ELEC4410

Control System Design

Lecture 7: Introduction to MIMO Systems

School of Electrical Engineering and Computer Science
The University of Newcastle

Outline

- MIMO Systems

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- Transfer Matrices, Poles and Zeros

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- Stability

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- Interaction, Decoupling and Diagonal Dominance

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References: Control System Design, Goodwin, Graebe & Salgado.
Multivariable Feedback Control: Analysis and Design, Skogestad & Postlethwaite.

MIMO Systems

- Up to now we have assumed that a control problem can be reduced to controlling a single "control variable" with a single "manipulated variable". The two are assumed to be related via some simple (linear) dynamics, for example, a transfer function

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s), \quad \text{where } \mathbf{Y}, \mathbf{G}, \mathbf{U} : \mathbb{C} \mapsto \mathbb{C}.$$

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- However, in most cases, a system has more than one manipulated variable and more than one control input, and the **interactions** between these are such that the model cannot be further reduced.
- A system in which the input and the output are **vectors**, rather than scalars, is a system with Multiple Inputs and Multiple Outputs (a MIMO system), sometimes also called a **multivariable system**.

Example: Control of an Aircraft

Wilbur Wright said in 1901:

> *Men know how to construct airplanes. Men also know how to build engines. Inability to* ***balance*** *and* ***steer*** *still confronts students on the flying problem. When this one feature has been worked out, the age of flying will have arrived, for all other difficulties are of minor importance.*

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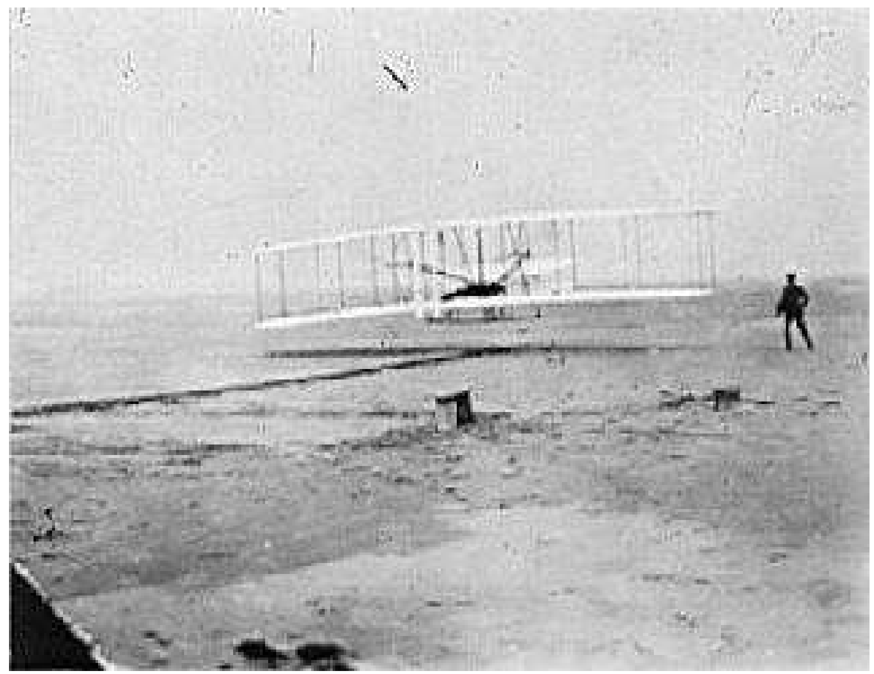
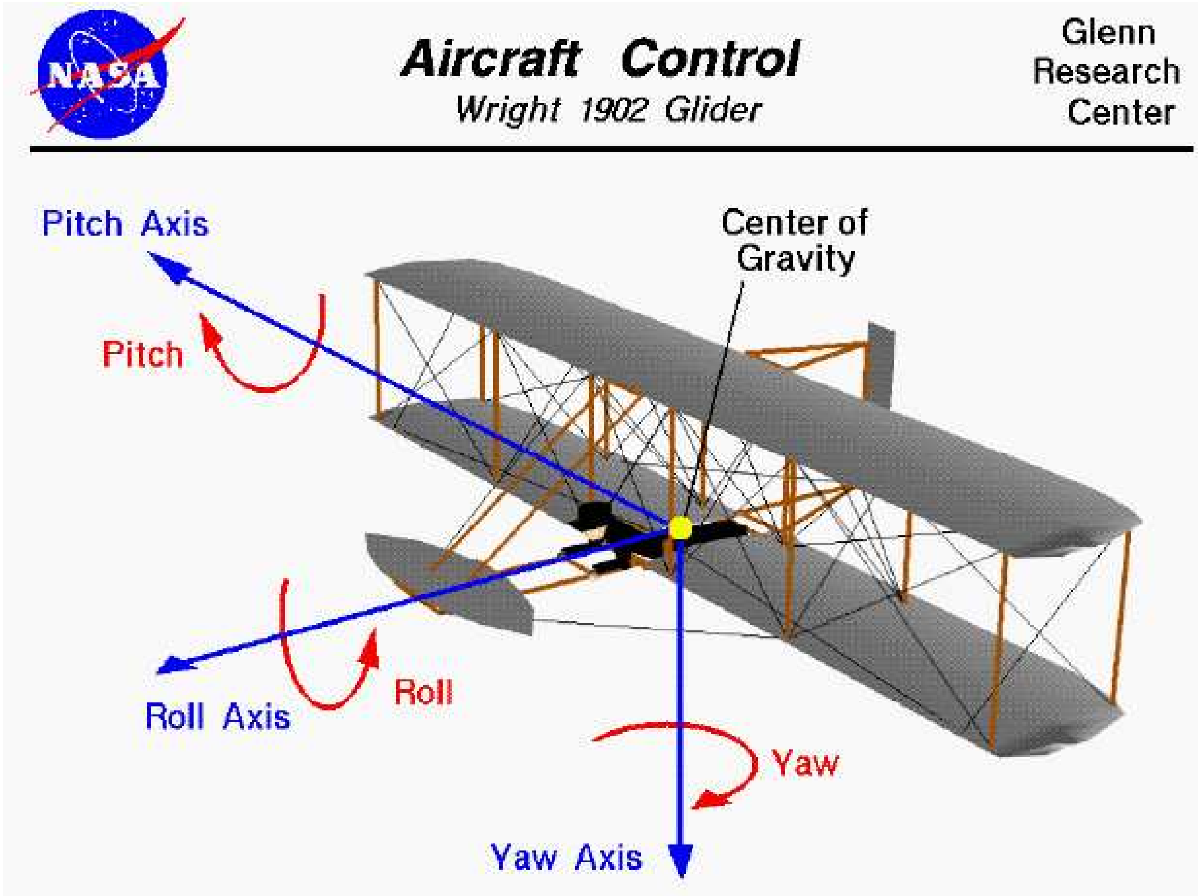
The Wright Brothers solved the control problem and flew the Kitty Hawk on december 17, 1903.



Example: Control of an Aircraft

The Wright Brothers had to solve a MIMO control problem!



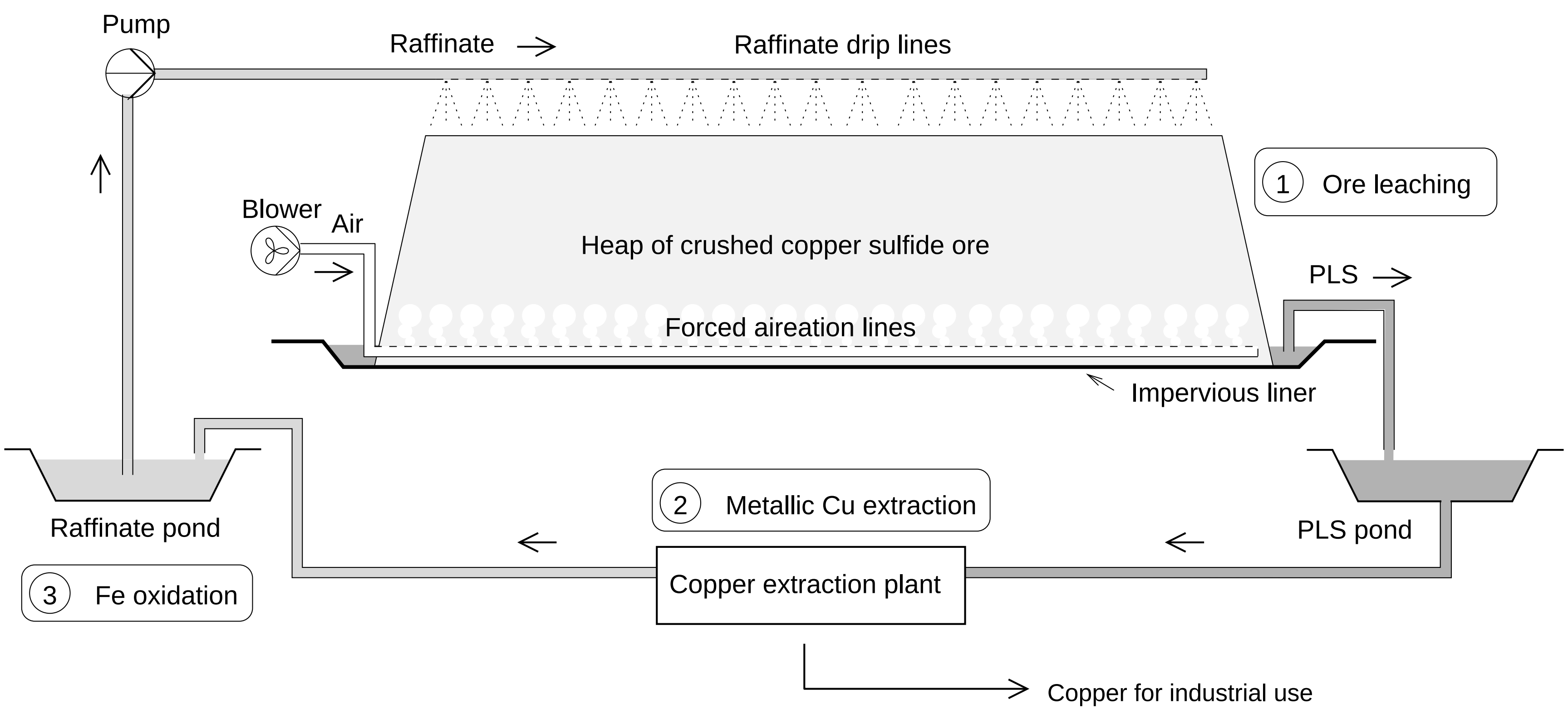
Example: Copper Heap Bioleaching

Heap bioleaching is a process used to extract copper and other metals from large amounts of heaped ore with low grade content, based on the action of chemolithotrophic bacteria.

