partial V_1}{\partial x_2} = [x_2-\alpha(x_1)]^T[g_2(x_1)g_2^T(x_1)]^{-1/2}\,,$$

which satisfies (28) when $\varepsilon = 0$, and completes the proof. ☐

Now we show that the control law (8) achieves semiglobal (in ε) asymptotic stability of the system (1). We use V_1 from (27) in the closed-loop system (9) to obtain

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2\,,\\ \varepsilon\dot{x}_2 &= -[g_2(x_1)g_2^T(x_1)]^{1/2}[x_2-\alpha(x_1)] + \varepsilon f_2(x_1,x_2)\,.\end{aligned} \tag{29}$$

It is convenient to introduce the change of coordinates $\eta = x_2 - \alpha(x_1)$, under which (29) is rewritten as

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)\alpha(x_1) + g_1(x_1)\eta\,,\\ \varepsilon\dot{\eta} &= -[g_2(x_1)g_2^T(x_1)]^{1/2}\eta + \varepsilon\phi(x_1,\eta)\,,\end{aligned} \tag{30}$$

where $\phi(x_1,\eta) \triangleq f_2(x_1,\eta+\alpha(x_1)) - \frac{\partial\alpha}{\partial x_1}[f_1(x_1)+g_1(x_1)\alpha(x_1)+g_1(x_1)\eta]$. This system is in standard singular perturbation form (Kokotović et al. 1986). On the slow manifold $\eta = 0$ the system reduces to the slow subsystem

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\alpha(x_1)\,, \tag{31}$$

which, by Assumption 2 (i) is GAS. Convergence to the slow manifold is governed by the fast (boundary-layer) subsystem

$$\frac{d\eta}{d\tau} = -[g_2(x_1)g_2^T(x_1)]^{1/2}\eta\,, \qquad \tau = \frac{t}{\varepsilon}\,, \tag{32}$$

with x_1 treated as a constant vector. The fast subsystem (32) is exponentially stable by Assumption 1.

These stability properties of the slow and fast subsystems (31) and (32) guarantee semiglobal (in ε) boundedness of the solutions of (30) (Christofides & Teel 1996). This means that for each pair of positive numbers (R,ρ), there exists $\varepsilon_R > 0$ such that for all $\varepsilon \in (0,\varepsilon_R]$ the solutions of (30) starting in the ball B_R enter in finite time the ball B_ρ. Assumption 2 (ii) will guarantee the ultimate convergence to the origin.

Because the change of coordinates $\eta = x_2 - \alpha(x_1)$ is C^1 with C^1 inverse, the stability properties of (30) will be the same as those of (29).

Lemma 6.2 (Semiglobal (in ε) Asymptotic Stability). Under Assumptions 1 and 2, to each $R > 0$ there corresponds an $\varepsilon_R > 0$ such that for all $\varepsilon \in (0,\varepsilon_R]$ the equilibrium $(x_1,x_2) = (0,0)$ of (30) is asymptotically stable and its basin of attraction contains B_R.

Proof. We first analyse the stability properties of the system (30) in the region B_δ, given by Assumption 2 (ii). Introduce the Lyapunov function $W(x_1,\eta) = V_0(x_1) + \frac{1}{2}\eta^T\eta$. Its derivative along the solutions of (30) in B_δ satisfies

$$\begin{aligned}
\dot{W} &= -\frac{1}{\varepsilon}\eta^T[g_2(x_1)g_2^T(x_1)]^{1/2}\eta + \eta^T\phi(x_1,\eta) + \frac{\partial V_0}{\partial x_1}[f_1(x_1) + g_1(x_1)\alpha(x_1) + g_1(x_1)\eta]\,, \\
&\le -\frac{1}{\varepsilon}\eta^T[g_2(x_1)g_2^T(x_1)]^{1/2}\eta + k_1\|\eta\|^2 + k_2\|\eta\|\,\|\alpha(x_1)\| + \frac{\partial V_0}{\partial x_1}f_1(x_1) - \|\alpha(x_1)\|^2 - \alpha(x_1)\eta\,, \\
&\le -\frac{1}{\varepsilon}\eta^T[g_2(x_1)g_2^T(x_1)]^{1/2}\eta + k_1\|\eta\|^2 + (1+k_2)\|\eta\|\,\|\alpha(x_1)\| - \frac{1}{2}\|x_1\|^2 - \frac{1}{2}\|\alpha(x_1)\|^2\,,
\end{aligned} \tag{33}$$

where for the second line we have used (12) and (11), and for the third (10).

