Robustness of Zero-Shifting via Generalized Sampled-Data Hold Functions

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Abstract— **In this paper we study robustness and sensitivity properties of a sampled-data feedback system with a Generalized Sampled-data Hold Function (GSHF). We argue that shifting nonminimum phase zeros using GSHF control can lead to difficulties unless the zero is outside the closed loop bandwidth.**

Keywords— **Sampled-data systems, Generalized sampled-data hold functions, Robustness, Sensitivity analysis, Nonminimum-phase zeros, Feedback limitations.**

I. Introduction

Nonminimum-phase (NMP) zeros of a linear time invariant plant impose inherent design limitations that cannot be overcome by any linear time invariant controller [1], [2], [3]. This fact suggests that more general compensation schemes, such as periodic linear time-varying control, may prove useful in controlling NMP systems. Sampled-data (SD) control, wherein an analog plant is controlled by a digital compensator through the use of periodic sample and hold, is one class of periodic controllers.

In a SD system, the zeros of the discretized plant, unlike the poles, bear no straightforward relationship to the zeros of the original analog plant (e.g., [4], [5]). In particular, use of a Generalized Sampled-data Hold Function (GSHF) with a linear time invariant digital controller allows the zeros of the discretized plant to be placed arbitrarily [6], [7]. Hence it is tempting to conclude that design limitations due to NMP zeros of an analog plant may be circumvented by assigning the zeros of the discretized plant to be minimum phase [8], [9], [10].

On the other hand, several authors have pointed out potential disadvantages to the use of GSHF control. In [7, p.75] the authors note that "the control signal may become highly irregular". In [6], the author notes that systems with GSHF control can sometimes exhibit intersample ripple. Furthermore, the authors of [11] present analyses and simulations which suggest that systems with GSHF controllers are prone to robustness difficulties in addition to poor intersample behavior. Hence the potential utility of GSHF control in overcoming linear time invariant design limitations is still a matter of debate [11], [12].

Let us now consider a common procedure by which a digital compensator is designed in a SD system. Namely, one first discretizes the analog plant at an appropriate sample rate and then designs the compensator so that the discretized feedback system has desirable properties.[1] As a consequence, the behavior of the analog signals in the resulting hybrid feedback system will be as desired *at the sampling instants*. One then may simulate the hybrid system to verify that the intersample behavior is acceptable. If the plant is discretized with a zero-order hold (ZOH), and if an appropriate sample rate and anti-aliasing filter are used, then this is very often the case.

As noted above, when the plant is discretized using a GSHF hold, its zeros can be placed arbitrarily. In [14] the authors showed that design limitations imposed by NMP zeros of the analog plant remain present when the plant is discretized with a GSHF hold, *even if the discretized plant is minimum phase.* One of the contributions of the present paper is to expand on the implications of this fact by considering the following situation: Suppose that the analog plant has a NMP zero within the desired closed loop bandwidth, but the discretized plant does not. Suppose also that the discrete closed loop system possesses feedback properties that would be unachievable if the discretized plant also had a problematic NMP zero. Then, as we show in Section III, these feedback properties *cannot* also be present in the intersample behavior of the hybrid system.

A more intriguing question, whose analysis is the core of the paper, is whether the use of GSHF control to relocate zeros is responsible for sensitivity and robustness difficulties in the resulting feedback system above and beyond those due to the NMP zero of the analog plant. It was argued in [11] that the poor robustness properties of GSHF control are due to the way in which components of the high frequency plant response are aliased down into the baseband to form the frequency response of the discretized plant. We investigate this phenomenon in detail by developing a framework in which the robustness difficulties associated with zero-shifting may be studied quantitatively (Sections IV-VI). A key concept introduced is that of a *fidelity function*, which measures differential sensitivity and robustness of the discretized feedback system against modeling uncertainty in the *analog* plant. As illustration, we analyze in Section VII an example that originally appeared in [10]. Conclusions are presented in Section VIII.

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[1] There are of course other digital design methods, including continuous based synthesis, and direct digital design [13]. Our development here focuses on the discretized synthesis approach since this is the main approach inherent in the original development of GSHF control [6]. Our results apply to *any* digital controller (no matter what the design), although we have not explored interpretations.

II. Background

Consider the single-input single-output SD feedback system of Figure 1, where $P(s)$ and $F(s)$ are the transfer functions of the analog plant and anti-aliasing filter, $C_d(z)$ is the transfer function of the digital controller, $r(t), d(t)$, and $n(t)$ are the command, disturbance, and noise signals, $u(t)$ is the control input, and $y(t)$ is the system output. Denote the sampling period by T and the associated sampling and Nyquist frequencies by $\omega_s \triangleq 2\pi/T$ and $\omega_N \triangleq \pi/T$, respectively. The term *baseband* will denote the frequency range $\Omega_N \triangleq (-\omega_N, \omega_N]$.

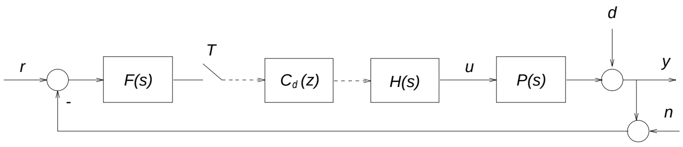

Fig. 1. Sampled-data Feedback System

We denote the open and closed right halves of the s-plane by ORHP and CRHP, respectively; the open and closed unit disks by $D \triangleq \{z : |z| < 1\}$ and $\bar{D} \triangleq \{z : |z| \leq 1\}$, and their complements by D^c and $\bar{D}^c$, respectively. A rational function of s (respectively z) is *minimum phase* if it has no zeros in the ORHP (respectively, in $\bar{D}^c$). Otherwise, it is NMP, and the corresponding zeros are termed NMP *zeros*.

We shall assume that the plant, prefilter, and controller are each free of unstable hidden modes. In addition, suppose that $P(s) = P_0(s)e^{-s\tau}$, where $P_0(s)$ is rational and proper and $\tau \geq 0, F(s)$ is strictly proper, rational, and has no CRHP poles or zeros, and $C_d(z)$ is rational and proper. The hold function is a GSHF [6] defined by

$$u(t) = h(t - kT)u_k, \quad kT \leq t < (k+1)T ,$$

where $h(t)$ satisfies the mild technical assumptions stated in [15], [14]. Then to this hold there is an associated frequency response function $H(s) = \int_0^T e^{-st}h(t)dt$.

The transfer function of the discretized connection of plant, hold, and prefilter is the *discretized plant*, and is denoted by $(FPH)_d(z)$. As shown in [16], our assumptions on $P(s), H(s)$, and $F(s)$ suffice to guarantee that $(FPH)_d(z)$ satisfies the following well-known formula

$$(FPH)_d(e^{sT}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F_k(s)P_k(s)H_k(s) , \qquad (1)$$

where we have introduced the notation $F_k(\cdot)$ to represent $F(\cdot + jk\omega_s)$. This notation will be used frequently.

Exponential and $\mathcal{L}_2$ input-output stability of the system in Figure 1 follow from the results of [17] and [18] under the nonpathological sampling hypothesis for GSHF systems of [15].

Assuming that the hybrid system of Figure 1 is stable, one may use classical results from hybrid system theory to calculate the steady state response of the output $y(t)$ to a sinusoidal input signal of frequency ω. This response consists of a fundamental component with frequency ω plus infinitely many harmonics at frequencies separated from the fundamental by an integer multiple of the sampling frequency (cf. (1)). The fundamental component is governed by the *fundamental sensitivity and complementary sensitivity functions*

$$S_{\text{fun}}(s) \triangleq 1 - \frac{1}{T}P(s)H(s)C_d(e^{sT})S_d(e^{sT})F(s) \qquad (2)$$

and

$$T_{\text{fun}}(s) \triangleq \frac{1}{T}P(s)H(s)C_d(e^{sT})S_d(e^{sT})F(s) , \qquad (3)$$

where

$$S_d(z) \triangleq \frac{1}{1 + (FPH)_d(z)C_d(z)} \quad \text{and} \quad T_d(z) \triangleq 1 - S_d(z)$$

are the *discrete sensitivity and complementary sensitivity functions* [14].