Example: Copper Heap Bioleaching

Heap bioleaching is a complex MIMO process with many interacting variables, among many others:

- Raffinate concentration
- Temperature and pH gradient
- Bacteria population
- Oxygen flow



Transfer Matrices

- When a MIMO system can be represented by a LTI model, we can use an external representation that extends the idea of a transfer function: a **transfer matrix function**

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_q(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1p} \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ G_{q1}(s) & G_{q2}(s) & \cdots & G_{qp} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \\ \vdots \\ U_p(s) \end{bmatrix}$$

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- We can still write

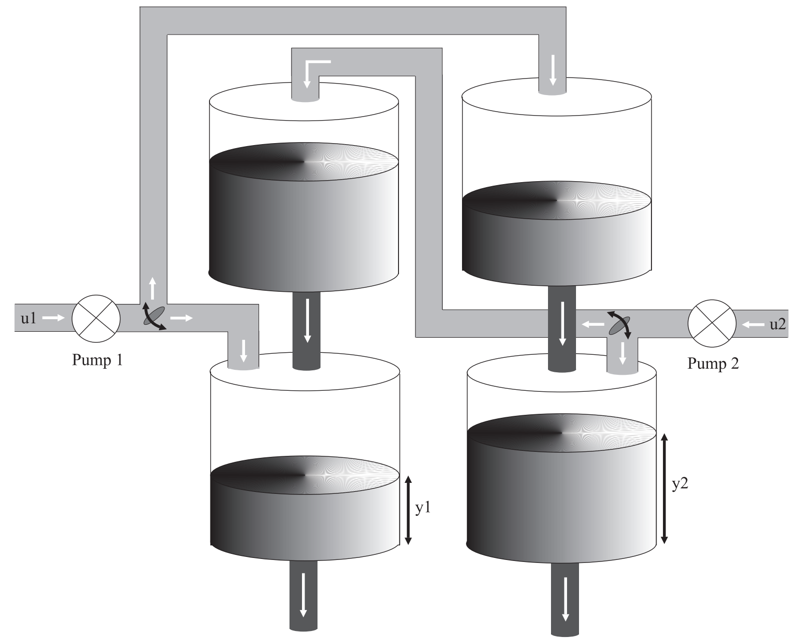
$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s),$$

but now $\mathbf{Y} \in \mathbb{C}^q$, $\mathbf{U} \in \mathbb{C}^p$, and $\mathbf{G} \in \mathbb{C}^{q\times p}$. Besides **gain** and **phase** in $\mathbf{G}(s)$, for MIMO systems also **directions** play a fundamental role.

Example: Four Tank Apparatus

The Four Tank Apparatus is a laboratory system useful to study MIMO systems.

It is a system with
two inputs (the flows u_1 and u_2 provided by the pumps)
and two outputs (the levels y_1 and y_2 of the two lower tanks).



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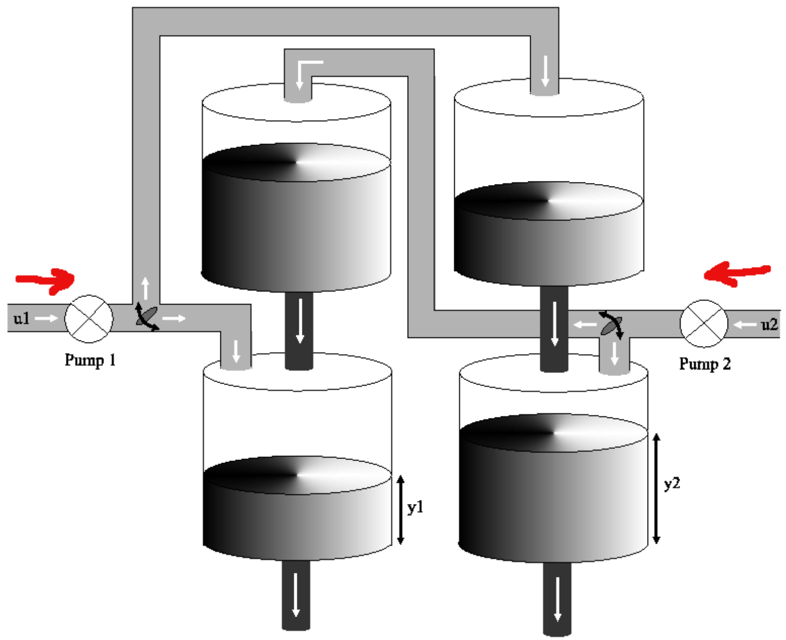

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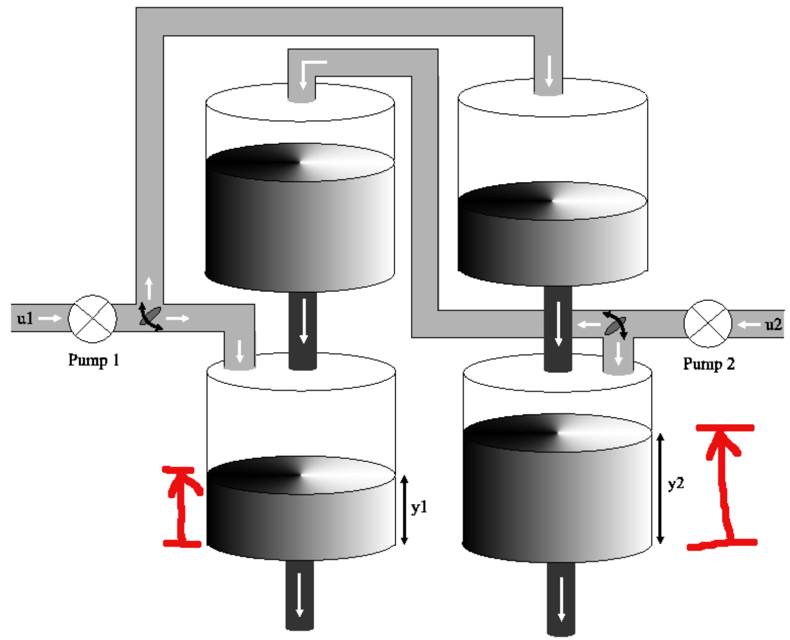

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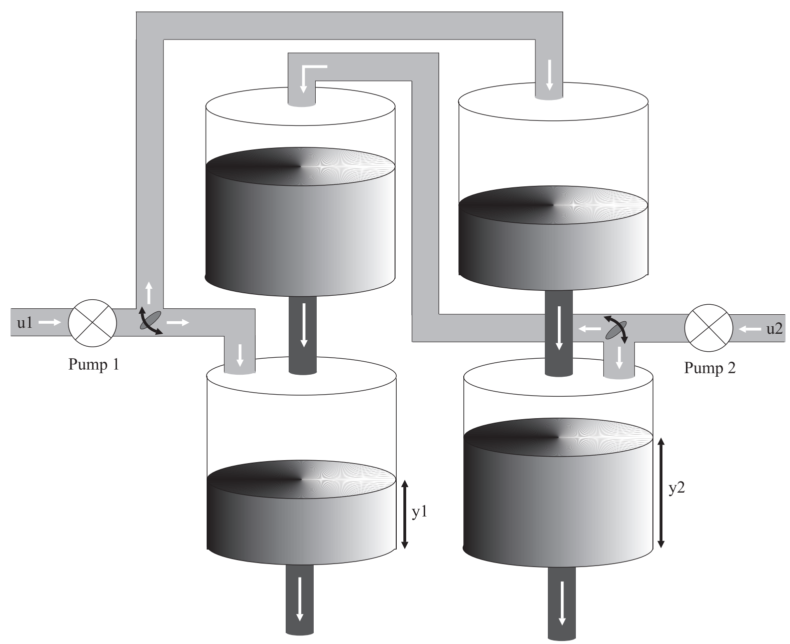

At a suitable operating point, the system can be described by the transfer matrix

$$\mathbf{G}(s) = \begin{bmatrix} \frac{3.7\gamma_1}{62s+1} & \frac{3.7(1-\gamma_2)}{(23s+1)(62s+1)} \\ \frac{4.7(1-\gamma_1)}{(30s+1)(90s+1)} & \frac{4.7\gamma_2}{90s+1} \end{bmatrix}$$

Poles of a MIMO System

- The poles of a multivariable system are the poles of the elements of the transfer matrix.

Example. Take the transfer matrix of the four tank apparatus,

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Then the poles of $\mathbf{G}(s)$ lie at

$$p_1 = -1/62, \quad p_2 = -1/23, \quad p_3 = -1/30, \quad p_4 = -1/90.$$

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$\square$

- It is less straightforward to tell the multiplicity of the poles. One way to do it, if the system is square, is via the computation of the **determinant** of $\mathbf{G}(s)$.

Poles of a MIMO System

- The multiplicity of the poles of a MIMO system, **if it has the same number of inputs and outputs**, can be found from the determinant of the transfer matrix.

