Feedback Limitations in Nonlinear Systems: From Bode Integrals to Cheap Control

M.M. Seron, J.H. Braslavsky, P.V. Kokotović and D.Q. Mayne

Abstract— **Feedback limitations of nonlinear systems are investigated using the cheap control approach. The main result is that in the limit, when the control effort is free, the smallest achievable L_2 norm of the output is equal to the least amount of control energy (L_2 norm) needed to stabilize the unstable zero dynamics. This nonlinear result is structurally similar to an earlier linear result by Qiu & Davison (1993), which, in turn, is connected with a Bode-type integral derived by Middleton (1991).**

Keywords— **Nonlinear systems, Cheap optimal control, Performance limitations, Zero Dynamics, Nonminimum phase systems.**

I. Introduction

In this note we investigate and quantify obstacles limiting the performance of nonlinear systems. We clarify which structural properties of the nonlinear system prevent any feedback controller from reducing to zero the L_2 norm of the output.

For linear systems such questions have been answered by classical and modern results, extensively reviewed in a recent monograph [1]. However, most of these linear results are formulated in the frequency domain and are not directly applicable to nonlinear systems.

Feedback limitations of linear systems have also been approached via "cheap" optimal control, which we now extend to nonlinear systems. The key idea is that an optimal controller will be close to its *ideal performance* if the control effort is cheap, and will attain it, if the control effort is free. The penalty on the control effort is thus scaled by a small parameter $\varepsilon > 0$ and the ideal performance is evaluated in the limit as $\varepsilon \to 0$.

Although the linear cheap control problem is well understood [2], [3], we revisit it in Section II for two reasons. First, we use it to introduce a singular perturbation technique which we subsequently apply to the nonlinear problem. This parallel development makes the structural similarities of the two problems more apparent. Second, we point out to a heretofore unnoticed connection between Bode-type integrals and linear cheap control. Specifically, we show that a T-integral formula derived by Middleton [4] reduces to a limiting cheap control formula derived by Qiu & Davison [3]. The new insight gained is that the value of the T-integral is, in fact, the least amount of energy (L_2 norm) required to stabilize the unstable zeros of the linear system.

Because of this insight, the nonlinear cheap control analysis in Section III focuses on the unstable zero dynamics — the nonlinear counterpart of nonminimum-phase (NMP) zeros. For simplicity, we assume that the zero dynamics are *antistable* (asymptotically stable in reverse time) and affinely driven by the output, and that the relative degree of the nonlinear system is one. In this way we avoid complicated derivations and gain in clarity. Our main result is that in the limit, as $\varepsilon \to 0$, the cheap control problem reduces to the minimum energy (L_2 norm) problem in which the system output is used to stabilize the nonlinear zero dynamics. This result is a nonlinear counterpart of the Qiu-Davison formula, through which it is also connected with Middleton's integral formula.

II. Feedback Limitations in Linear Systems

A. Cheap Optimal Control

The cheap control problem for a stabilizable and detectable linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad && x \in \mathbb{R}^n, u \in \mathbb{R}^m, \\ y(t) &= Cx(t)\ , && y \in \mathbb{R}^m, \end{aligned} \tag{1}$$

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consists in finding a stabilizing state feedback control which minimizes the functional

$$J_\varepsilon = \frac{1}{2}\int_0^\infty \left[y^{\mathrm{T}}(t)y(t) + \varepsilon^2 u^{\mathrm{T}}(t)u(t)\right]dt \tag{2}$$

when $\varepsilon > 0$ is small.

As $\varepsilon \to 0$, the optimal value J_ε^* tends to J_0^*, the *ideal performance*. Kwakernaak & Sivan proved that $J_0^* = 0$ *if and only if the system* (1) *is minimum phase and right invertible* [2, Theorem 3.14]. Qiu & Davison derived an explicit formula as a function of the NMP zeros [3]. We will re-derive the Qiu-Davison formula under the assumption that the system (1) has relative degree one ($\operatorname{rank} CB = m$). Then, a change of coordinates $x \mapsto (y, z)$ exists which transforms (1) into the form

$$\begin{aligned} \dot{y} &= A_1 y + A_2 z + B_1 u, \quad && B_1 = CB\ , \\ \dot{z} &= B_0 y + A_0 z, && y \in \mathbb{R}^m, z \in \mathbb{R}^{n-m}\ , \end{aligned} \tag{3}$$

The zeros of $C(sI - A)^{-1}B$ are the eigenvalues of A_0. We assume that all the zeros are NMP, that is, $-A_0$ is Hurwitz.