The functions (2)-(3) play a role similar to the usual sensitivity and complementary sensitivity functions for analog systems. In particular, $S_{\text{fun}}(s)$ governs the fundamental component of the intersample response to disturbances, and $T_{\text{fun}}(s)$ plays this role for the response to noise. Furthermore, $S_{\text{fun}}(s)$ and $T_{\text{fun}}(s)$ satisfy interpolation constraints at poles and zeros of the plant and compensator which translate into Bode and Poisson integrals. These integrals quantify important tradeoffs in SD design [14].

Of particular interest in this paper, is the tradeoff between values of $|S_{\text{fun}}(j\omega)|$ in different frequency ranges depending upon the relative location of the NMP zeros of the plant and hold function: Suppose that ξ is a real NMP zero of $P(s)$ or $H(s)$. Then, if we require that

$$|S_{\text{fun}}(j\omega)| \leq \alpha \ \ , \forall \omega \in \Omega \triangleq [0, \omega_0) ,$$

it necessarily follows that

$$\sup_{\omega > \omega_0} |S_{\text{fun}}(j\omega)| \geq (1/\alpha)^{\frac{\Theta(\xi,\Omega)}{\pi - \Theta(\xi,\Omega)}} \left|B_p^{-1}(\xi)\right|^{\frac{\pi}{\pi - \Theta(\xi,\Omega)}} , \qquad (4)$$

where

$$B_p(s) \triangleq \prod_{i=1}^{N_p} \frac{p_i - s}{\bar{p}_i + s}$$

denotes the Blaschke product of the set $\{p_1, p_2, \ldots, p_{N_p}\}$ of ORHP poles of $P(s)$; and

$$\Theta(\xi, \Omega) \triangleq -\sphericalangle \frac{\xi - j\omega_0}{\xi + j\omega_0}$$

is the negative of the phase lag contributed by the all-pass function $(\xi - s)/(\xi + s)$ at the upper end point of the interval Ω.

The bound (4) shows that if disturbance attenuation is required throughout a frequency interval in which the NMP zero contributes significant phase lag, then disturbances will be greatly amplified at some higher frequency. Similar bounds can be found for complex NMP zeros [14].

We shall also need a bound corresponding result to (4) that holds for the discrete sensitivity function $S_d(z)$ [19]: Assume that $S_d(z)$ is stable and suppose that

$$\left|S_d(e^{j\omega T})\right| \leq \alpha, \quad \forall \omega \in \Omega = [0, \omega_0),$$

where $\omega_0 \leq \omega_N$. Then if $\nu = e^{\xi T}$, with $\xi > 0$,

$$\sup_{\omega \in [\omega_0, \omega_N)} \left|S_d(e^{j\omega T})\right| \geq (1/\alpha)^{\frac{\Theta_d(\xi,\Omega)}{\pi - \Theta_d(\xi,\Omega)}} \times \left|B_\rho^{-1}(\nu)\right|^{\frac{\pi}{\pi - \Theta_d(\xi,\Omega)}} \left|S_d(\nu)\right|^{\frac{\pi}{\pi - \Theta_d(\xi,\Omega)}}, \quad (5)$$

where

$$B_\rho(z) \triangleq \prod_{i=1}^{N_p} \frac{z - \rho_i}{1 - \bar{\rho}_i z}$$

denotes the Blaschke product of the set $\{\rho_i = e^{jp_i T}; i = 1, \ldots, N_p\}$ of poles of $(FPH)_d(z)$ lying in $\bar{D}^c$; and

$$\begin{aligned} \Theta_d(\xi, \Omega) &\triangleq -\sphericalangle \prod_{k=-\infty}^{\infty} \frac{\xi - j(\omega_0 - k\omega_s)}{\xi + j(\omega_0 + k\omega_s)}, \\ &= -\sphericalangle \frac{\sinh\left((\xi - j\omega_0)\frac{T}{2}\right)}{\sinh\left((\xi + j\omega_0)\frac{T}{2}\right)}, \end{aligned}$$

is the negative of the *sum* of the phase lags contributed by the all-pass terms $(\xi - s)/(\xi + s)$ at each of the points $\omega_0 + k\omega_s, k = 0, \pm 1, \pm 2, \ldots$, that are mapped to the upper end point of the interval Ω. Clearly then, $\Theta_d(\xi, \Omega) \geq \Theta(\xi, \Omega)$.

If ν is a NMP zero of the discretized plant, then $S_d(\nu) = 1$ in (5), and $|S_d(e^{j\omega T})|$ is guaranteed to have a peak greater than one. Since $\Theta_d(\xi, \Omega) \geq \Theta(\xi, \Omega)$ it follows that the lower bound on this peak is *guaranteed* to be greater than that given by (4) in the analog case.

Finally, we shall need the discrete version of the Bode sensitivity integral [19]. For a fixed sampling period, this integral implies a nontrivial sensitivity tradeoff even if no bandwidth constraint is imposed. Assume that $S_d(z)$ is stable and that $(FPH)_d(z)C_d(z)$ is strictly proper. Suppose in addition that

$$|S_d(e^{j\omega T})| \leq \beta, \quad 0 \leq \omega \leq \omega_0 < \omega_N.$$

Then necessarily

$$\sup_{\omega_0 < \omega < \omega_N} |S_d(e^{j\omega T})| \geq \left(\frac{1}{\beta}\right)^{\frac{\omega_0}{\omega_N - \omega_0}} \left(\prod_{i=1}^{N_p} e^{p_i T}\right)^{\frac{\omega_N}{\omega_N - \omega_0}}. \quad (6)$$

It follows from (6) that if $\beta < 1$, then there must exist a peak in sensitivity whose magnitude exceeds one.

III. Gedanken Experiment No.1: Analog Performance

Consider the following scenario. We wish to design a digital compensator for an analog plant having a problematic NMP zero. Suppose that a GSHF is used so that the discretized plant is minimum phase, or has NMP zeros only at less problematic locations. Then one can design a digital controller so that the discrete sensitivity function satisfies the specification

$$\left|S_d(e^{j\omega T})\right| \leq \beta_1, \quad 0 \leq \omega \leq \omega_1 \quad (7)$$

$$\left|S_d(e^{j\omega T})\right| \leq \gamma, \quad \omega_1 < \omega \leq \omega_N, \quad (8)$$

where $\beta_1 < 1$ and γ satisfies the lower bound (6) imposed by the discrete Bode sensitivity integral. On the other hand, the intersample behavior of the hybrid system must satisfy constraints due to the analog NMP zero. We now present a Gedanken experiment whose result shows that these constraints manifest themselves as limitations upon the ability of the analog response to approximate that of the discretized system.

Gedanken Experiment No.1: Suppose that we wish to design a digital controller for an analog plant. Then the following three questions (among others) are of interest:
(A1) Is the nominal response of the discretized system satisfactory? Equivalently, is the response of the SD system satisfactory *at the sampling instants*?
(A2) Does the nominal analog response approximate that of the discrete system, so that a satisfactory discrete response corresponds to satisfactory intersample behavior?
(A3) Is the analog response *insensitive* to plant uncertainty, disturbances, and sensor noise?

Clearly it is desirable that the answers to all three questions be affirmative. *The proposed experiment is to determine whether affirmative answers to all three of these questions can be obtained simultaneously.*

We shall consider that the answer to (A1) is affirmative if the discrete sensitivity and complementary sensitivity functions are well behaved. Specifically, we require that $S_d(e^{j\omega T})$ satisfy bounds of the form (7)-(8). It follows from the identity $S_d(e^{sT}) + T_d(e^{sT}) = 1$ that if the bounds (7)-(8) are satisfied, then $|T_d(e^{j\omega T})|$ is also bounded.

To quantify the answer to (A2), define the *fidelity function.*

$$\begin{aligned} S_{\text{fid}}(s) &\triangleq S_{\text{fun}}(s) - S_d(e^{sT}) \\ &= -T_{\text{fun}}(s) + T_d(e^{sT}). \end{aligned} \quad (9)$$

If $|S_{\text{fid}}(j\omega)| \ll 1$, then at frequency ω the fundamental component of the analog response to disturbances, noise, and commands will closely approximate that of the discretized system.