  Example. Consider again the four tank apparatus. Then

$$\det G(s) = \det \begin{bmatrix} \frac{3.7\gamma_1}{62s+1} & \frac{3.7(1-\gamma_2)}{(23s+1)(62s+1)} \\ \frac{4.7(1-\gamma_1)}{(30s+1)(90s+1)} & \frac{4.7\gamma_2}{90s+1} \end{bmatrix}$$

$$= \frac{3.7}{(62s+1)} \frac{4.7}{(90s+1)} \left(\gamma_1\gamma_2 - \frac{(1-\gamma_1)(1-\gamma_2)}{(30s+1)(23s+1)} \right)$$

$$= \frac{3.7 \times 4.7}{(62s+1)(90s+1)} \left(\frac{\gamma_1\gamma_2(30s+1)(23s+1) - (1-\gamma_2)(1-\gamma_1)}{(30s+1)(23s+1)} \right)$$

We see that all the poles of $G(s)$ have **multiplicity 1**. □

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&= \frac{3.7}{(62s+1)} \frac{4.7}{(90s+1)} \left(\gamma_1\gamma_2 - \frac{(1-\gamma_1)(1-\gamma_2)}{(30s+1)(23s+1)} \right) \\
&= \frac{3.7 \times 4.7}{(62s+1)(90s+1)} \left(\frac{\gamma_1\gamma_2(30s+1)(23s+1) - (1-\gamma_2)(1-\gamma_1)}{(30s+1)(23s+1)} \right)
\end{aligned}
$$

  We see that all the poles of $G(s)$ have **multiplicity 1**. □

- If the system is **not** square, then the multiplicity of the poles of $G(s)$ can be found from a minimal state space representation.

Poles of a MIMO System

Example. Consider the transfer matrix

$$\mathbf{G}(s) = \begin{bmatrix} \frac{4}{(s+1)(s+2)} & \frac{-1}{(s+1)} \\ \frac{2}{(s+1)} & \frac{-1}{2(s+1)(s+2)} \end{bmatrix}$$

We can see that it has poles at $p_1 = -1$ and $p_2 = -2$. To find their multiplicity we compute the determinant of $\mathbf{G}(s)$,

$$\begin{aligned} \det \mathbf{G}(s) &= \frac{-4}{2(s+1)^2(s+2)^2} + \frac{2}{(s+1)^2} \\ &= \frac{2}{(s+1)^2}\left(1 - \frac{1}{(s+2)^2}\right) = \frac{2\left[(s+2)^2 - 1\right]}{(s+1)^2(s+2)^2} \\ &= \frac{2(s+3)(s+1)}{(s+1)^2(s+2)^2} = . \end{aligned}$$

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Thus there is one pole at $s = -1$ and two poles at $s = -2$. □

Zeros of a MIMO system

There are many definitions of zeros for MIMO systems. The most useful one that of a **transmission zero**, which can be (loosely) defined as a pole of the inverse plant (for square plants).

Example. Consider again the transfer matrix of the previous Example

$$\mathbf{G}(s) = \begin{bmatrix} \frac{4}{(s+1)(s+2)} & \frac{-1}{(s+1)} \\ \frac{2}{(s+1)} & \frac{-1}{2(s+1)(s+2)} \end{bmatrix}$$

Because **det**$[\mathbf{G}(s)^{-1}] =$ **det**$[\mathbf{G}(s)]^{-1}$, we can obtain the zeros of $\mathbf{G}(s)$ as the zeros of **det**$[\mathbf{G}(s)]$. From the previous Example

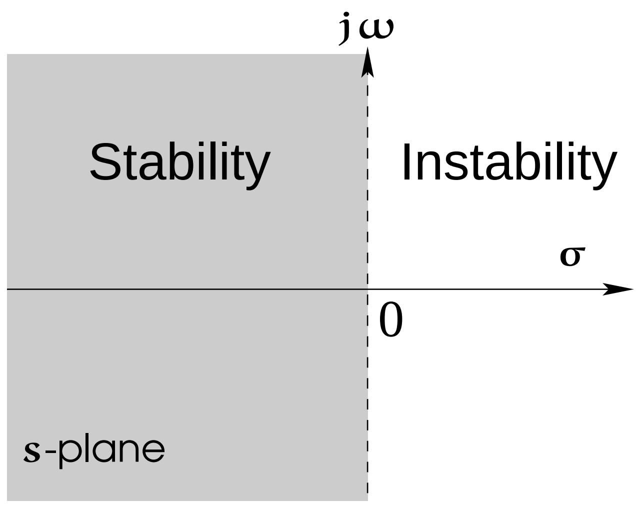
$$\textbf{det}\, \mathbf{G}(s) = \frac{2(s+3)}{(s+1)(s+2)^2}.$$

Thus $\mathbf{G}(s)$ has a zero at $s = -3$. Again, more sophisticated methods should be used if the transfer matrix is not square. □

Stability of MIMO Systems

Stability for MIMO systems is as for SISO systems.

- A continuous-time multivariable system is stable if and only if all its poles lie in the **left** half plane.



(a) Continuous-time

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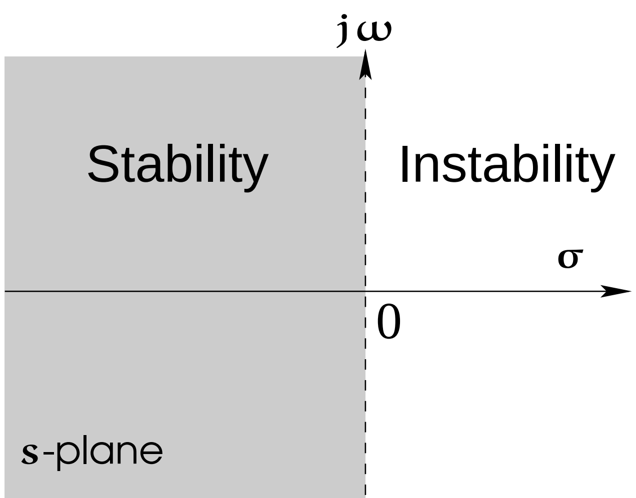
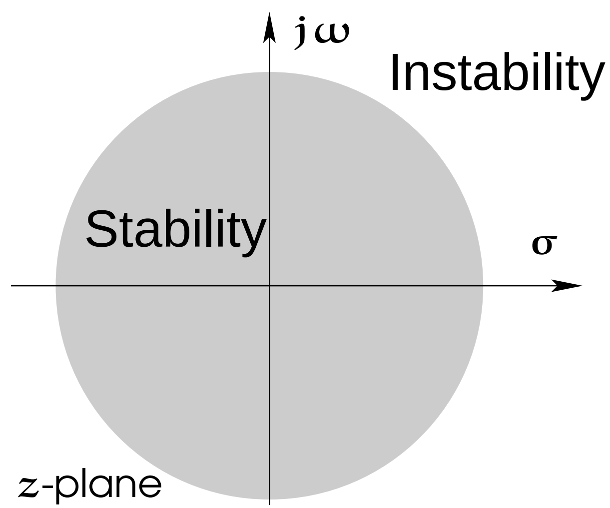


(c) Continuous-time



(d) Discrete-time

Minimum Phase MIMO Systems

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Example. Consider again the four tank apparatus system. It is not difficult to show that the system has two multivariable zeros that satisfy **det**$[G(s)] = 0$ at the roots of

$$(23s + 1)(30s + 1) - \eta = 0, \quad \text{where} \quad \eta = \frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_1\gamma_2}.$$

The system is nonminimum phase if $\eta > 1 \Leftrightarrow (\gamma_1 + \gamma_2) < 1$. □

Interaction and Decoupling

- One of the most challenging aspects of the control of MIMO systems is the interaction between different inputs and outputs.
  In general, each input will have an effect on every output of the system (outputs are coupled). Take for example a 2×2 system

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

$$= \begin{bmatrix} G_{11}(s)U_1(s) + G_{12}(s)U_2(s) \\ G_{12}(s)U_1(s) + G_{22}(s)U_2(s) \end{bmatrix}$$

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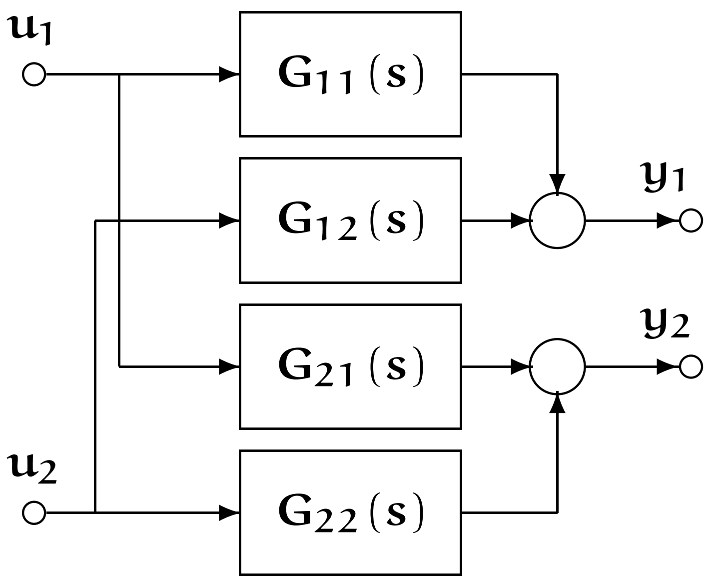
- When the MIMO system is such that each input only affects one particular output, different from the outputs affected by other inputs, the system is decoupled or noninteracting.

Interaction and Decoupling

Because of coupling, in MIMO systems signals can interact in unexpected ways.