We now minimize the functional (2) for the system (3). The associated algebraic Riccati equation is

$$\begin{aligned} \begin{bmatrix} A_1 & A_2 \\ B_0 & A_0 \end{bmatrix}^{\mathrm{T}} P(\varepsilon) + P(\varepsilon)\begin{bmatrix} A_1 & A_2 \\ B_0 & A_0 \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ = \frac{1}{\varepsilon^2}P(\varepsilon)\begin{bmatrix} B_1B_1^{\mathrm{T}} & 0 \\ 0 & 0 \end{bmatrix}P(\varepsilon)\ . \end{aligned} \tag{4}$$

To find the limit of the positive definite solution $P(\varepsilon)$ as $\varepsilon \to 0$, we must resolve the singularity at $\varepsilon = 0$. Following Jameson & O'Malley [5] we seek $P(\varepsilon)$ in the form of a series in ε,

$$P(\varepsilon) = \begin{bmatrix} \varepsilon P_1 & \varepsilon P_2 \\ \varepsilon P_2^{\mathrm{T}} & P_0 + \varepsilon P_3 \end{bmatrix} + O(\varepsilon^2)\ , \tag{5}$$

where P_0, P_1, P_2 and P_3 are independent of ε. The substitution of (5) into (4) yields

$$I - P_1 B_1 B_1^{\mathrm{T}} P_1 + O(\varepsilon) = 0\ , \tag{6}$$

$$B_0^{\mathrm{T}} P_0 - P_1 B_1 B_1^{\mathrm{T}} P_2 + O(\varepsilon) = 0\ , \tag{7}$$

$$A_0^{\mathrm{T}} P_0 + P_0 A_0 - P_2^{\mathrm{T}} B_1 B_1^{\mathrm{T}} P_2 + O(\varepsilon) = 0\ . \tag{8}$$

Setting $\varepsilon = 0$, we find $P_1 = (B_1B_1^{\mathrm{T}})^{-1/2}$ from (6), and $P_2 = (B_1B_1^{\mathrm{T}})^{-1/2}B_0^{\mathrm{T}}P_0$ from (7), and, substituting in (8), we obtain

$$A_0^{\mathrm{T}} P_0 + P_0 A_0 = P_0 B_0 B_0^{\mathrm{T}} P_0\ . \tag{9}$$

Because the system (3) is stabilizable and $-A_0$ is Hurwitz, the unique positive definite P_0 exists. By equating to zero the $O(\varepsilon)$ terms (not shown in (6)-(8)) we find $P_3 = P_0B_0(B_1B_1^{\mathrm{T}})^{-1/2}B_0^{\mathrm{T}}P_0$.

Finally, the optimal value $J_\varepsilon^* = V(y, z; \varepsilon)$ is calculated to be

$$\begin{aligned} J_\varepsilon^* &= \frac{1}{2}z^{\mathrm{T}}P_0 z + \frac{\varepsilon}{2}\left[y + B_0^{\mathrm{T}}P_0 z\right]^{\mathrm{T}}\left(B_1B_1^{\mathrm{T}}\right)^{-1}\left[y + B_0^{\mathrm{T}}P_0 z\right] + O(\varepsilon^2) \\ &= V_0(z) + \varepsilon V_1(y, z) + O(\varepsilon^2)\ . \end{aligned} \tag{10}$$

Its limit as $\varepsilon \to 0$

$$V_0(z) = \frac{1}{2}z^{\mathrm{T}}P_0 z \tag{11}$$

defines the *ideal performance* of system (3).