Since the discrete frequency response is periodic in ω, it is clearly not possible (nor desirable) that $S_{\text{fun}}(j\omega)$ and $T_{\text{fun}}(j\omega)$ closely approximate the discrete responses at all frequencies. Hence, we shall consider that the answer to (A2) is affirmative if fidelity is achieved over a low frequency range:

$$|S_{\text{fid}}(j\omega)| \leq \beta_2, \quad 0 \leq \omega \leq \omega_2 < \omega_N. \quad (10)$$

Finally, as discussed in [14], it is necessary to keep the fundamental sensitivity and complementary sensitivity functions bounded at all frequencies to prevent large intersample response to disturbances and noise, as well as to prevent poor differential sensitivity and stability robustness. Hence an affirmative answer to (A3) will require that $S_{\text{fun}}(j\omega)$ and $T_{\text{fun}}(j\omega)$ satisfy upper bounds of the form

$$|S_{\text{fun}}(j\omega)| \le M_S(\omega), \quad \text{and} \quad |T_{\text{fun}}(j\omega)| \le M_T(\omega) \qquad (11)$$

at all frequencies.

It follows immediately from (4) that the analog NMP zero imposes a limitation upon our ability to achieve affirmative answers to all of questions (A1)-(A3).

Lemma III.1: Suppose that the hybrid feedback system is stable and that (7) and (10) both hold. Define $\omega^* \triangleq \min\{\omega_1, \omega_2\}$, and $\Omega^* \triangleq [0, \omega^*)$. If the analog plant has a NMP zero at ξ, it follows that

$$\sup_{\omega > \omega^*} |S_{\text{fun}}(j\omega)| \ge \left(\frac{1}{\beta_1 + \beta_2}\right)^{\frac{\Theta(\xi,\Omega^*)}{\pi - \Theta(\xi,\Omega^*)}} \left|B_p^{-1}(\xi)\right|^{\frac{\pi}{\pi - \Theta(\xi,\Omega^*)}} \qquad (12)$$

Thus, if $|S_d(e^{j\omega T})|$ is made small over a wide frequency band relative to the location of the NMP zero, then $|S_{\text{fun}}(j\omega)|$ cannot closely approximate the discrete response over this band without incurring large peaks at higher frequencies. These peaks, in turn, will tend to compromise the bounds (11). We see that there exists a tradeoff between the quality of the response at sampling instants and that of the intersample behavior that cannot be removed by GSHF control.

IV. Gedanken Experiment No.2: Discrete Response

In the present section, we shall argue that use of GSHF control to shift zeros so that $|S_d(e^{j\omega T})|$ can be made small over a wide frequency range may lead to unacceptable robustness difficulties *even if no requirement is imposed upon the analog response.* The source of these difficulties is the necessity to maintain stability robustness against the contribution of high frequency aliases to the discrete plant response.

Consider again the formula (1). Typically, the anti-aliasing filter has a monotonically decreasing Bode gain plot, and thus tends to diminish the contribution of the high frequency plant behavior to the discretized system response. The effect of the hold response in (1) is identical to that of the anti-aliasing filter, and shows that the hold response plays an equally important role in determining the effect of the high frequency plant behavior upon the discretized system. In this regard, it is instructive to compare the responses of the ZOH and a GSHF taken from Example 2 in [6]; these responses are plotted in Figure 2. Note that the frequency response of *this particular* GSHF has larger gain at high frequencies than does that of the ZOH. Hence it follows that the frequency response of a plant discretized with this GSHF will depend more heavily upon the high frequency characteristics of the analog plant than if the plant were discretized with a ZOH.

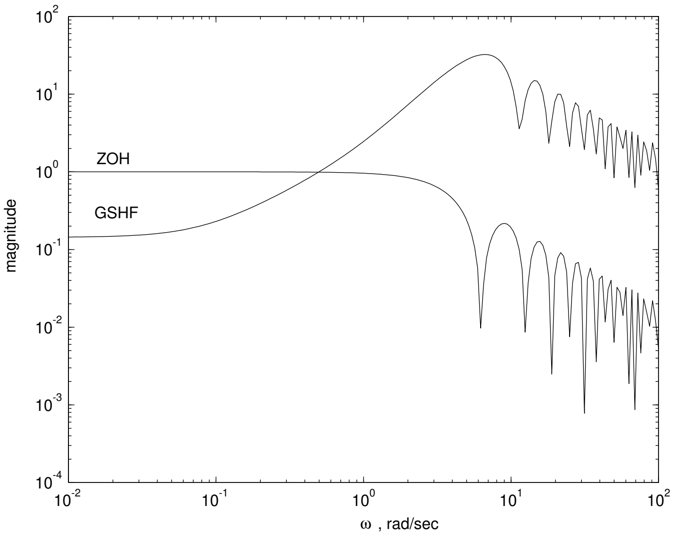

Fig. 2. Frequency Responses of a ZOH and a GSHF

To explore this phenomenon further, we rewrite (1) as

$$(FPH)_d(e^{sT}) = \frac{1}{T}F(s)P(s)H(s) + \frac{1}{T}\Delta_d(s), \qquad (13)$$

where $\Delta_d(s) \triangleq \sum_{k\neq 0} F_k(s)P_k(s)H_k(s)$.

The decomposition (13) separates the frequency response of the discretized plant into a fundamental component, $\left(\frac{1}{T}\right) F(s)P(s)H(s)$, plus the term $\Delta_d(s)$, which represents the *net* effect of aliases from other frequencies. If $s = j\omega, |\omega| \in \omega_N$, then the former term represents the baseband contribution to the discrete frequency response, and the latter term represents the net contribution of high frequency aliases.

It follows from (13) that if for some value of $s, P(s) = 0$ but $(FPH)_d(e^{sT}) \neq 0$, then *necessarily* the response of the discretized plant at $z = e^{sT}$ must depend upon the response of the analog plant at one or more of the frequencies $s + jk\omega_s, k \neq 0$. As a corollary, the response of the discretized system will be potentially sensitive to uncertainty in the analog plant at these frequencies. This fact is significant in that uncertainty in the plant model generally increases at higher frequencies. Hence if a strong dependence upon high frequency plant behavior is required to shift a zero, then one might suspect that the sensitivity and robustness of the resulting design would be poor. (See also the discussion in [20, §10.5].) We now propose another Gedanken experiment whose result will clarify this issue.

Gedanken Experiment No.2: Suppose that we wish to design a digital controller for an analog plant. Then the following two questions (among others) are of interest:

(D1) Is the nominal response of the discrete system satisfactory?

(D2) Is the *discrete* response insensitive to uncertainty in the analog plant?

Clearly, it is desirable that the answers to both questions be affirmative. *The proposed experiment is to determine whether affirmative answers to both of these questions can be obtained simultaneously.*

By way of contrast with Gedanken Experiment No.1, we are now concerned solely with the response of the system *at the sampling instants.*

The only requirement related to the analog system is that the discrete behavior must be robust against uncertainty in the analog plant.

Let us now consider the problems of achieving small differential sensitivity and robust stability against linear time invariant uncertainty in the *analog* plant. As the source of uncertainty is modeling error in the analog plant, these problems are more interesting from an engineering standpoint than are their discrete counterparts.

Motivated by the discussion surrounding (13), we shall consider separately uncertainty in the two terms on the right hand side (RHS) of (13). In particular since uncertainty in the analog plant tends to increase with frequency, it follows that for $\omega \in \Omega_N$ uncertainty in the term $F(j\omega)P(j\omega)H(j\omega)$ will tend to be dominated by uncertainty in the term $\Delta_d(j\omega)$ due to the high frequency aliases.

To state the desired formulas for differential sensitivity and stability robustness, we shall need the following definitions.

Definition IV.1: (High gain sensitivity and complementary sensitivity functions) In the limit as $|C_d(e^{sT})| \to \infty$, $S_{\text{fun}}(s) \to S_{\text{HG}}(s)$ and $T_{\text{fun}}(s) \to T_{\text{HG}}(s)$, where

$$S_{\text{HG}}(s) \triangleq \begin{cases} \frac{\left(\frac{1}{T}\Delta_d(s)\right)}{(FPH)_d(e^{sT})}, & (FPH)_d(e^{sT}) \neq 0 \\ 1, & (FPH)_d(e^{sT}) = 0 \end{cases} \tag{14}$$

and $T_{\text{HG}}(s) \triangleq 1 - S_{\text{HG}}(s)$.