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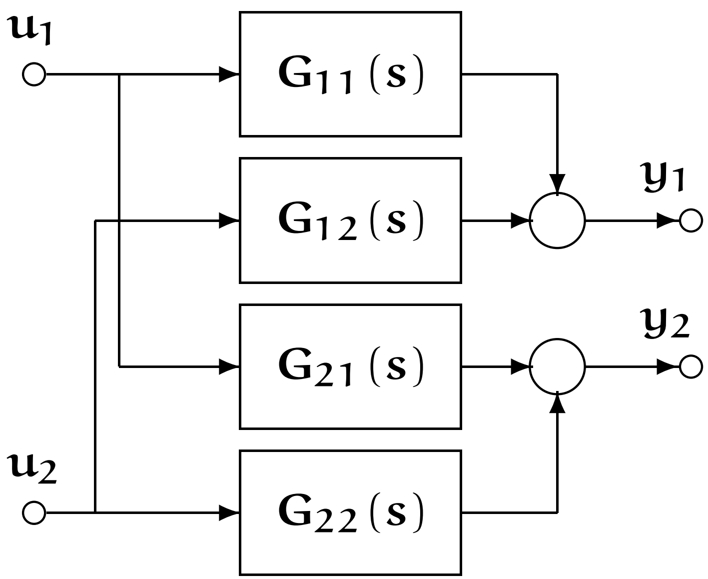

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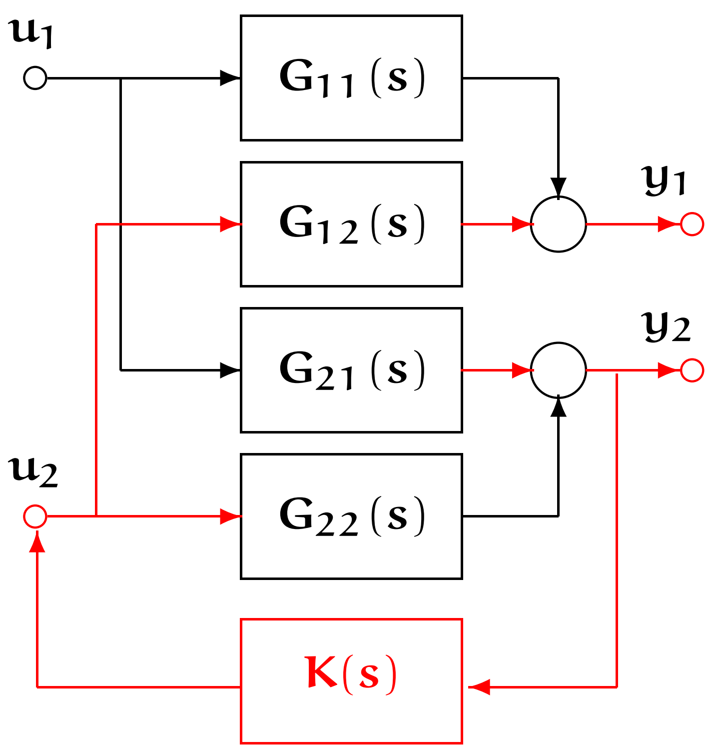

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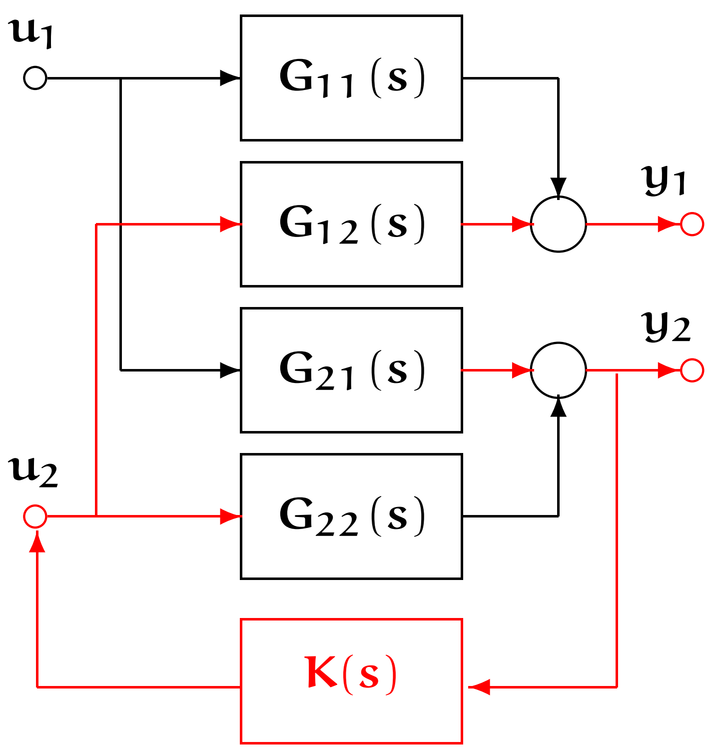

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Otherwise, there is coupling, and the manipulation of any input may affect all outputs. Suppose for example that we close a single loop between y_2 and u_2.

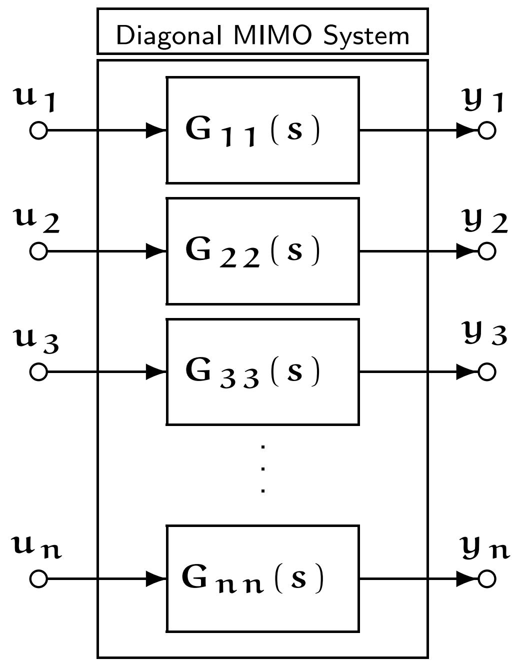
Both outputs will be affected!

Interaction and Decoupling

If $\mathbf{G}(s)$ is a diagonal matrix, the MIMO system is totally decoupled. The MIMO control design problem then reduces to several SISO control design problems.

For example, we could implement IMC for each loop separately.

$$K_i(s) = \frac{Q_i(s)}{1 - Q_i(s)G_{ii}(s)}$$



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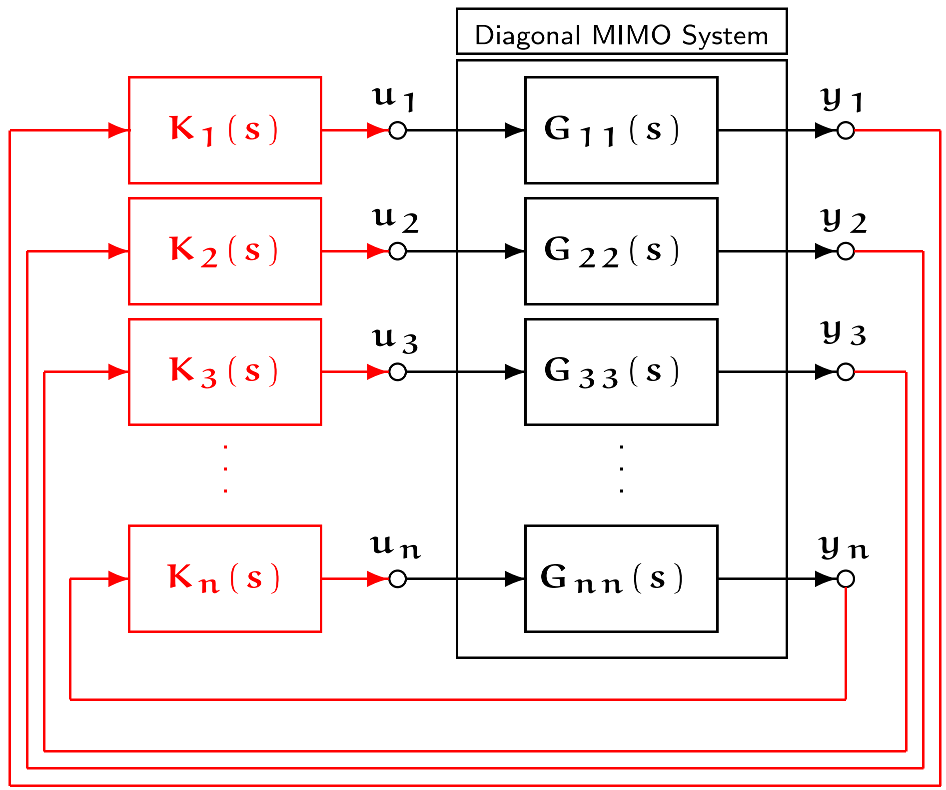

Diagonal Dominance

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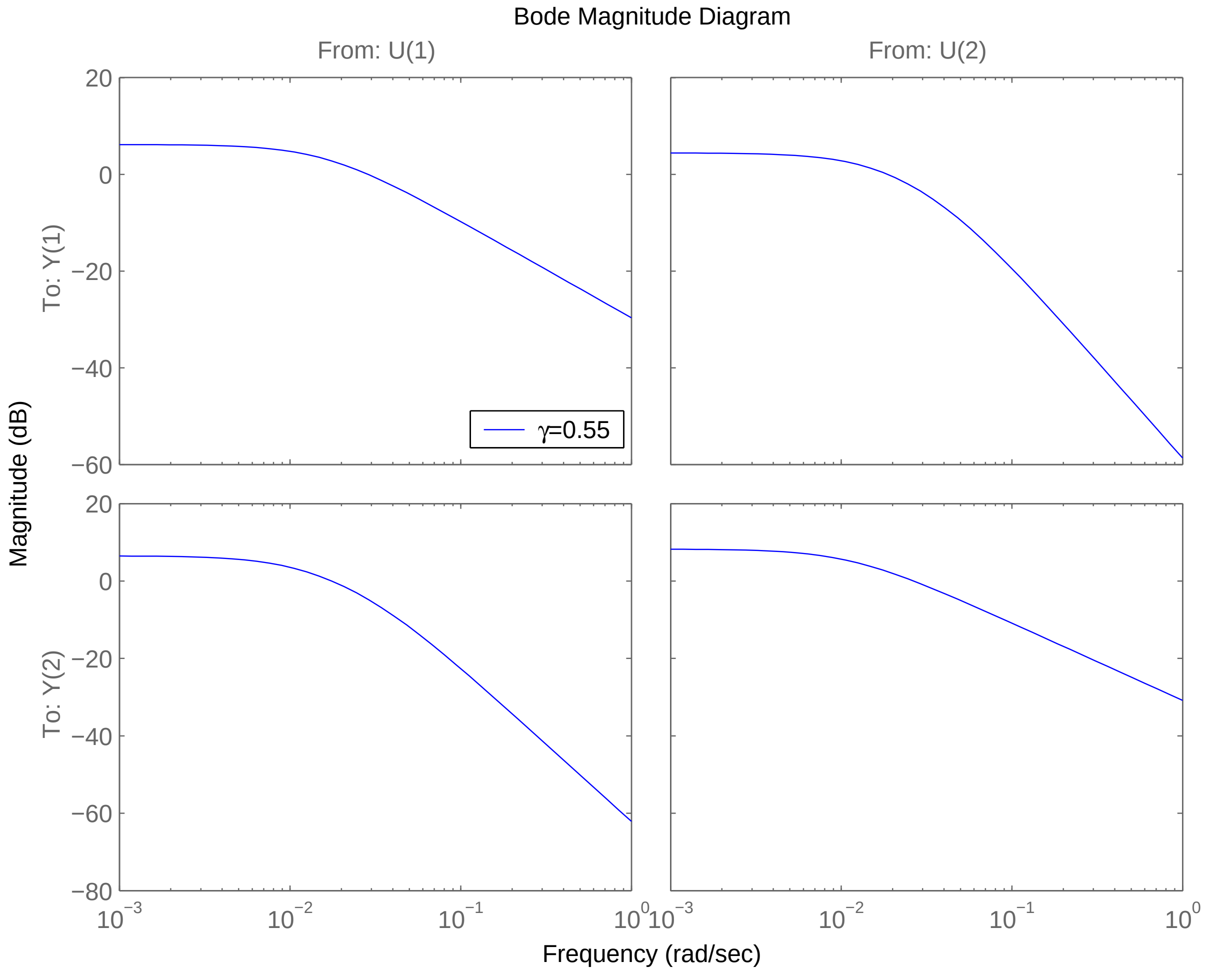
- Loosely, a diagonally dominant plant has a transfer matrix in which the transfer functions on the diagonal are greater in magnitude than the off-diagonal elements.