We interpret this result via the *minimum energy problem*: Find y to stabilize $\dot{z} = A_0 z + B_0 y$ and to minimize

$$J = \frac{1}{2}\int_0^\infty y^{\mathrm{T}}(t)y(t)\,dt. \tag{12}$$

Here the system output y acts as the control variable for the zero dynamics. The optimal value is $V_0(z)$ given in (11). This means that *the limiting optimal value for the full system is equal to the least amount of energy required to stabilize the zero-dynamics subsystem.*

B. Singular Perturbation Analysis

When the optimal feedback control

$$u^* = -\frac{1}{\varepsilon} B_1^{\mathrm T}(B_1B_1^{\mathrm T})^{-1/2}\left[y + B_0^{\mathrm T}P_0 z + O(\varepsilon)\right]$$

is applied to (3), the resulting closed-loop system is in the standard singular perturbation form

$$\begin{aligned} \varepsilon\dot y &= -(B_1B_1^{\mathrm T})^{1/2}\left[y + B_0^{\mathrm T}P_0z\right] + O(\varepsilon)\ , \\ \dot z &= B_0 y + A_0 z\ . \end{aligned} \tag{13}$$

For a slow-fast analysis [6] we set $\varepsilon = 0$ and get $y = -B_0^{\mathrm T}P_0z$. Then, substituting in the z-equation, we find that the slow subsystem of (13) is the optimal zero-dynamics subsystem

$$\dot z = -P_0^{-1}A_0^{\mathrm T}P_0 z\ . \tag{14}$$

These slow dynamics evolve in the *slow invariant subspace* $y + B_0^{\mathrm T}P_0z = 0$, while the fast dynamics represent the convergence of $y + B_0^{\mathrm T}P_0z$ to zero. The two parts of the optimal value are: $V_0(z)$ *slow*, and $\varepsilon V_1(y,z)$ *fast*. We note that the eigenvalues of (14) are the mirror image of the NMP zeros of (3), a fact known from [2, Theorem 3.12].

C. Cheap Control and the Bode T-Integral

We now apply the above results to the problem of regulating the output y of (3) to a constant setpoint r. With the feedforward term $\bar u = -B_1^{-1}(A_1 - A_2A_0^{-1}B_0)r$ we place the equilibrium of (3) at $y = r$. Introducing the error variables $e = y - r$, $\tilde z = z + A_0^{-1}B_0 r$, and $\tilde u = u - \bar u$, we rewrite (3) as

$$\begin{aligned} \dot e &= A_1 e + A_2\tilde z + B_1\tilde u\ , \\ \dot{\tilde z} &= A_0\tilde z + B_0 e\ . \end{aligned} \tag{15}$$

Then the cheap control problem is the same as for (3), but with $(e,\tilde z)$ replacing (y,z). The ideal performance is $V_0(\tilde z)$. We are interested in $V_0(\tilde z(0))$ for the initial condition $e(0) = r, \tilde z(0) = A_0^{-1}B_0 r$, which corresponds to the state y being transferred from 0 to r.

Qiu-Davison Formula. For the initial condition $e(0) = r, \tilde z(0) = A_0^{-1}B_0 r$, the ideal performance of (15) is

$$V_0(\tilde z(0)) = \tfrac12 r^{\mathrm T}Hr\ , \qquad \text{where} \qquad \operatorname{trace} H = 2\sum_{i=1}^{k}\frac{1}{\alpha_i}\ , \tag{16}$$

and $\alpha_1,\dots,\alpha_k$ are the unstable eigenvalues of A_0, that is the NMP zeros of system (15). This formula is obtained from (11) and (9)

$$\begin{aligned} V_0(\tilde z(0)) &= \tfrac12 \tilde z^{\mathrm T}(0)P_0\tilde z(0) \\ &= \tfrac12 r^{\mathrm T}B_0^{\mathrm T}(A_0^{\mathrm T})^{-1}P_0A_0^{-1}B_0 r \triangleq \tfrac12 r^{\mathrm T}Hr\ . \\ \operatorname{trace} H &= \operatorname{trace}\left[P_0^{1/2}A_0^{-1}B_0B_0^{\mathrm T}(A_0^{\mathrm T})^{-1}P_0^{1/2}\right] \\ &= \operatorname{trace}\left[P_0^{1/2}A_0^{-1}P_0^{-1/2} + P_0^{-1/2}(A_0^{\mathrm T})^{-1}P_0^{1/2}\right] \\ &= 2\ \operatorname{trace} A_0^{-1} = 2\sum_{i=1}^{k}\frac{1}{\alpha_i}\ . \end{aligned}$$

How is this result related to the performance limitations defined by well-known Bode integrals? An integral formula for the complementary sensitivity function T derived by Middleton [4] is

$$\frac{1}{\pi}\int_0^\infty \log|T(j\omega)|\ \frac{d\omega}{\omega^2} + \frac{1}{2K_v} = \sum_{i=1}^{k}\frac{1}{\alpha_i}\ , \tag{17}$$

where K_v is the *velocity constant*. The glaring similarity of the Middleton formula (17) and the Qiu-Davison formula (16) has, apparently, remained unnoticed.