At a given frequency ω, not necessarily in the baseband, $S_{\text{HG}}(j\omega)$ is a measure of the contribution of aliases from other frequencies to the frequency response of the discretized plant. By (13), it is therefore a measure of the difference between the analog and discrete responses. Further design interpretations are presented in [14, §3]. For now, we use $S_{\text{HG}}(s)$ in the following result.

Proposition IV.1: At each value of s

$$S_{\text{fid}}(s) = S_{\text{HG}}(s) T_d(e^{sT}). \tag{15}$$

Proof: It is straightforward to show that the *relative difference* between the discrete command response and the fundamental component of the analog response is given by

$$\frac{T_{\text{fun}}(s) - T_d(e^{sT})}{T_d(e^{sT})} = -S_{\text{HG}}(s). \tag{16}$$

Together, (9) and (16) yield the desired result. ■

We now use $S_{\text{fid}}(s), S_{\text{HG}}(s)$, and $T_{\text{HG}}(s)$ to describe the relative differential sensitivity of the discrete command and control response to uncertainty in the analog plant. (Recall that the discrete control response is governed by the transfer function $C_d(z)S_d(z)$.) For purposes of comparison, we first state the corresponding result for uncertainty in the *discretized* plant.

Lemma IV.2: (Differential sensitivity to the discretized plant) For each $\omega \in \Omega_N$,

$$\frac{(FPH)_d(e^{j\omega T})}{T_d(e^{j\omega T})} \frac{\partial T_d(e^{j\omega T})}{\partial (FPH)_d(e^{j\omega T})} = S_d(e^{j\omega T})$$

$$\frac{(FPH)_d(e^{j\omega T})}{C_d(e^{j\omega T}) S_d(e^{j\omega T})} \frac{\partial C_d(e^{j\omega T}) S_d(e^{j\omega T})}{\partial (FPH)_d(e^{j\omega T})} = -T_d(e^{j\omega T}).$$

Proof: Straightforward calculation. ■

Whether $S_d(e^{j\omega T})$ and $T_d(e^{j\omega T})$ also correctly describe sensitivity with respect to analog plant variations depends upon how closely the discrete and analog responses approximate one another.

Proposition IV.3: (Differential sensitivity to the analog plant) For each $\omega \in \Omega_N$ and $k = 0, \pm 1, \pm 2, \ldots$

$$\frac{P_k(j\omega)}{T_d(e^{j\omega T})} \frac{\partial T_d(e^{j\omega T})}{\partial P_k(j\omega)} = T_{\text{HG}}(j(\omega + k\omega_s)) S_d(e^{j\omega T}) \tag{17}$$

$$\frac{\Delta_d(j\omega)}{T_d(e^{j\omega T})} \frac{\partial T_d(e^{j\omega T})}{\partial \Delta_d(j\omega)} = S_{\text{HG}}(j\omega) S_d(e^{j\omega T}) \tag{18}$$

$$\begin{aligned} \frac{P_k(j\omega)}{C_d(e^{j\omega T}) S_d(e^{j\omega T})} \frac{\partial C_d(e^{j\omega T}) S_d(e^{j\omega T})}{\partial P_k(j\omega)} &= \\ -T_{\text{HG}}(j\omega) T_d(e^{j\omega T}) &= -T_{\text{fun}}(j\omega) \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{\Delta_d(j\omega)}{C_d(e^{j\omega T}) S_d(e^{j\omega T})} \frac{\partial C_d(e^{j\omega T}) S_d(e^{j\omega T})}{\partial \Delta_d(j\omega)} &= \\ -S_{\text{HG}}(j\omega) T_d(e^{j\omega T}) &= -S_{\text{fid}}(j\omega). \end{aligned} \tag{20}$$

Proof: Straightforward calculation. ■

Consider a frequency $\omega \in \Omega_N$ and suppose that $S_{\text{HG}} \simeq 0$ and $T_{\text{HG}}(j\omega) \simeq 1$, so that the *net* contribution of high frequency aliases to the discretized response is relatively small (cf. (14)). It then follows from (17) and (19) that $S_d(e^{j\omega T})$ and $T_d(e^{j\omega T})$ accurately describe sensitivity to baseband ($k = 0$) variations in the analog plant.

Furthermore, from (18) and (20) it follows that sensitivity to the *net* contribution of the high frequency aliases will be small. On the other hand, as $\omega \to \infty, T_{\text{HG}}(j\omega) \to 0$ and thus sensitivity to variations in individual components of the high frequency plant response ($k \neq 0$) will also become small. It is important to note, however, that large peaks in $S_{\text{HG}}(j\omega)$ and $T_{\text{HG}}(j\omega)$ will cause sensitivity to analog variations to be much worse than that to discrete variations. Finally, note that differential sensitivity of the control response is governed by $T_{\text{fun}}(s)$ and $S_{\text{fid}}(s)$. As we now show, these functions are also related to stability robustness.

Assume that the feedback system of Figure 1 is nominally stable. Consider uncertainty in the discretized plant due to analog plant uncertainty of the form

$$P'(s) \triangleq P(s)\left(1 + W(s)\Delta(s)\right), \tag{21}$$

where $\Delta(s)$ is stable and proper and $W(s)$ is a stable weighting function used to represent frequency dependence of the modeling error. It was shown in [14] that a *necessary* condition for the system to remain stable for all $\Delta(s)$ satisfying

$$|\Delta(j\omega)| < 1 \quad \forall \omega \in \mathbb{R} \tag{22}$$

is that

$$|W(j\omega)T_{\rm fun}(j\omega)| \leq 1 \quad \forall \omega \in \mathbb{R} \ . \tag{23}$$

Typically, $|W(j\omega)|$ will become unbounded at high frequencies, and so it is necessary that $|T_{\rm fun}(j\omega)| \to 0$ sufficiently rapidly as $\omega \to \infty$. The derivation of (23) in [14] ignores the effect of aliases in (13); we now use the results of [21] to develop a stronger necessary condition that does take aliases into account.

Lemma IV.4: Assume that $\Delta(s)$ in (21) is arbitrary save for the bound (22). Define

$$\underline{w} = \inf_{\omega \notin \Omega_N} |W(j\omega)| \tag{24}$$

Then a *necessary* condition for robust stability is that

$$|W(j\omega)T_{\rm fun}(j\omega)| + \underline{w}|S_{\rm fid}(j\omega)| \leq 1 \quad \forall \omega \in \Omega_N \ . \tag{25}$$

Proof: The results of [21] establish that the feedback system will remain stable for all $\Delta(s)$ satisfying (22) if and only if the condition

$$\sup_{\omega \in \Omega_N} \sum_{k=-\infty}^{\infty} |T_{\rm fun}(s+jk\omega_s)W(s+jk\omega_s)| \leq 1 \tag{26}$$

is satisfied (cf. [22, Chapter 6]). Condition (25) follows immediately from (26) and (9), since

$$\begin{aligned}\sum_{k=-\infty}^{\infty} &|T_{\rm fun}(j\omega+jk\omega_s)W(j\omega+jk\omega_s)| \geq |T_{\rm fun}(j\omega)W(j\omega)| \\ &+ \frac{\underline{w}}{T}\sum_{k\neq 0}|F_k(j\omega)P_k(j\omega)H_k(j\omega)S_d(e^{j\omega T})C_d(e^{j\omega T})| \\ &\geq |T_{\rm fun}(j\omega)W(j\omega)| + \underline{w}|S_{\rm fid}(j\omega)| \ .\end{aligned}$$

■

Since relative uncertainty in the analog plant (21) typically becomes large at high frequencies, and since the Nyquist frequency is usually chosen to be around 5 times the desired closed loop bandwidth, it is reasonable to assume that $\underline{w}$ in (24) is greater than 1. Hence (25) requires that $|S_{\rm fid}(j\omega)| < 1$ over the baseband. This fact is significant since, as we shall see in the next section, $S_{\rm fid}(s)$ must satisfy a Poisson integral relation.

V. Interpolation Constraints and an Integral Relation

We now develop a set of interpolation constraints and an integral relation that must be satisfied by the fidelity function, $S_{\rm fid}(s)$. We first require an additional assumption that will hold generically.

Assumption 1: If ξ is a CRHP zero of $P(s)$ or $H(s)$, then $e^{\xi T}$ is *not* a zero of $(FPH)_d(z)$.