Diagonal Dominance

Take the four tank apparatus with $\gamma_1 = \gamma_2 = \gamma$. We see that the higher the value of γ, the more the leading diagonal dominates.



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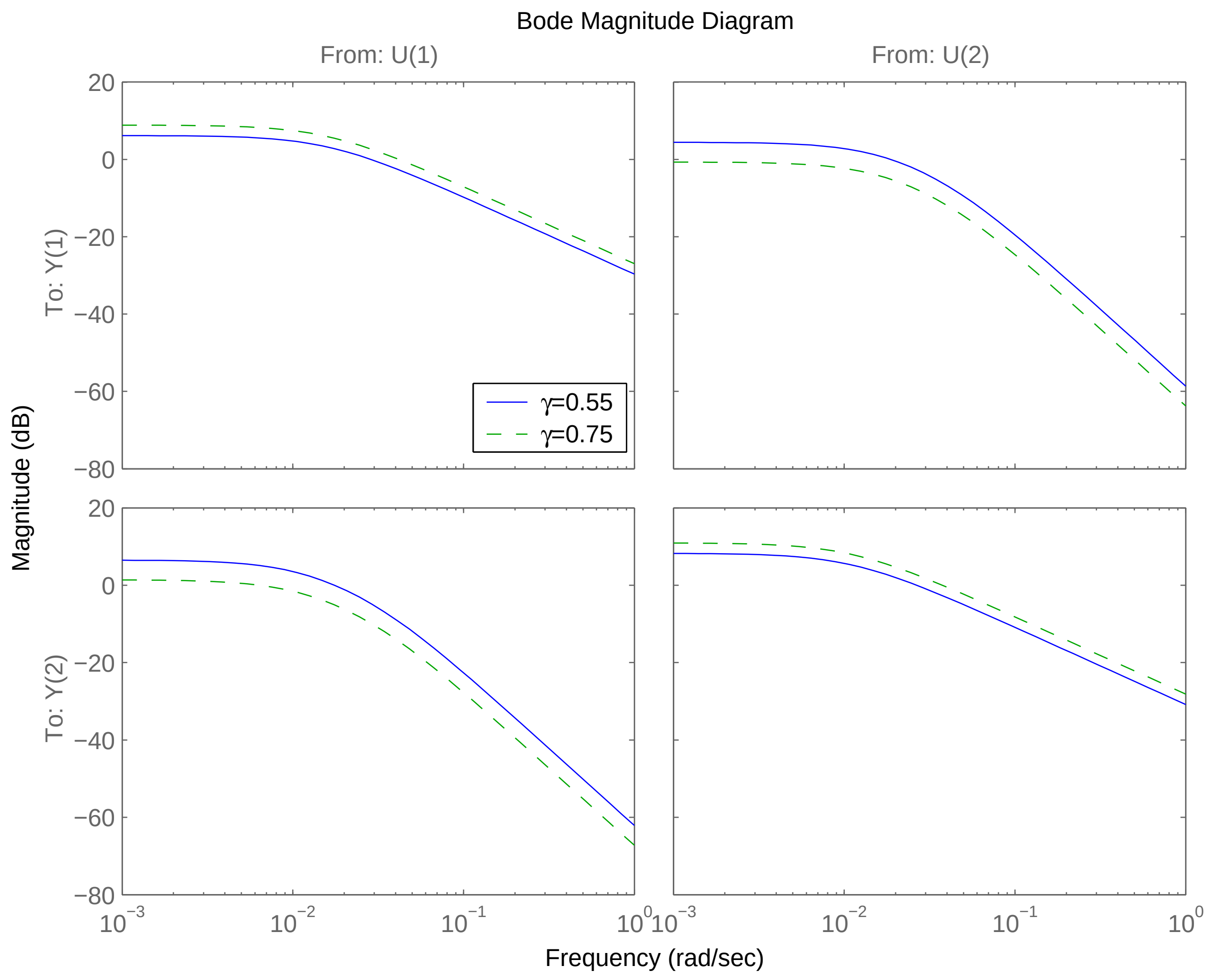

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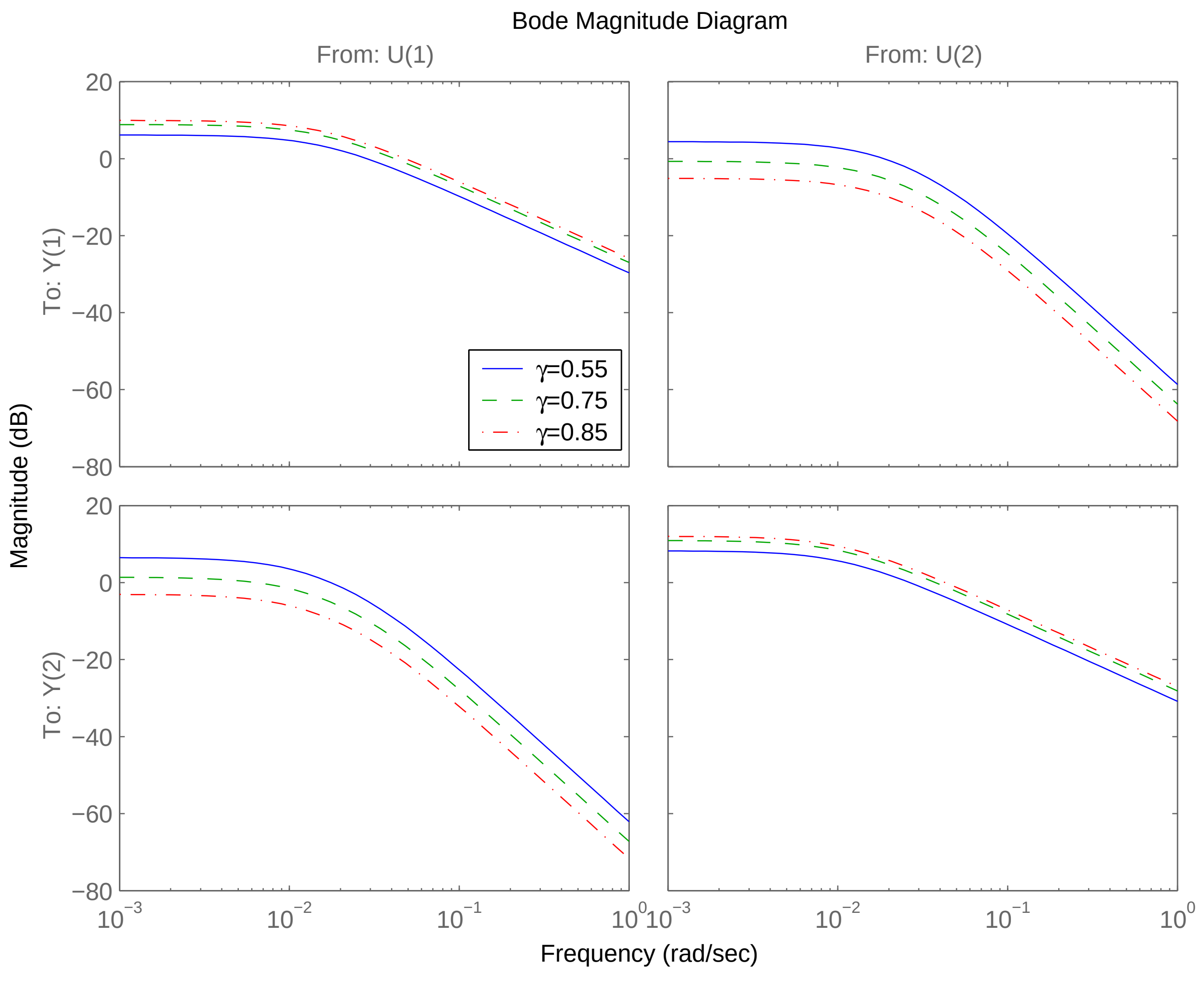