Proposition (Cheap Control and Bode T-Integral) Let G and K be the single-input single-output transfer functions of a plant and a minimum phase controller, respectively. Suppose that $T = GK(1+GK)^{-1}$ is stable and $T(0) = 1$. Then

$$\frac{1}{\pi}\int_0^\infty \log|T(j\omega)|\ \frac{d\omega}{\omega^2} + \frac{1}{2K_v} = \lim_{\varepsilon\to 0}\frac12\int_0^\infty e^2(t)\,dt\ ,$$

where $e(t) = y(t) - r$ is the error in transferring the system from rest to the setpoint $r = 1$ using the cheap optimal control. ∘

A dual result relates the classical Bode integral for the sensitivity function S [7] with the minimum energy (L_2 norm) required to transfer the system to rest from the initial condition originating from a unit impulse at the input [8]. Similar relations have been derived in linear filtering [9].

III. Feedback Limitations in Nonlinear Systems

A. Cheap Optimal Control

We now proceed to the main subject of this note: the lowest attainable L_2 norm of the output y of the nonlinear system

$$\begin{aligned} \dot y &= f(y,z) + g(y,z)u, \qquad && y,\ u\in\mathbb R^m, \\ \dot z &= f_0(z) + g_0(z)y, && z\in\mathbb R^{n-m}, \end{aligned} \tag{18}$$

where $f(0,0) = 0$, $f_0(0) = 0$. The following assumption implies that the relative degree of (18) is one and that its zero dynamics are governed by $\dot z = f_0(z)$ [10].

Assumption 1: There exists $\gamma > 0$ such that the smallest singular value of $g(y,z)$ is greater than or equal to γ for all y and z. ∘

For the nonlinear system (18), the cheap optimal control problem consists in finding a feedback control u which guarantees asymptotic stability and minimizes the functional (2) when $\varepsilon > 0$ is small. It is well-known (e.g., [11, p. 91]) that this problem has a solution if there exists a positive semidefinite optimal value $V(y,z;\varepsilon)$ which satisfies the Hamilton-Jacobi-Bellman equation (HJBE)

$$\begin{aligned} &\frac{\partial V}{\partial y}f(y,z) + \frac{\partial V}{\partial z}\left[f_0(z) + g_0(z)y\right] + \frac12 y^{\mathrm T}y \\ &\qquad - \frac{1}{2\varepsilon^2}\frac{\partial V}{\partial y}g(y,z)g^{\mathrm T}(y,z)\frac{\partial^{\mathrm T}V}{\partial y} = 0\ , \quad V(0,0,\varepsilon) = 0\ , \end{aligned} \tag{19}$$

and such that the feedback control

$$u = -\frac{1}{\varepsilon^2}g^{\mathrm T}(y,z)\frac{\partial^{\mathrm T}V}{\partial y} \tag{20}$$

asymptotically stabilizes (18). We are interested in the limit of $V(y,z;\varepsilon)$ as $\varepsilon\to 0$. At $\varepsilon = 0$ the HJBE (19) has a singularity, which we try to resolve by seeking a solution in the form

$$V(y,z;\varepsilon) = V_0(z) + \varepsilon V_1(y,z) + O(\varepsilon^2)\ . \tag{21}$$

Its substitution in (19) yields

$$\begin{aligned} &\frac{\partial V_0}{\partial z}\left[f_0(z) + g_0(z)y\right] + \frac12 y^{\mathrm T}y \\ &\qquad - \frac12\frac{\partial V_1}{\partial y}g(y,z)g(y,z)^{\mathrm T}\frac{\partial^{\mathrm T}V_1}{\partial y} + O(\varepsilon) = 0\ . \end{aligned} \tag{22}$$

A closer examination of (22) shows that the first two terms represent the Hamiltonian for the minimum energy problem: Find y to asymptotically stabilize the zero-dynamics subsystem

$$\dot z = f_0(z) + g_0(z)y \tag{23}$$

and to minimize the cost (12). We therefore make the following assumption.