For the given R, let now γ denote the lowest value of the smallest singular value of $g_2(x_1)$ for all x_1 in B_R. Because of Assumption 1, γ is a positive number, and moreover, we have that $-\eta^T[g_2(x_1)g_2^T(x_1)]^{1/2}\eta \le -\gamma\|\eta\|^2$ at each η. Using this in (33) gives

$$\dot{W} \le -\frac{1}{2}\|x_1\|^2 - \frac{1}{2}\begin{bmatrix}\|\alpha(x_1)\| \\ \|\eta\|\end{bmatrix}^T \begin{bmatrix} 1 & -(1+k_2) \\ -(1+k_2) & 2\left(\frac{\gamma}{\varepsilon} - k_1\right)\end{bmatrix} \begin{bmatrix}\|\alpha(x_1)\| \\ \|\eta\|\end{bmatrix}. \tag{34}$$

The matrix on the right hand side of (34) is positive definite for sufficiently small values of ε. This is for example true for every $\varepsilon \in (0, \varepsilon_a]$, where $\varepsilon_a = \gamma/[2k_1 + (1+k_2)^2]$. Thus $\dot{W} \le 0$ in B_δ.

Now pick $c > 0$ such that the level set $\Omega_c = \{(x_1,x_2) : W(x_1,x_2) \le c\} \subset B_\delta$. Hence, by LaSalle's invariance theorem (e.g., Khalil 1996), every solution starting in Ω_c converges to the largest invariant set contained in the set $\{(x_1,x_2) : \dot{W} = 0\}$, which here is just the origin. We have thus shown that the origin is an asymptotically stable equilibrium for (30) with a basin of attraction that contains Ω_c.

Next choose $\rho > 0$ such that $B_\rho \subset \Omega_c$. Then, Theorem 1 in (Christofides & Teel 1996) guarantees the existence of $\varepsilon_R \in (0, \varepsilon_a]$ such that for all $\varepsilon \in (0, \varepsilon_R]$ the solutions of (30) starting in B_R enter B_ρ in finite time, and thereafter converge to $(x_1,\eta) = (0,0)$. This shows that the basin of attraction of the origin of (30) in fact contains B_R, which proves the claim of the lemma. □

Finally, we show that the control law (8) yields the cost (15).

Lemma 6.3 (Boundedness of the Cost). Under Assumptions 1 and 2, and for each initial state $(x_1(0), x_2(0))$, the control $\hat{u}$ from (8) incurs the cost $J_\varepsilon(x_1(0), x_2(0), \hat{u}) = V_0(x_1(0)) + O(\varepsilon)$.

Proof. In the η-coordinates the control law (8) is $\hat{u} = \frac{1}{\varepsilon}\, g_2[g_2g_2^T]^{-1/2}\eta$, and the cost (2) is

$$\begin{aligned}
J_\varepsilon &= \frac{1}{2}\int_0^\infty \left(\|x_1\|^2 + \|(\eta + \alpha(x_1))\|^2 + \varepsilon^2\|\hat{u}\|^2\right) dt\,, \\
&= \frac{1}{2}\int_0^\infty \left(\|x_1\|^2 + \|\alpha(x_1)\|^2 + 2\alpha^T(x_1)\eta + 2\|\eta\|^2\right) dt\,.
\end{aligned} \tag{35}$$

Using the identity $-\dot{V}_0 = \frac{1}{2}\left(\|x_1\|^2 + \|\alpha(x_1)\|^2 + 2\alpha^T(x_1)\eta\right)$, which follows from (10) and (30), and the asymptotic stability of the (30), which implies that $V_0(x(\infty)) = V_0(0) = 0$, we can express (35) as

$$J_\varepsilon = V_0(x_1(0)) + \int_0^\infty \|\eta\|^2\, dt\,. \tag{36}$$

Thus, to prove (15), it remains to show that $\int_0^\infty \|\eta\|^2\, dt$ is $O(\varepsilon)$. From the η-equation in (30) we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\eta^T\eta\right) = \eta^T\dot\eta &= -\frac{1}{\varepsilon}\eta^T(g_2 g_2^T)^{1/2}\eta + \eta^T\phi \\
&\leq -\frac{\gamma}{\varepsilon}\eta^T\eta + \eta^T\phi\,, && \text{using } \gamma \text{ from the previous proof,} \\
&\leq -\frac{\gamma}{2\varepsilon}\eta^T\eta + \frac{\varepsilon}{2\gamma}\phi^T\phi\,, && \text{by Young's inequality,}
\end{aligned}$$

and therefore,

$$\frac{\gamma}{\varepsilon}\int_0^\infty \eta^T\eta\, dt \leq \|\eta(0)\|^2 + \frac{\varepsilon}{\gamma}\int_0^\infty \phi^T\phi\, dt\,. \tag{37}$$

Inequality (34), and the (uniform in ε) boundedness of closed-loop trajectories, imply that x_1, α and η are in $L_2(0,\infty)$, and so is ϕ, because of Assumption 2(ii). Hence, from (37), we conclude that $\int_0^\infty \|\eta\|^2\, dt$ is $O(\varepsilon)$, and thus (36) yields (15), and concludes the proof. □

Proof of Proposition 3.1. Follows from Definition 3.1 and the concatenation of the proofs of Lemmata 6.1, 6.2 and 6.3. □

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