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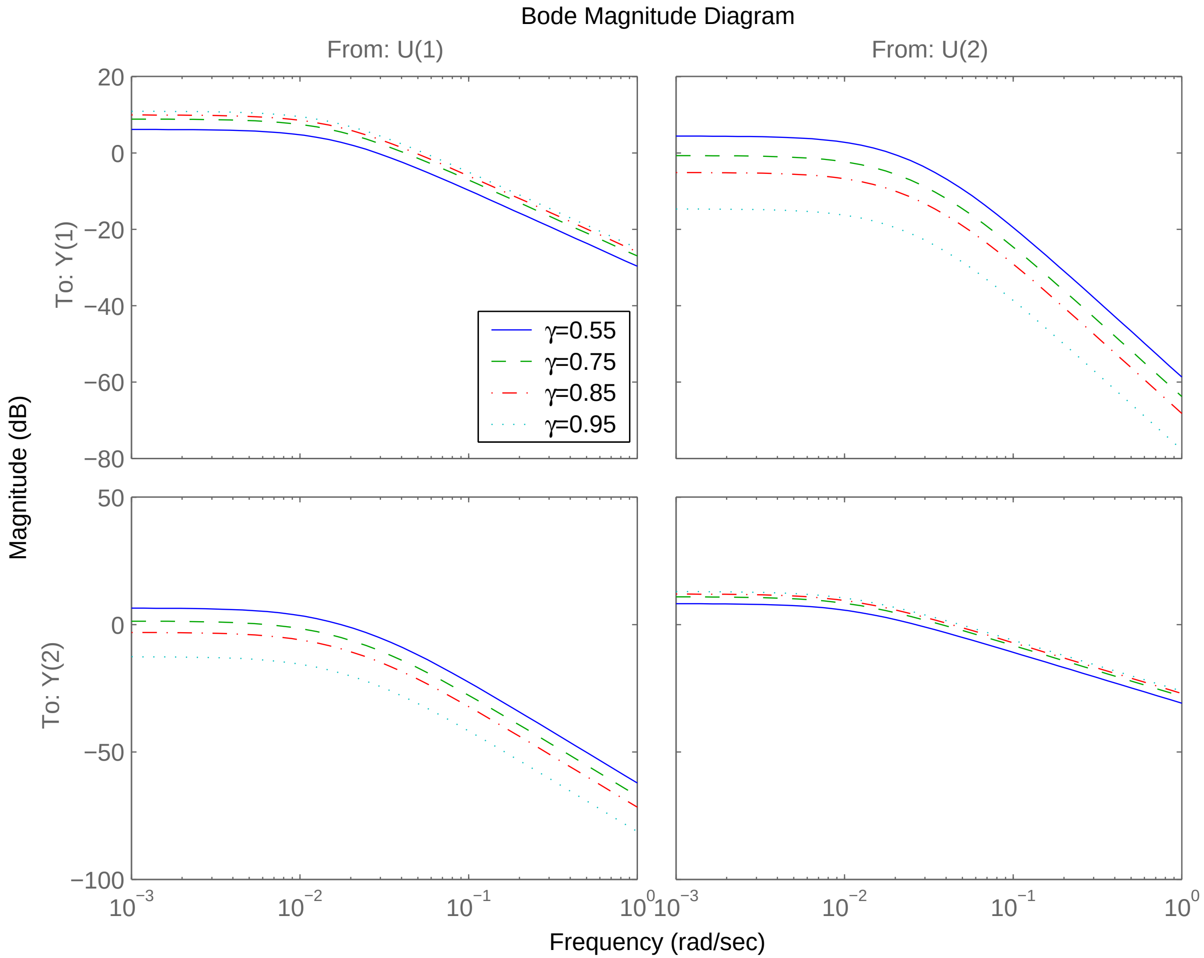
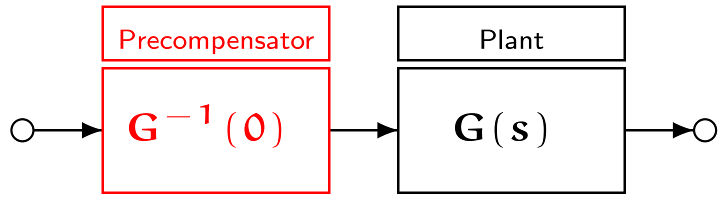


Diagonal Dominance

A plant can sometimes be **made** to be diagonally dominant, at least at some critical frequencies.

- One possibility is to achieve decoupling at DC by making $\mathbf{G}(0)$ diagonal using a static pre-compensator at the input of the plant $\mathbf{P} = \mathbf{G}^{-1}(0)$, so that $\mathbf{PG}(0) = \mathbf{I}$.



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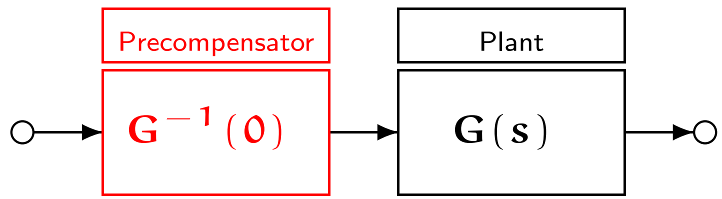


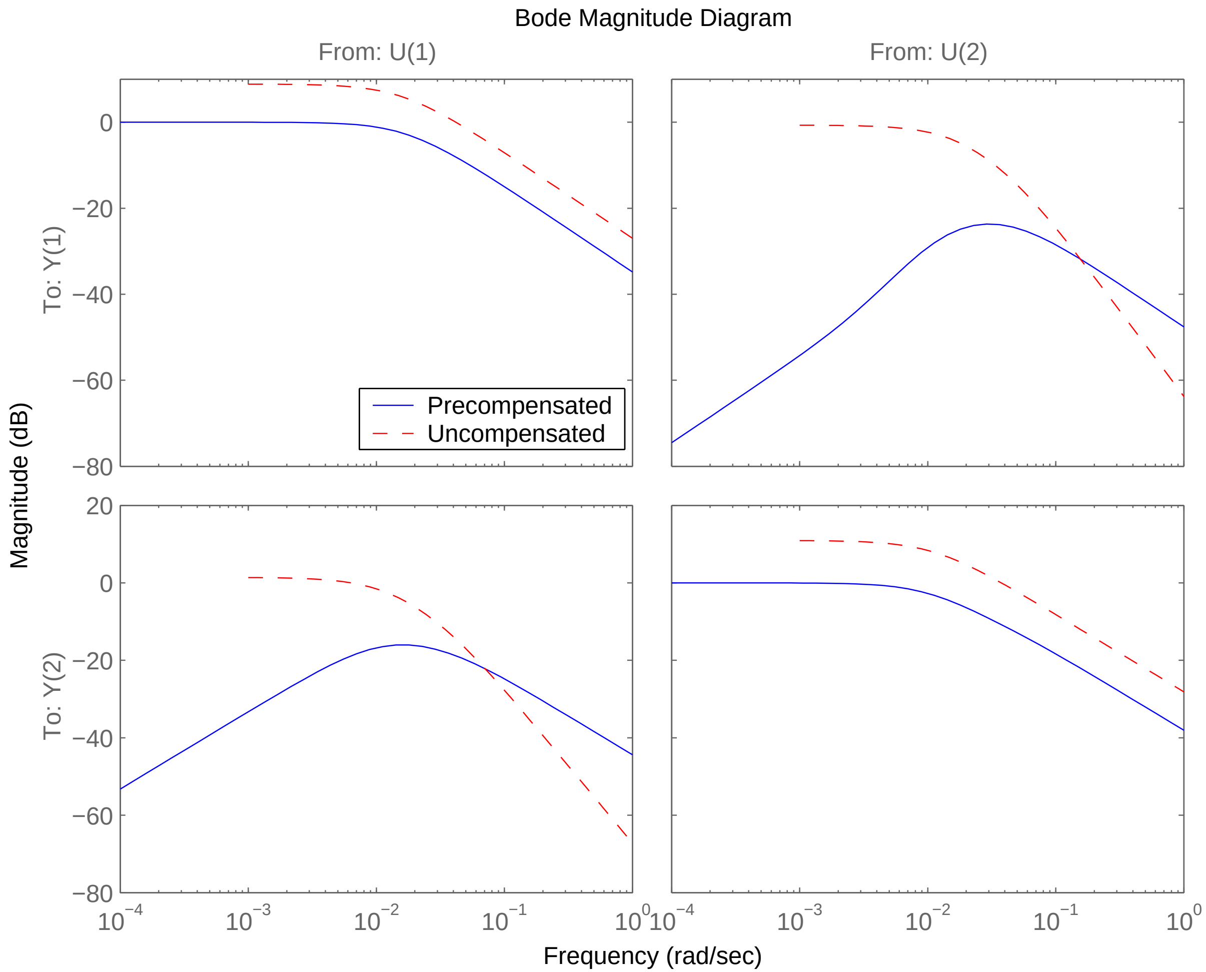
- For the four tank example, the pre-compensator is

$$\mathbf{P} = \begin{bmatrix} 3.7\gamma & 3.7(1-\gamma) \\ 4.7(1-\gamma) & 4.7\gamma \end{bmatrix}^{-1}$$

$$= \frac{1}{3.7 \times 4.7(2\gamma - 1)} \begin{bmatrix} 4.7\gamma & 3.7(\gamma - 1) \\ 4.7(\gamma - 1) & 3.7\gamma \end{bmatrix}$$

Diagonal Dominance

The figure below shows the Bode magnitudes of the four tank apparatus, with and without precompensation for DC decoupling, for $\gamma = 0.75$.



Decentralised Control and the RGA

If a plant transfer matrix is diagonally dominant, it may be possible to design a good controller by considering each input-output pair as a separate loop. This approach is sometimes called **decentralised control**.

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- One way of choosing the pairing is via the **relative gain array (RGA)** Λ, given by

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$$\boldsymbol{\Lambda} = \mathbf{G}(0)\ .*\ \ \mathbf{G}^{-1}(0)^{\mathrm{T}}$$

  where $.*$ denotes element-wise multiplication.
- Each row and column in the RGA always sums to 1, and the values are independent of units (e.g., A or mA, m or mm).