Assumption 2: The zero dynamics of (18) are *antistable* ($\dot{z} = -f_0(z)$ is asymptotically stable), and there exists a positive definite function $V_0(z)$ that satisfies the HJBE

$$\frac{\partial V_0}{\partial z} f_0(z) - \frac{1}{2}\frac{\partial V_0}{\partial z} g_0(z) g_0^{\mathrm{T}}(z) \frac{\partial^{\mathrm{T}} V_0}{\partial z} = 0 \ , \quad V_0(0) = 0 \ , \tag{24}$$

and such that the feedback control

$$y = \alpha_0(z) \triangleq -g_0^{\mathrm{T}}(z) \frac{\partial^{\mathrm{T}} V_0}{\partial z} \tag{25}$$

achieves global asymptotic stability (GAS) of (23). ○

With $V_0(z)$ so defined, we return to (22), add and subtract $\frac{1}{2}\|\alpha_0(z)\|^2 = \frac{1}{2}\|g_0^{\mathrm{T}} \partial^{\mathrm{T}} V_0/\partial z\|^2$, and, using (24), obtain

$$\frac{1}{2}\left[y - \alpha_0(z)\right]^{\mathrm{T}}\left[y - \alpha_0(z)\right] - \frac{1}{2}\frac{\partial V_1}{\partial y} g(y,z) g(y,z)^{\mathrm{T}} \frac{\partial^{\mathrm{T}} V_1}{\partial y} = O(\varepsilon) \ . \tag{26}$$

Defining $\eta = y - \alpha_0(z)$, $\tilde{g}(\eta, z) = g(\eta + \alpha_0(z), z)$, and $\tilde{V}_1(\eta, z) = V_1(\eta + \alpha_0(z), z)$, and letting $\varepsilon = 0$, we rewrite (26) as

$$\frac{1}{2}\eta^{\mathrm{T}}\eta - \frac{1}{2}\frac{\partial \tilde{V}_1}{\partial \eta} \tilde{g}(\eta, z) \tilde{g}^{\mathrm{T}}(\eta, z) \frac{\partial^{\mathrm{T}} \tilde{V}_1}{\partial \eta} = 0 \ . \tag{27}$$

This is the HJBE for the cheap control problem: Find u to stabilize the system $\dot{\eta} = \tilde{g}(\eta, z) u$, where z is treated as a constant vector, and to minimize the functional $\frac{1}{2}\int_0^\infty \left[\eta^{\mathrm{T}}\eta + \varepsilon^2 u^{\mathrm{T}} u\right] dt$. The optimal value for this problem is $\varepsilon\tilde{V}_1(\eta, z)$ and the optimal control is

$$u = -\frac{1}{\varepsilon}\tilde{g}^{\mathrm{T}}\frac{\partial^{\mathrm{T}} \tilde{V}_1}{\partial \eta} = -\frac{1}{\varepsilon}\tilde{g}^{\mathrm{T}}(\tilde{g}\tilde{g}^{\mathrm{T}})^{-1/2}\eta \, . \tag{28}$$

Thus, the original cheap control problem (18), (2) has decomposed in two subproblems: a minimum energy problem for the zero-dynamics subsystem (23), and a cheap control problem to rapidly regulate $\eta = y - \alpha_0(z)$ to zero.

B. Singular Perturbation Analysis

We treat (28) as an approximation of the optimal control (20) and apply it to (18). In the (η, z)-coordinates, the resulting feedback system is

$$\begin{aligned} \varepsilon\dot{\eta} &= -[\tilde{g}(\eta, z)\tilde{g}^{\mathrm{T}}(\eta, z)]^{1/2}\eta + \varepsilon\phi(\eta, z) \ , \\ \dot{z} &= f_0(z) + g_0(z)\alpha_0(z) + g_0(z)\eta \ , \end{aligned} \tag{29}$$

where $\phi(\eta, z) = f(\eta + \alpha_0(z), z) - \dot{\alpha}_0(z)$.