Proposition V.1 (Interpolation Constraints) Suppose that the SD feedback system is stable, and that $P(s), F(s), H(s)$, and $C_d(z)$ satisfy all assumptions stated in Section II as well as Assumption 1. Then the following conditions are satisfied:
(i) If ζ is a CRHP zero of $P(s)$, then $S_{\rm fid}(\zeta) = T_d(e^{\zeta T})$.
(ii) If γ be a CRHP zero of $H(s)$, then $S_{\rm fid}(\gamma) = T_d(e^{\gamma T})$.
(iii) If $a \in D^c$ is a zero of $C_d(z)$, define $a_k \triangleq \frac{1}{T}\log(a) + jk\omega_s, k = 0, \pm1, \pm2, \ldots$ Then $S_{\rm fid}(a_k) = 0$,$\forall k$.
(iv) If p is a CRHP pole of $P(s)$, define $p_k \triangleq p + jk\omega_s, k = \pm1, \pm2, \ldots$. Then $S_{\rm fid}(p) = 0$ and $S_{\rm fid}(p_k) = 1$.
(v) If δ is a CRHP zero of $\Delta_d(s)$, then $S_{\rm fid}(\delta) = 0$.
(vi) $S_{\rm fid}(s)$ has no CRHP zeros other than those given in (iii), (iv), and (v).

Proof: Conditions (i) and (ii) follow from Assumption 1, and the identity

$$S_{\rm fid}(s) = \frac{\frac{1}{T}\Delta_d(s)}{(FPH)_d(e^{sT})}T_d(e^{sT}). \tag{27}$$

Condition (iii) follows from the identity

$$S_{\rm fid}(s) = \frac{1}{T}\Delta_d(s)S_d(e^{sT})C_d(e^{sT}). \tag{28}$$

Condition (iv) follows from (27). Condition (v) follows from (28). Finally, (28) shows that the zeros of $S_{\rm fid}(s)$ are restricted to those of $\Delta_d(s), S_d(e^{sT})$ and $C_d(e^{sT})$, and condition (vi) follows. ■

Introduce the notation

$\{\zeta_1, \ldots, \zeta_{N_\zeta}\}$,	for the NMP zeros of $P(s)$
$\{\gamma_1, \ldots, \gamma_{N_\gamma}\}$,	for the NMP zeros of $H(s)$
$\{a_1, \ldots, a_{N_a}\}$,	for the NMP zeros of $C_d(s)$
$\{\delta_1, \ldots, \delta_{N_\delta}\}$,	for the NMP zeros of $\Delta_d(s)$
$\{p_1, \ldots, p_{N_p}\}$,	for the ORHP poles of $P(s)$, and
$\{\delta_1, \ldots, \delta_{N_p}\}$,	for the poles of $(FPH)_d(z)$ in $\bar{D}^c$.

Define associated Blaschke products

$$\begin{aligned} B_a(s) &\triangleq \prod_{i=1}^{N_a}\prod_{k=-\infty}^{\infty}\frac{a_i - s - jk\omega_s}{\bar{a}_i + s + jk\omega_s} \\ &= \prod_{i=1}^{N_a}\frac{\sinh\left((a_i - s)\frac{T}{2}\right)}{\sinh\left((\bar{a}_i + s)\frac{T}{2}\right)} \ , \\ B_p(s) &\triangleq \prod_{i=1}^{N_p}\frac{p_i - s}{\bar{p}_i + s} \ , \\ B_\rho(s) &\triangleq \prod_{i=1}^{N_p}\frac{\sinh\left((p_i - s)\frac{T}{2}\right)}{\sinh\left((\bar{p}_i + s)\frac{T}{2}\right)} \ , \\ B_\delta(s) &\triangleq \prod_{i=1}^{N_\delta}\frac{\delta_i - s}{\bar{\delta}_i + s} \ . \end{aligned}$$

Finally, note that we may factor $\Delta_d(s)$ as $\Delta_d(s) = \Delta_m(s)B_\delta(s)e^{-s\tau_\Delta}$ for some $\tau_\Delta \geq 0$, where $\Delta_m(s)$ satisfies the Poisson integral relation [23].

Using these definitions yields

Theorem V.2: Let $\xi = x + jy$ equal a NMP zero of either $P(s)$ or $H(s)$. Then

$$\begin{aligned}\int_0^\infty \log|S_{\rm fid}(j\omega)|\Psi(\xi,\omega)d\omega &= \pi x(N_c T + \tau_\Delta) \\ &+ \pi\log\left|B_a^{-1}(\xi)\right| + \pi\log\left|B_p^{-1}(\xi)\right| \\ &+ \pi\log\left|B_\delta^{-1}(\xi)\right| + \pi\log\left|T_d(e^{\xi T})\right| \ ,\end{aligned} \tag{29}$$

where

$$\Psi(\xi,\omega) \triangleq \frac{x}{x^2+(y-\omega)^2} + \frac{x}{x^2+(y+\omega)^2} \,.$$

Proof: Immediate from the factorization $S_{\text{fid}}(s) = S_m(s)B_a(s)B_p(s)B_\rho(s)B_\delta(s)e^{-s(\tau_\Delta+N_cT)}$, where $S_m(s)$ satisfies the Poisson integral relation [23], together with the identities $|S_{\text{fid}}(j\omega)| = |S_m(j\omega)|$ and $S_{\text{fid}}(\xi) = T_d(\xi)$. ■

The integral (29) imposes a constraint upon values of $|S_{\text{fid}}(j\omega)|$. Analysis of design implications is deferred to the next section.

VI. Result of Gedanken Experiment No.2

Suppose that the analog plant has at least one NMP zero and is subject to large modeling uncertainty at high frequencies. We now use the Poisson integrals for $S_d(e^{j\omega T})$ and $S_{\text{fid}}(j\omega)$ to show that there exists a limit upon the ability of a SD feedback system to satisfy, with affirmative answers, questions (D1) and (D2) of the second Gedanken experiment. In particular, we shall show that there exists a tradeoff between achieving both high performance in the discrete system and stability robustness against uncertainty in the analog plant model. The severity of the tradeoff is determined by the location of the analog NMP zero, and is *independent of whether or not the discretized plant is minimum phase.* Furthermore, unlike the limitations revealed by Gedanken experiment No.1, this tradeoff exists *even if no performance requirements are imposed upon the intersample behavior.*

To demonstrate this tradeoff, we shall assume that the discrete sensitivity function satisfies the performance specification

$$|S_d(e^{j\omega T})| \leq \beta, \quad \forall\omega \in \Omega \triangleq [0,\omega_0)\,, \tag{30}$$
$$|S_d(e^{j\omega T})| \leq \gamma, \quad \forall\omega \notin \Omega\,, \tag{31}$$

where $\omega_0 < \omega_N$, $\beta < 1$, and γ is at least as large as the RHS of (6).

We also require that the system be robustly stable against modelling uncertainty of the form $P'(s) \triangleq P(s)\,(1 + W(s)\Delta(s))$, where $\Delta(s)$ is arbitrary save for the bound (22), and $|W(j\omega)| \to \infty$ as $\omega \to \infty$. Let $\underline{w}$ be given by (24). It follows from Lemma IV.4 and (23) that the conditions

$$|S_{\text{fid}}(j\omega)| < \frac{1}{\underline{w}} \quad \forall\omega \in \Omega_N \tag{32}$$

and

$$|T_{\text{fun}}(j\omega)| < \frac{1}{\underline{w}} \quad \forall\omega \notin \Omega_N \tag{33}$$

are both *necessary* for robust stability.

The main result of this section, Proposition VI.3, will show that if conditions (30),(31), and (32) are all satisfied, then there exists a constraint upon $|T_{\text{fun}}(j\omega)|$ at frequencies outside the baseband. This constraint may prevent condition (33) from being satisfied; as a result, the feedback system may not be robustly stable against high frequency modelling uncertainty. We first state and prove two preliminary lemmas.