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so

$$\Lambda = \begin{bmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{bmatrix}, \quad \text{with } \lambda = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-1}$$

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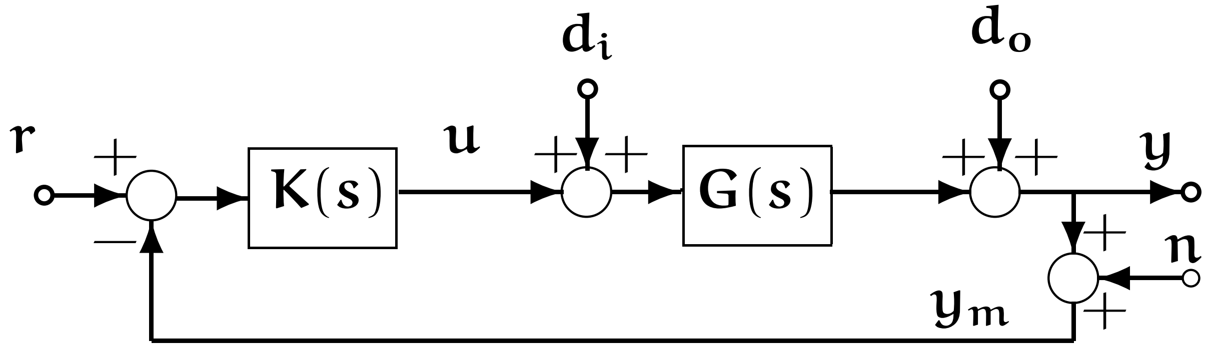
If $\gamma_1 = \gamma_2 = \gamma$, and $\gamma = 1$, the RGA rule suggests the pairing (u_1, y_1) and (u_2, y_2). On the other hand, if $\gamma = 0$, the pairing suggested is the opposite, (u_1, y_2), (u_2, y_1). □

MIMO Sensitivity Functions

- Consider a closed-loop system



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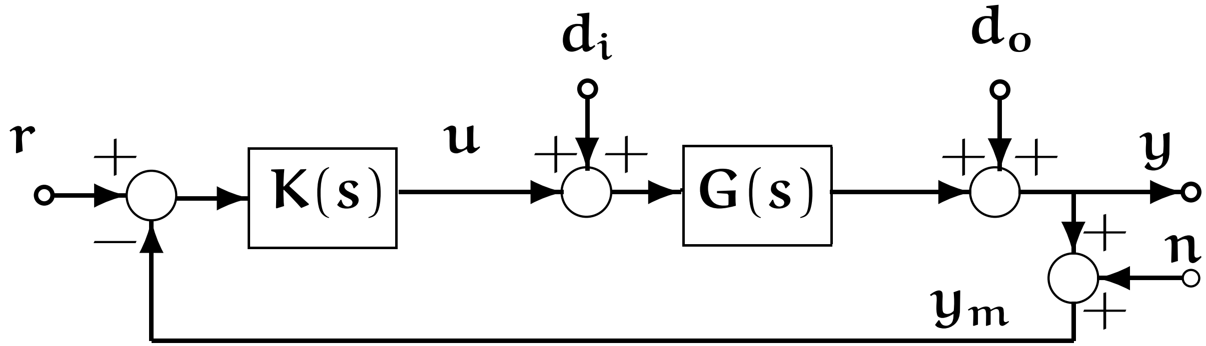

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- As for SISO systems, we define

$$\mathbf{S} = (\mathbf{I} + \mathbf{GK})^{-1} \quad : \text{The (nominal) Sensitivity}$$

$$\mathbf{T} = (\mathbf{I} + \mathbf{GK})^{-1}\mathbf{GK} \quad : \text{The (nominal) Complementary Sensitivity}$$

Sensitivities are very useful to specify and analyse performance and robustness of feedback control systems.

Performance and Robustness

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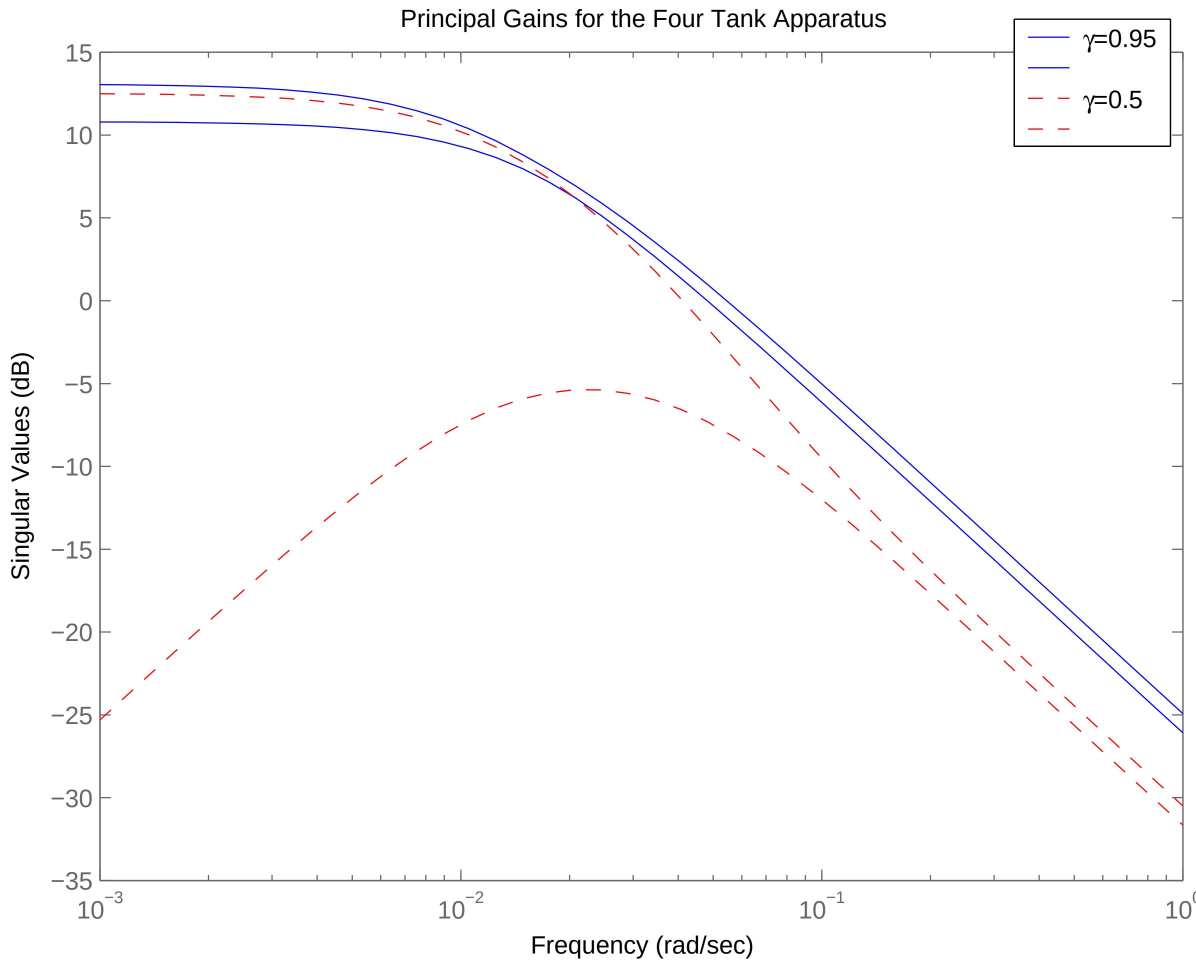
- However, it is certainly difficult to specify desired performance for $n \times n$ scalar transfer functions, if say we have a n-input n-output plant.
- We often obtain more useful results if we consider the **principal gains** rather than the gains of each element in the transfer function matrix.
- The principal gains are the **singular values** of the complex transfer function matrix.

$$\sigma_i(j\omega) = \sqrt{\textbf{eig}\left[G(j\omega)G^H(j\omega)\right]}$$

where $(.)^H$ denotes the conjugate transpose.

Performance and Robustness

The figure shows the principal gains for the four tank example for $\gamma = 0.95$ and $\gamma = 0.5$.



Performance and Robustness

If we have a result for a SISO (single-input single-output) system requiring an upper bound on a certain gain, then it is likely to generalise to a MIMO (multivariable) system as requiring an upper bound on the corresponding **maximum principal gain**.

$$\sigma_{max}(j\omega) = \max_i \sigma_i(j\omega)$$

For the SISO case good tracking in some bandwidth $0 \leq \omega \leq B$ requires $T \approx 1$. This in turn requires $S \approx 0$ in the corresponding bandwidth.

For the MIMO case this becomes the requirement $\sigma_{max}(S) \approx 0$ within that range of frequencies. This is equivalent to the requirement that

$$T \approx I, \quad \text{for } \omega \in [0, B]$$

In MATLAB the principal gains can be computed with the function `sigma`.

MIMO Control Design

The generalisation of IMC design methodology for **square** MIMO systems is straightforward. The parameterisation of all controllers that yield a stable closed-loop system for a stable plant with nominal model $\mathbf{G}_0(s)$ is given by

$$\mathbf{K}(s) = [\mathbf{I} - \mathbf{Q}(s)\mathbf{G}_0(s)]^{-1}\mathbf{Q}(s) = \mathbf{Q}(s)[\mathbf{I} - \mathbf{G}_0(s)\mathbf{Q}(s)]^{-1},$$

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Example. Consider again the 2-input, 2-output plant represented by

$$\mathbf{G}(s) = \begin{bmatrix} \frac{4}{(s+1)(s+2)} & \frac{-1}{(s+1)} \\ \frac{2}{(s+1)} & \frac{-1}{2(s+1)(s+2)} \end{bmatrix}$$

MIMO Control Design

Example (continuation).

- As seen before, this plant is stable and minimum phase, with poles at $s = -2, -2, -1$ and a zero at $s = -3$. Because the only zero is in the left half plane, the plant is relatively easy to control.