This system is in standard singular perturbation form [6]. We investigate its stability properties. In the slow manifold $\eta = 0$ the system reduces to the slow subsystem

$$\dot{z} = f_0(z) + g_0(z)\alpha_0(z) \ , \tag{30}$$

which, by Assumption 2, is GAS. Convergence to the slow manifold is governed by the fast (boundary-layer) subsystem

$$\frac{d\eta}{d\tau} = -[\tilde{g}(\eta, z)\tilde{g}^{\mathrm{T}}(\eta, z)]^{1/2}\eta \ , \qquad \tau = \frac{t}{\varepsilon} \ , \tag{31}$$

with z treated as a constant vector. The fast subsystem (31) is globally exponentially stable by Assumption 1.

As shown in [12], these stability properties of the slow and fast subsystems (30) and (31) guarantee semiglobal (in ε) boundedness of the solutions of (29). For each pair of positive numbers (R, ρ), there exists $\varepsilon_R > 0$ such that for all $\varepsilon \in (0, \varepsilon_R]$ the solutions of (29) starting in the ball B_R of radius R enter in finite time the ball B_ρ.

However, this is not sufficient for asymptotic stability of the equilibrium $(\eta, z) = (0, 0)$: the system $\varepsilon\dot{\eta} = -\eta + \varepsilon z$, $\dot{z} = -z^3 + \eta$, has slow and fast subsystems as above, but for $\varepsilon > 0$ its equilibrium is unstable [12]. As suggested in [13], we need to restrict the interconnection $\phi(\eta, z)$.

Lemma (Asymptotic Stability) Suppose $\|\phi(\eta, z)\| \leq k_1\|\eta\| + k_2\|\alpha_0(z)\|$ in B_δ for some $\delta > 0$. Then, under Assumptions 1 and 2, to each $R > 0$ there corresponds an $\varepsilon_R > 0$ such that for all $\varepsilon \in (0, \varepsilon_R]$ the equilibrium $(\eta, z) = (0, 0)$ of (29) is asymptotically stable and its basin of attraction contains B_R.

Proof: Using (24) and (25), the derivative of the Lyapunov function $W(\eta, z) = \frac{1}{2}\eta^{\mathrm{T}}\eta + V_0(z)$ along the solutions of (29) in B_δ satisfies

$$\dot{W} \leq -\frac{1}{2}\begin{bmatrix} \|\eta\| \\ \|\alpha_0(z)\| \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 2\left(\frac{\gamma}{\varepsilon} - k_1\right) & 1 + k_2 \\ 1 + k_2 & 1 \end{bmatrix} \begin{bmatrix} \|\eta\| \\ \|\alpha_0(z)\| \end{bmatrix} \tag{32}$$

where γ is given by Assumption 1.

Thus, $\dot{W} \leq 0$ for all $\varepsilon \in (0, \varepsilon_a]$, where $\varepsilon_a = \gamma/[(1 + k_2)^2 + 2k_1]$. Pick c such that $\Omega_c = \{(\eta, z) : W(\eta, z) \leq c\} \subset B_\delta$. Then, by LaSalle's invariance theorem, every solution of (29) starting in Ω_c converges to the largest invariant set contained in $E = \{(\eta, z) : \dot{W} = 0\} \subset \{(\eta, z) : \eta = 0, \alpha_0(z) = 0\}$. Because, by Assumption 2, $y = \alpha_0(z)$ asymptotically stabilizes (23), this invariant set is just $(\eta, z) = (0, 0)$, which is then asymptotically stable with a basin of attraction that contains Ω_c.

Next, choose $\rho > 0$ such that $B_\rho \subset \Omega_c$. Then, Theorem 1 in [12] guarantees the existence of $\varepsilon_R \in (0, \varepsilon_a]$ such that for all $\varepsilon \in (0, \varepsilon_R]$ the solutions of (29) starting in B_R enter B_ρ in finite time, and thereafter converge to $(\eta, z) = (0, 0)$. ■

C. Limitations to Nonlinear Ideal Performance

Thus far we have established that $V_0 + \varepsilon V_1$ satisfies the HJBE (19) with an $O(\varepsilon)$ error and that the corresponding control, given by (28),

$$u = -\frac{1}{\varepsilon^2} g^{\mathrm{T}} \frac{\partial^{\mathrm{T}}(V_0 + \varepsilon V_1)}{\partial y} = -\frac{1}{\varepsilon}\tilde{g}^{\mathrm{T}}\frac{\partial^{\mathrm{T}}\tilde{V}_1}{\partial \eta} = -\frac{1}{\varepsilon}\tilde{g}^{\mathrm{T}}(\tilde{g}\tilde{g}^{\mathrm{T}})^{1/2}\eta$$

asymptotically stabilizes (18), written in closed-loop form as (29).