Lemma VI.1: Assume that the feedback system in Figure 1 is stable. Let ζ denote a NMP zero of $P(s)$, and suppose that $S_{\text{fid}}(s)$ satisfies the bound (32). Then

$$\sup_{\omega>\omega_N} |S_{\text{fid}}(j\omega)| \geq F_S(\underline{w},\zeta)|T_d(e^{\zeta T})|^{\frac{\pi}{\pi-\Theta(\zeta,\Omega_N)}} \tag{34}$$

where

$$F_S(\underline{w},\zeta) \triangleq (\underline{w})^{\frac{\Theta(\zeta,\Omega_N)}{\pi-\Theta(\zeta,\Omega_N)}}. \tag{35}$$

Proof: This result is a corollary to Theorem V.2. Note first that the first four terms on the RHS of (29) are nonnegative, and thus may be ignored. The bound (34) follows by imposing (32), exponentiating both sides, and rearranging the result. ■

To illustrate Lemma VI.1, consider Figure 3, which contains plots of (35) vs. the location of a real NMP zero for various values of $\underline{w}$. For a fixed value of $\underline{w}$, we see that $F_S(\underline{w},\zeta)$ will be large whenever the NMP zero lies well below the Nyquist frequency. It follows from (34) that $|S_{\text{fid}}(j\omega)|$ will have a large peak outside the baseband *unless* the value of $|T_d(e^{\zeta T})|$ is sufficiently small.

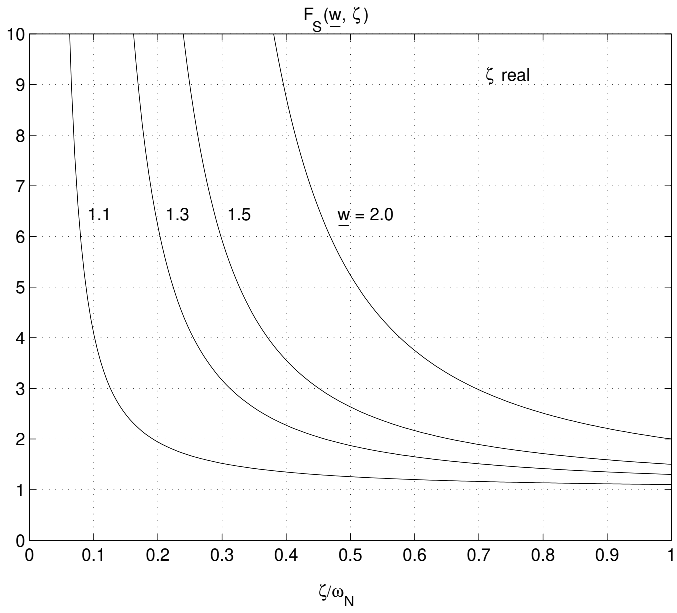

Fig. 3. Plot of (35) vs. ζ/ω_N for various values of $\underline{w}$

We now show that imposing aggressive performance specifications upon the discrete sensitivity function will tend to force the value of $|T_d(e^{\zeta T})|$ to be nearly unity. Our next result shows that satisfying such specifications imposes a constraint upon the value of $T_d(e^{sT})$ at any point outside the unit circle. For the purpose of generality, we state the following result for an arbitrary point $e^{\xi T}$ outside the unit circle. In applying the lemma, we shall be interested in the case that $e^{\xi T}$ is the image of a NMP zero of the analog plant; i.e., the case in which $\xi = \zeta$.

Lemma VI.2: Assume that $S_d(z)$ is stable and that $S_d(e^{j\omega T})$ satisfies the bounds (30) - (31). Consider $\nu = e^{\xi T}$, where $\xi \in$ ORHP. Then

$$|T_d(e^{\xi T})| \geq F_T(\xi, \beta, \gamma, \omega_0), \tag{36}$$

where

$$F_T(\xi, \beta, \gamma, \omega_0) \triangleq 1 - \beta^{\frac{\Theta_d(\xi,\Omega)}{\pi}} \gamma^{\frac{\pi - \Theta_d(\xi,\Omega)}{\pi}}. \tag{37}$$

Proof: It follows from Corollary 5 that

$$\begin{aligned} |T_d(e^{\xi T}) - 1| &\leq \beta^{\frac{\Theta_d(\xi,\Omega))}{\pi}} \gamma^{\frac{\pi - \Theta_d(\xi,\Omega}{\pi}} |B_\rho(e^{\xi T})| \\ &\leq \beta^{\frac{\Theta_d(\xi,\Omega)}{\pi}} \gamma^{\frac{\pi - \Theta_d(\xi,\Omega)}{\pi}}, \end{aligned} \tag{38}$$

from which the result follows. ■

Consider a fixed value of ξ. It follows from (38) that the value of $T_d(e^{\xi T})$ will converge to unity as the bound (30) upon discrete sensitivity converges to zero. To illustrate, consider Figure 4, which contains plots of the lower bound (36) vs. the ratio ξ/ω_0 for a real ξ and various values of β. As Figure 4 shows, the rate at which $T_d(e^{\xi T}) \to 1$ as $\beta \to 0$ depends upon the ratio ξ/ω_0; i.e., upon the location of the point $e^{\xi T}$ relative to the discrete frequency interval over which sensitivity reduction is demanded.

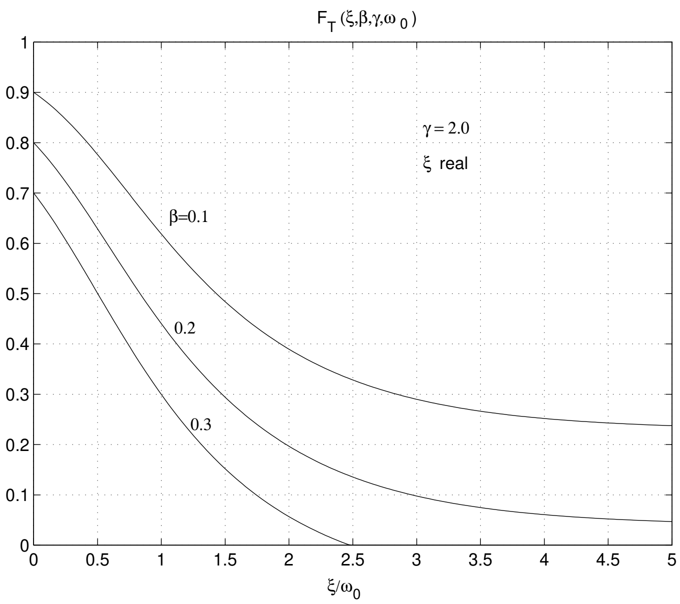

Fig. 4. Lower Bound (37) on $|T_d(e^{\xi T})|$ as a Function of the Level of Sensitivity Reduction

Proposition VI.3: Assume that the feedback system in Figure 1 is stable. Assume that $S_d(e^{j\omega T})$ satisfies (30) - (31), and that $S_{\text{fid}}(s)$ satisfies (32). Let ζ denote a NMP zero of $P(s)$. Then

$$\begin{aligned} \sup_{\omega > \omega_N} |T_{\text{fun}}(j\omega)| \geq F_S(\underline{w}, \zeta) F_T(\zeta, \beta, \gamma, \omega_0)^{\frac{\pi}{\pi - \Theta(\zeta, \Omega_N)}} \\ - (1 + \gamma) \end{aligned} \tag{39}$$

Proof: The triangle inequality in (9) yields

$$|T_{\text{fun}}(j\omega)| \geq |S_{\text{fid}}(j\omega)| - |T_d(e^{j\omega T})|. \tag{40}$$

Using (34) and (36) in (40) gives

$$\begin{aligned} \sup_{\omega > \omega_N} |T_{\text{fun}}(j\omega)| \geq F_S(\underline{w}, \zeta) |T_d(e^{\zeta T})|^{\frac{\pi}{\pi - \Theta(\zeta, \Omega_N)}} \\ - |T_d(e^{j\omega T})| \end{aligned}$$

Finally, the result follows by noting that the bound (31) together with the triangle inequality imply that $|T_d(e^{j\omega T})| \leq 1 + \gamma$. ■

To illustrate Proposition VI.3, consider first Figure 5, which contains plots of the lower bound (39) vs. the ratio ζ/ω_N for $\beta = 0.1$ and various values of ω_0.[2] Suppose that the frequency interval over which discrete sensitivity reduction is demanded is relatively large with respect to the location of a real analog NMP zero ($\zeta/\omega_N < \omega_0/\omega_N$). Then, as illustrated in Figure 5, there will necessarily exist a large peak in $|T_{\text{fun}}(j\omega)|$ at some frequency outside the baseband. On the other hand, if the zero lies outside the frequency range in which the specification is imposed, then the lower bound (39) is vacuous. In fact, there is a rather abrupt demarcation between these two cases, indicated by the almost vertical plots in Figure 5.