Our analysis has decomposed the optimal cheap control problem in two separate subproblems: a minimum energy problem for asymptotic stabilization of the zero-dynamics subsystem (23), and a cheap control problem for asymptotic stabilization of the boundary-layer subsystem (31). The solutions of (29), as well as those of (18), (20), begin by rapidly ("instantaneously" as $\varepsilon \to 0$) converging to the zero-dynamics manifold $\eta = y - \alpha_0(z) = 0$. After that, they settle in a slow motion during which the zero dynamics are controlled by $y = \alpha_0(z)$. The cost of the rapid transient vanishes as $\varepsilon \to 0$. What remains is the cost $V_0(z)$ of the slow motion needed to stabilize the unstable zero dynamics. The overall cost cannot be reduced below $V_0(z)$. We are now ready to state and prove our main result.

Theorem (Ideal Performance) Under the conditions of the Lemma, for every initial state $(y(0), z(0))$ for which the cheap control problem (18), (2) has a solution, the optimal value satisfies

$$J_\varepsilon^* = V(y(0), z(0); \varepsilon) = V_0(z(0)) + O(\varepsilon) \, , \tag{33}$$

and thus the ideal performance is $V_0(z(0))$, the optimal value of the minimum energy problem for the zero-dynamics subsystem (23) controlled by the output y.

Proof: We prove that the cost produced by the control (28) satisfies (33). Since the optimal control cannot produce a larger cost, (33) will therefore hold for the optimal value. Let t_δ be the finite time such that $(\eta(t), z(t)) \in B_\delta$ for all $t \geq t_\delta$. By integrating (32) over $[t_\delta, \infty)$ we obtain

$$\int_{t_\delta}^\infty \left[\frac{1}{2}\|\alpha_0(z)\|^2 + (1 + k_2)\|\alpha_0(z)\|\,\|\eta\| + \left(\frac{\gamma}{\varepsilon_R} - k_1\right)\|\eta\|^2\right] dt \leq V_0(z(t_\delta)) + \frac{1}{2}\|\eta(t_\delta)\|^2 < \infty \, . \tag{34}$$

This, along with the boundedness of the solutions of (29) on $[0, t_\delta]$, guarantees that the cost

$$J_\varepsilon = \int_0^\infty \left[\frac{1}{2}\|\alpha_0(z)\|^2 + \alpha_0^{\mathrm{T}}(z)\eta + \|\eta\|^2\right] dt \tag{35}$$

is finite. Observing that (24) and (29) yield $\frac{1}{2}\|\alpha_0(z)\|^2 + \alpha_0^{\mathrm{T}}(z)\eta = -\dot{V}_0$, we express (35) as

$$J_\varepsilon = V_0(z(0)) + \int_0^\infty \|\eta\|^2\, dt\,. \tag{36}$$

Multiplying the η-equation in (29) by η^{T},

$$\eta^{\mathrm{T}}\dot{\eta} = -\frac{1}{\varepsilon}\eta^{\mathrm{T}}(\tilde{g}\tilde{g}^{\mathrm{T}})^{1/2}\,\eta + \eta^{\mathrm{T}}\phi \le -\frac{\gamma}{\varepsilon}\|\eta\|^2 + \eta^{\mathrm{T}}\phi\,,$$

and integrating we get

$$\frac{\gamma}{\varepsilon}\int_0^\infty \|\eta\|^2\, dt \le \frac{1}{2}\|\eta(0)\|^2 + \int_0^{t_\delta} \eta^{\mathrm{T}}\phi\, dt + \int_{t_\delta}^\infty \eta^{\mathrm{T}}\phi\, dt\,. \tag{37}$$

Since, by assumption, the bound $\|\phi\| \le k_1\|\eta\| + k_2\,\|\alpha_0\|$ holds on $[t_\delta, \infty)$, it follows from (34) and (37) that $\int_0^\infty \|\eta\|^2\, dt$ in (36) is $O(\varepsilon)$, which proves (33). ■

Thus, the lowest attainable L_2 norm of the output of (18) is the least amount of energy required to stabilize the unstable zero dynamics. Although essentially nonlinear, this result is structurally similar to its linear counterpart.