Consider next Figure 6, which illustrates that the sharp dependence of the bound (39) upon the relative location of ζ with respect to ω_0 remains even as β is varied. It appears that the length of the frequency interval in which we desire sensitivity reduction is a relatively more critical parameter than is the level of desired sensitivity reduction.

Proposition VI.3, as illustrated by Figures 5 and 6, motivates us to recommend that discrete design specifications should respect the bandwidth limitations imposed by the analog NMP zero, irrespective of whether the discrete plant is minimum phase. Violating this recommendation will necessarily lead to feedback designs that are unduly sensitive to high frequency errors in the analog plant model. For example, from Figure 5 we see that if $\zeta < \omega_0$, then sensitivity will be very large for parameters with values $\beta = 0.1, \gamma = 2$, and $\underline{w} = 2$.

VII. Example

We now illustrate the robustness difficulties described in this paper using an example that originally appeared in [10]. The plant is given by

$$P(s) = \frac{s - 5}{s^2 + 4s + 3}.$$

The authors of [10] desire that the closed loop bandwidth, ω_b, satisfy the lower bound $\omega_b \geq 15.3$ rad/sec. Note that the analog NMP zero lies well within the desired closed loop bandwidth. It is well known (cf. [1], [2]) that if this

[2] We include plots only for $\omega_0 \leq 0.2\omega_N$; this is consistent with the well-known design guideline stating that sampling frequency should be at least a factor of 10 times the desired closed loop bandwidth.

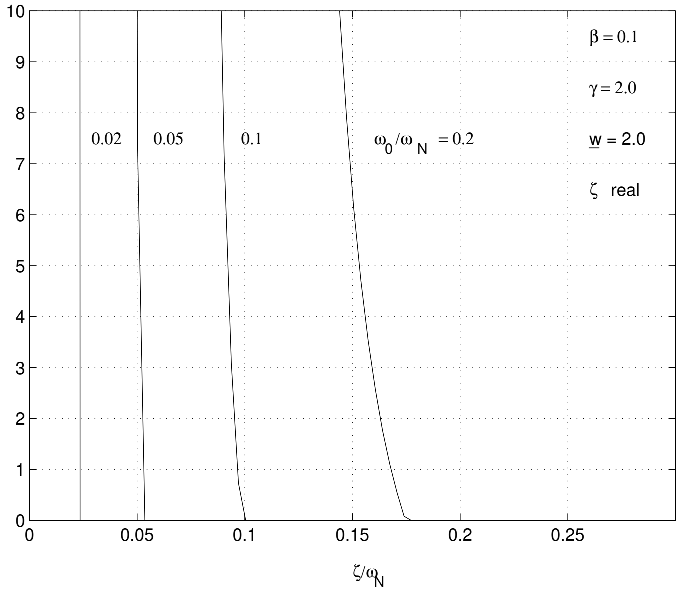

Fig. 5. Lower Bound on $|T_{\mathrm{fun}}(j\omega)|$ as a Function of the Sensitivity Reduction Interval

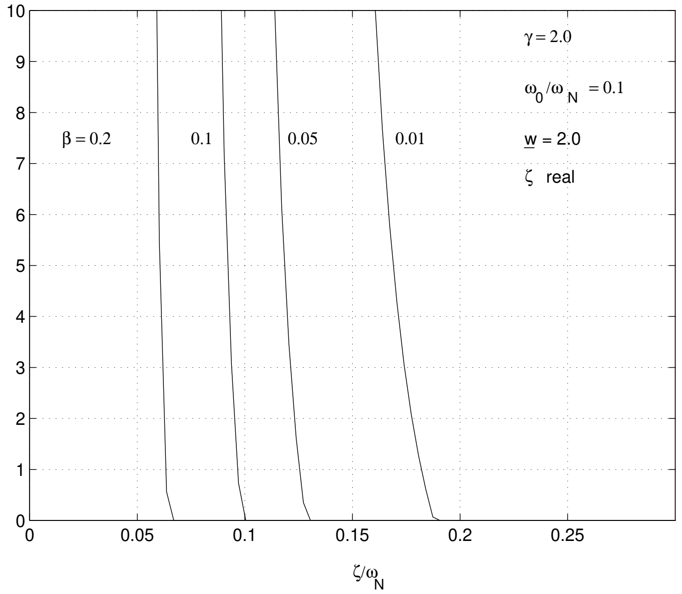

Fig. 6. Lower Bound on $|T_{\mathrm{fun}}(j\omega)|$ as a Function of the Level of Sensitivity Reduction

bandwidth is achieved with analog control, then the resulting closed loop system will have very poor sensitivity and robustness properties. The sampling period is chosen in [10] to be $T = 0.04$ seconds, so that the sampling frequency, $\omega_s = 157$ rad/sec, is ten times the desired bandwidth.

Using the GSHF

$$h(t) = \begin{cases} -1957, & 0 \leq t < 0.02 \\ 1707, & 0.02 \leq t < 0.04 \end{cases}$$

yields a minimum phase discretized plant. The controller is designed using the discrete-time version of the LQG/LTR methodology (cf. [24],[25]). In Figure 7 we plot $S_d(e^{j\omega T})$ and $T_d(e^{j\omega T})$ for the observer-based compensator obtained in [10] (with $q = 3$). (The discrete sensitivity function has a peak greater than one; this peak is consistent with (6).) Note that the closed loop bandwidth specification is achieved, and that both $S_d(e^{j\omega T})$ and $T_d(e^{j\omega T})$ are well behaved.

However, the results of the present paper lead us to expect that intersample behavior and robustness to analog plant uncertainty will be poor. Indeed, consider Figure 8, wherein we plot $S_{\mathrm{fun}}(j\omega)$ and $T_{\mathrm{fun}}(j\omega)$. Note that these functions differ significantly from their discrete counterparts over the baseband. This discrepancy is consistent with the plot of $S_{\mathrm{fid}}(j\omega)$, also shown in Figure 8.

Note that $|S_{\mathrm{fid}}(j\omega)|$ has a relatively large peak within the baseband. It follows from Lemma IV.4 that the system will have poor robustness against unstructured multiplicative plant uncertainty of the form (21). Indeed, it may be verified through simulation that the system is destabilized by a small time delay $P'(s) = P(s)e^{-\tau s}$, with $\tau = 0.0023$ seconds. This extreme sensitivity to small errors in the analog plant model is not apparent from the Bode plots of the discrete closed loop transfer functions (Figure 7).

One might conjecture that it is possible to improve this situation by designing a controller to decrease the peak in $|S_{\mathrm{fid}}(j\omega)|$ within the baseband while maintaining the same discrete response. The results of Section VI show that the potential for such improvements is limited. Indeed, imposing a bound (32) upon $|S_{\mathrm{fid}}(j\omega)|$ will tend to force a large peak in $|T_{\mathrm{fun}}(j\omega)|$ outside the baseband (with attendant robustness problems) *unless* the discrete performance specification (30) is relaxed.