It is also of interest to find a nonlinear analog of the property that, as $\varepsilon \to 0$, the finite poles of the optimal linear system converge to the mirror image of the NMP zeros. An expression of this property is that the zero dynamics are "as stable in the closed loop as they are unstable in the open loop". Using $V_0(z)$ as a Lyapunov function we reveal an analogous property of the nonlinear zero dynamics.

Corollary (Mirroring Property) For the open- and closed-loop zero dynamics, $V_0(z)$ satisfies

$$\frac{\partial V_0}{\partial z} f_0(z) = -\frac{\partial V_0}{\partial z}[f_0(z) + g_0(z)\alpha_0(z)]\,. \tag{38}$$

∘

This identity is immediate from (24) and (30).

D. Example

The essentially nonlinear character of the obtained results will be illustrated on the system

$$\begin{aligned} \dot{y} &= 6z^5 + u\,, \\ \dot{z} &= z^3 + y\,. \end{aligned} \tag{39}$$

Its linearization $\dot{y} = u$, $\dot{z} = y$ is not detectable so that the linear analysis does not apply.

Upon the substitution of $V = V_0 + \varepsilon V_1 + O(\varepsilon^2)$ in the HJBE (19), we obtain

$$\frac{\partial V_0}{\partial z}\left[z^3 + y\right] + \frac{1}{2}y^2 - \frac{1}{2}\left[\frac{\partial V_1}{\partial y}\right]^2 + O(\varepsilon) = 0\,. \tag{40}$$

The HJBE for the minimum energy stabilization of the zero-dynamics subsystem $\dot{z} = z^3 + y$ is

$$\frac{\partial V_0}{\partial z} z^3 - \frac{1}{2}\left[\frac{\partial V_0}{\partial z}\right]^2 = 0\,, \quad V_0(0) = 0\,. \tag{41}$$

The optimal value $V_0(z)$ and the corresponding control law are

$$V_0(z) = \frac{z^4}{2}, \quad y = \alpha_0(z) = -\frac{\partial V_0}{\partial z} = -2z^3\,.$$

The substitution of $V_0(z) = z^4/2$ and $\varepsilon = 0$ in (40) yields $\partial V_1/\partial y = y + 2z^3$, and, hence,

$$u = -\frac{1}{\varepsilon^2}\frac{\partial(V_0 + \varepsilon V_1)}{\partial y} = -\frac{1}{\varepsilon}\frac{\partial V_1}{\partial y} = -\frac{1}{\varepsilon}(y + 2z^3) = -\frac{1}{\varepsilon}\eta\,.$$

When this control is applied to (39), the closed-loop system is

$$\dot{\eta} = -\frac{1}{\varepsilon}\eta + 6z^2\eta\,, \qquad \dot{z} = -z^3 + \eta\,. \tag{42}$$

Our proof of asymptotic stability of (29) was in two steps: we first showed semiglobal boundedness and then local asymptotic stability. However, for (42) we give a direct proof using $W = \eta^2/2 + z^4/2$ as a Lyapunov function. Its derivative along the solutions of (42),

$$\dot{W} = -\begin{bmatrix}\eta \\ z^3\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\frac{1}{\varepsilon} - 6z^2 & -1 \\ -1 & 2\end{bmatrix}\begin{bmatrix}\eta \\ z^3\end{bmatrix},$$

is negative definite if $12z^2 < (2-\varepsilon)/\varepsilon$. For $\varepsilon < 1$ this inequality is satisfied if $z^2 \le (12\varepsilon)^{-1}$. Therefore, the invariant set $\{(\eta, z) : \eta^2 + z^4 \le (12\varepsilon)^{-2}\}$ belongs to the basin of attraction of the equilibrium $(\eta, z) = (0, 0)$ of (42). We conclude that the lowest achievable L_2-norm of y in (39) is $J_0^* = V_0(z) = z^4/2$.

IV. Conclusions

The main result in this note establishes that the ideal performance of a nonlinear system, that is the lowest attainable L_2 norm of its output, is the least amount of energy required to stabilize the zero dynamics. Thus, unstable zero dynamics represent a structural obstacle to attaining zero ideal performance. Although our results are for systems of relative degree one with a local growth condition, a similar structural obstacle is present in a wider class of nonlinear systems for which its quantification would require a more elaborate analysis.

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