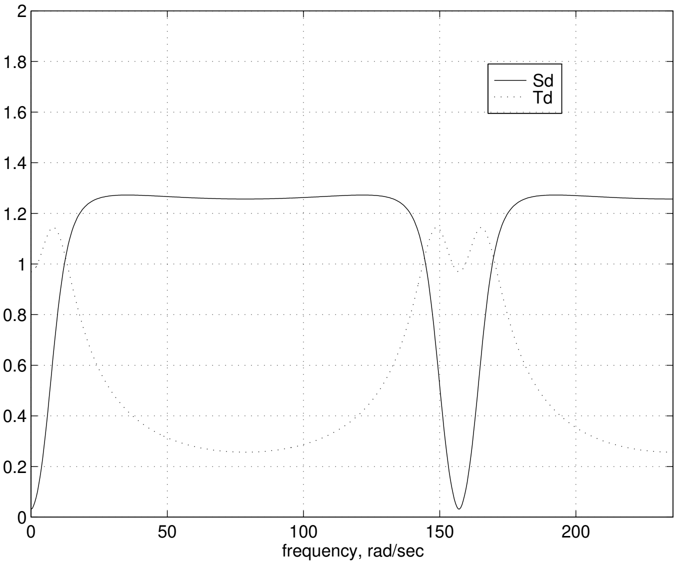

Fig. 7. Discrete Sensitivity and Complementary Sensitivity Functions

VIII. Conclusions

In this paper, we have discussed potential difficulties related to the zero shifting capabilities of a GSHF. In principle, the zero shifting capabilities of a GSHF appear to allow a designer to circumvent fundamental limitations in analog

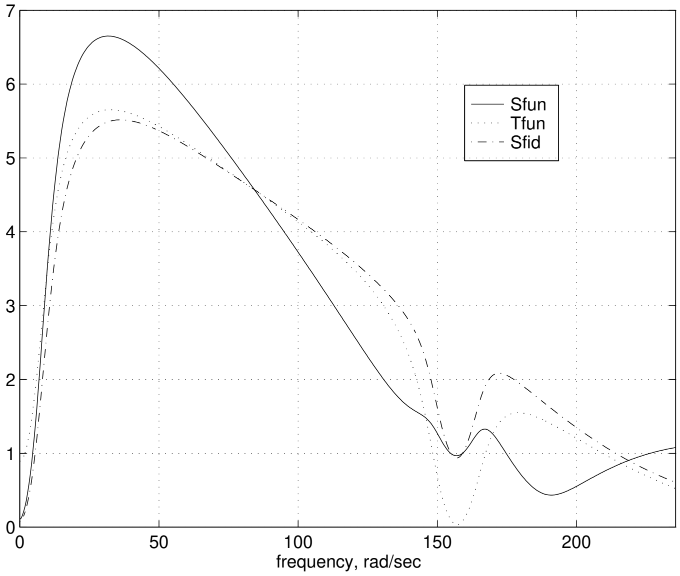

Fig. 8. Fundamental Sensitivity, Fundamental Complementary Sensitivity, and Fidelity Functions

control systems imposed by NMP plant zeros. On the other hand, several authors have noted that use of a GSHF may yield poor intersample response and result in sensitivity and robustness difficulties. We have explored these issues by considering two Gedanken experiments.

To obtain our first result, we considered a problem statement that simultaneously required good nominal discrete response, fidelity between the discrete and intersample responses, and robustness of the discrete response against analog plant uncertainty. We have shown that these objectives are mutually exclusive whenever the analog plant has a NMP zero that contributes significant phase lag within the desired closed loop bandwidth. On the other hand, if the NMP zero lies outside the desired closed loop bandwidth, then it poses no particular limitation to the use of an analog controller. As a consequence, the zero shifting abilities of GSHF control cannot remove design limitations on the analog response due to NMP zeros, and may lead to poor sensitivity of the analog response to model uncertainty.

We then proposed a second Gedanken experiment wherein no performance requirements are imposed upon the intersample behavior. Instead, we ask only that performance be good at sampling instants, and that discrete response be insensitive to unstructured uncertainty in the analog plant. Once again, we showed that these design goals are mutually exclusive whenever the analog plant has a NMP zero within the target closed loop bandwidth.

To summarize, we argue that the zero shifting capabilities of a GSHF should not be used for increasing the closed loop bandwidth beyond that achievable by an analog controller. Doing so will not remove the design limitations imposed by such a zero, and may result in systems with unacceptable intersample behavior and sensitivity.

References

[1] J.S. Freudenberg and D.P. Looze, "Right half plane poles and zeros and design tradeoffs in feedback systems", *IEEE Trans. on Automatic Control*, vol. 30, no. 6, pp. 555–565, June 1985.

[2] R.H. Middleton, "Trade-offs in linear control systems design", *Automatica*, vol. 27, no. 2, pp. 281–292, March 1991.

[3] M.M. Seron, J.H. Braslavsky, and G.C. Goodwin, *Fundamental Limitations in Filtering and Control*, CCES. Springer-Verlag, 1997.

[4] K.J. Åström, P. Hagander, and J. Sternby, "Zeros of sampled-data systems", *Automatica*, vol. 20, no. 1, pp. 31–38, 1984.

[5] T Hagiwara, "Analytic study on the intrinsic zeros of sampled-data systems", *IEEE Trans. on Automatic Control*, vol. 41, no. 2, pp. 261–263, 1996.

[6] P.T. Kabamba, "Control of linear systems using generalized sampled-data hold functions", *IEEE Trans. on Automatic Control*, pp. 772–783, September 1987.

[7] K.J. Åström and B. Wittenmark, *Computer-Controlled Systems: Theory and Design*, Prentice Hall, Englewood Cliffs NY, 2nd edition, 1990.

[8] E.-W. Bai and S. Dasgupta, "A note on generalized hold functions", *Systems and Control Letters*, vol. 14, pp. 363–368, 1990.

[9] W. Yan, B.D.O. Anderson, and R.R. Bitmead, "On the gain margin improvement using dynamic compensation based on generalized sampled-data hold functions", *IEEE Trans. on Automatic Control*, vol. 39, no. 11, pp. 2347, 1994.

[10] M.J. Er and B.D.O. Anderson, "Discrete-time loop transfer recovery via generalized sampled-data hold functions based compensator", *International Journal of Robust and Nonlinear Control*, vol. 4, no. 6, pp. 741–756, November-December 1994.

[11] A. Feuer and G.C. Goodwin, "Generalized sample hold functions: Frequency domain analysis of robustness, sensitivity, and intersample difficulties", *IEEE Trans. on Automatic Control*, vol. 39, no. 5, pp. 1042–1047, May 1994.

[12] M. Araki, "Recent developments in digital control theory", in *Proceedings of the 12th World Congress of the IFAC*, 1993, pp. IX–251.

[13] S. Hara, Y. Yamamoto, and H. Fujioka, "Modern and classical analysis-synthesis methods in sampled data control: A brief overview with numerical examples", in *Proc. 35th IEEE CDC*, Kobe, Japan, 1995, pp. 1251–1256.

[14] J.S. Freudenberg, R.H. Middleton, and J.H. Braslavsky, "Inherent design limitations for linear sampled-data feedback systems", *Int. J. of Control*, vol. 61, no. 6, pp. 1387–1421, June 1995.

[15] R.H. Middleton and J.S. Freudenberg, "Zeros and non-pathological sampling for generalized sampled-data hold functions", *Automatica*, vol. 31, no. 2, pp. 315–319, February 1995.

[16] J.H. Braslavsky, G. Meinsma, R.H. Middleton, and J.S. Freudenberg, "On a key sampling formula relating the Laplace and Z transforms", *Systems and Control Letters*, vol. 29, no. 4, pp. 181–190, 1997.

[17] T. Chen and B.A. Francis, "Input-output stability of sampled-data systems", *IEEE Trans. on Automatic Control*, vol. 36, no. 1, pp. 50–58, January 1991.

[18] B.A. Francis and T.T. Georgiou, "Stability theory for linear time-invariant plants with periodic digital controllers", *IEEE Trans. on Automatic Control*, vol. 33, no. 9, pp. 820–832, January 1988.

[19] H.-K. Sung and S. Hara, "Properties of sensitivity and complementary sensitivity functions in single-input single-output digital control systems", *Int. J. of Control*, vol. 48, no. 6, pp. 2429–2439, 1988.

[20] A. Feuer and G. Goodwin, *Sampling in Digital Signal Processing and Control*, Birkhäuser, 1996.

[21] G. Dullerud and K. Glover, "Robust stabilization of sampled-data systems to structured LTI perturbations", *IEEE Trans. on Automatic Control*, vol. 38, no. 10, pp. 1497–1508, October 1993.

[22] J.H. Braslavsky, *Frequency domain analysis of sampled-data control systems*, PhD thesis, Dept. of E&CE, The University of Newcastle, NSW 2308 Australia, 1995.

[23] K. Hoffman, *Banach spaces of analytic functions*, Dover, New York, 1962.

[24] J.M. Maciejowski, "Asymptotic recovery for discrete-time systems", *IEEE Trans. on Automatic Control*, vol. 30, no. 6, pp. 602–605, June 1985.

[25] Z. Zhang and J.S. Freudenberg, "Discrete-time loop transfer recovery for systems with nonminimum phase zeros and time delays", *Automatica*, vol. 29, no. 2, pp. 351–363, March 1993.