  To implement an IMC design, note that

$$\mathbf{G}^{-1}(s) = \frac{(s+2)^2}{2(s+3)} \begin{bmatrix} \frac{-1}{s+2} & 1 \\ -2 & \frac{4}{s+2} \end{bmatrix},$$

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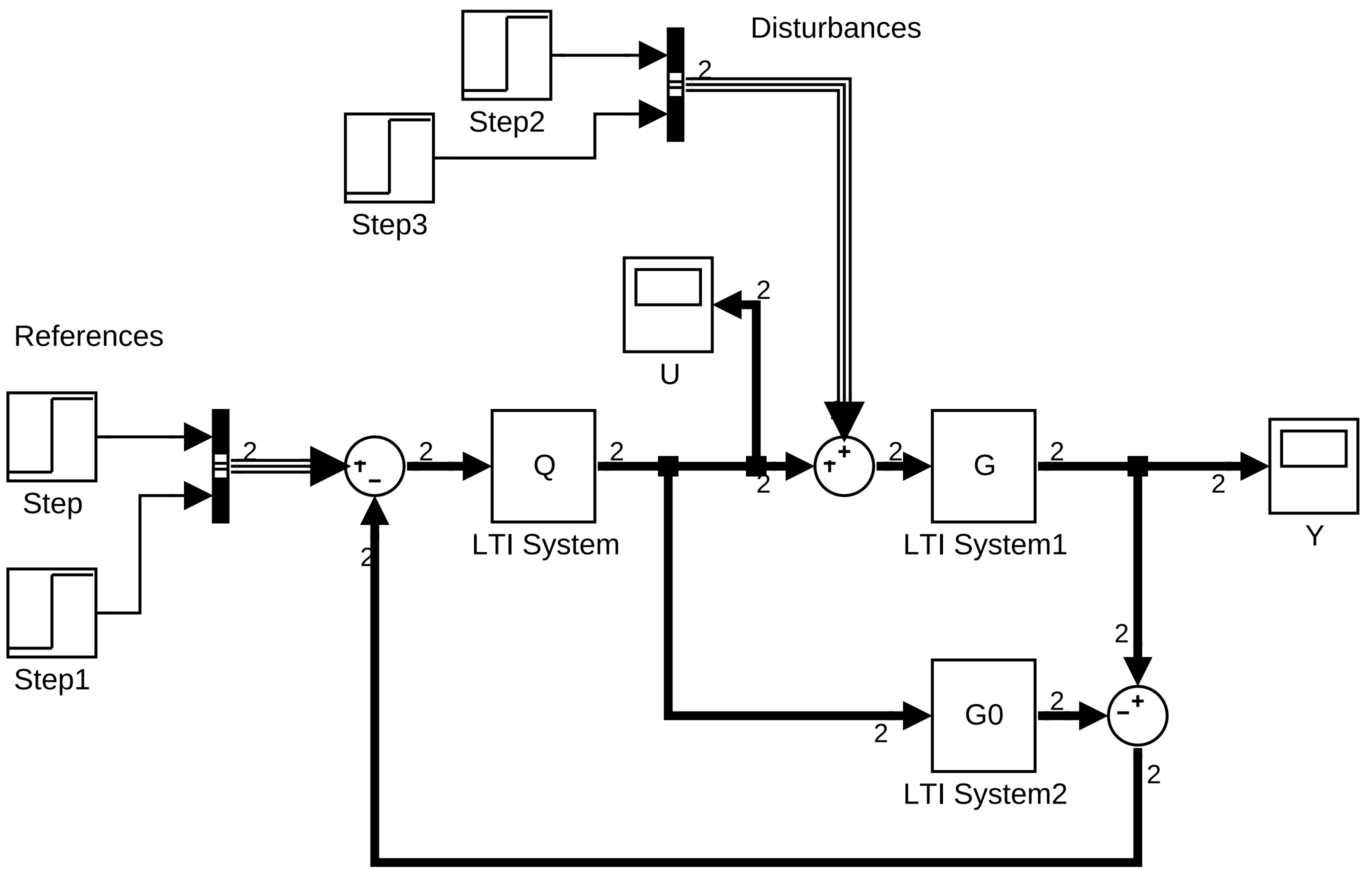
- So a sensible choice for $Q(s)$ might be

$$Q(s) = \frac{(s+2)^2}{2(s+3)(\lambda s+1)} \begin{bmatrix} \frac{-1}{s+2} & 1 \\ -2 & \frac{4}{s+2} \end{bmatrix}, \quad \text{for some } \lambda > 0.$$

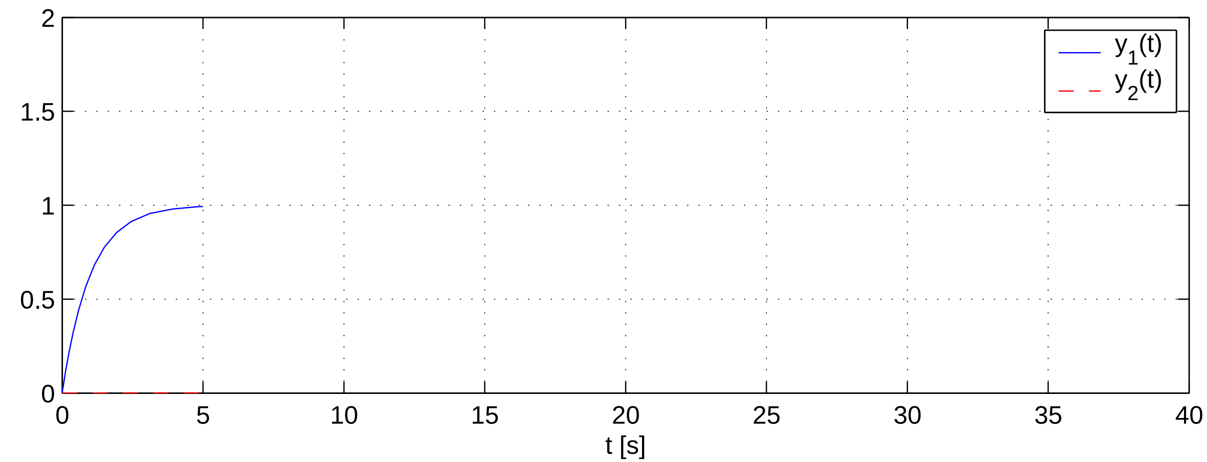
MIMO Control Design

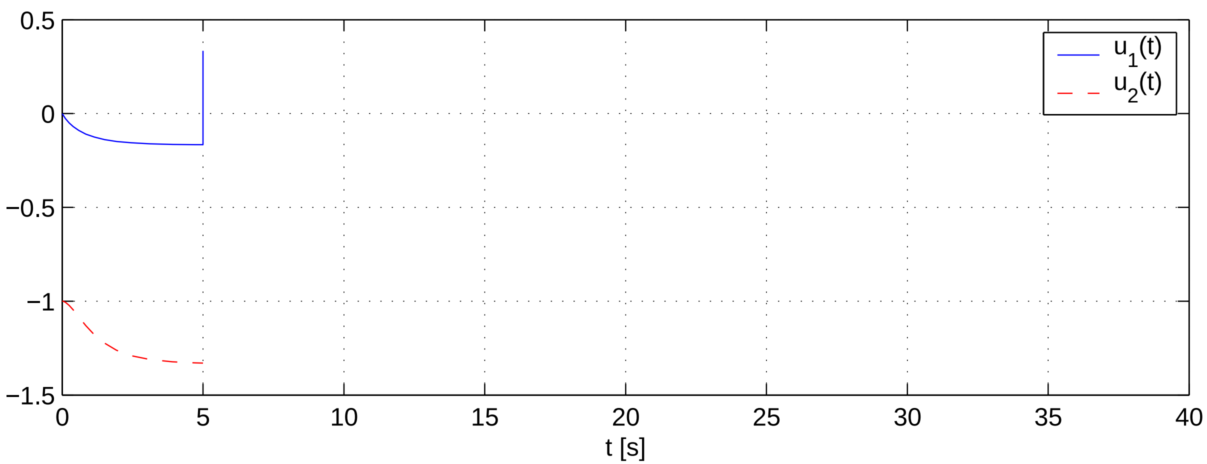
Example (continuation). We implement in Simulink this IMC design for $\lambda = 1$, including step references and input disturbances.

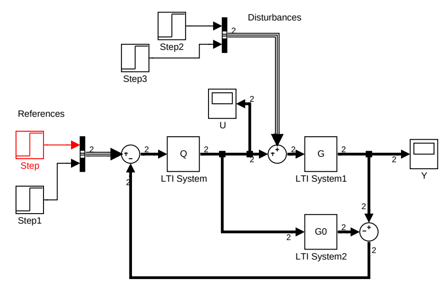


MIMO Control Design

Example (continuation). The plots show the response of the closed-loop system to input step references and disturbances.







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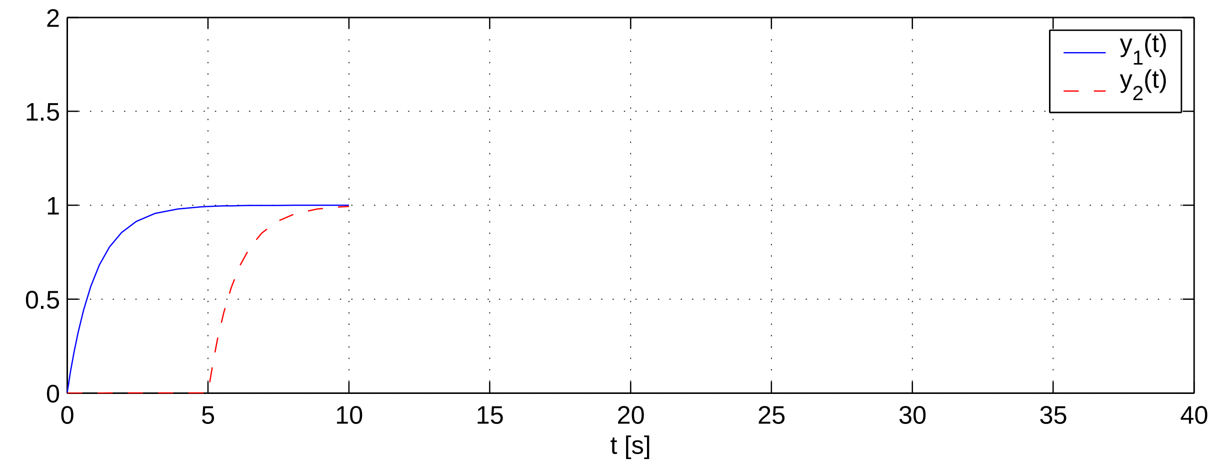

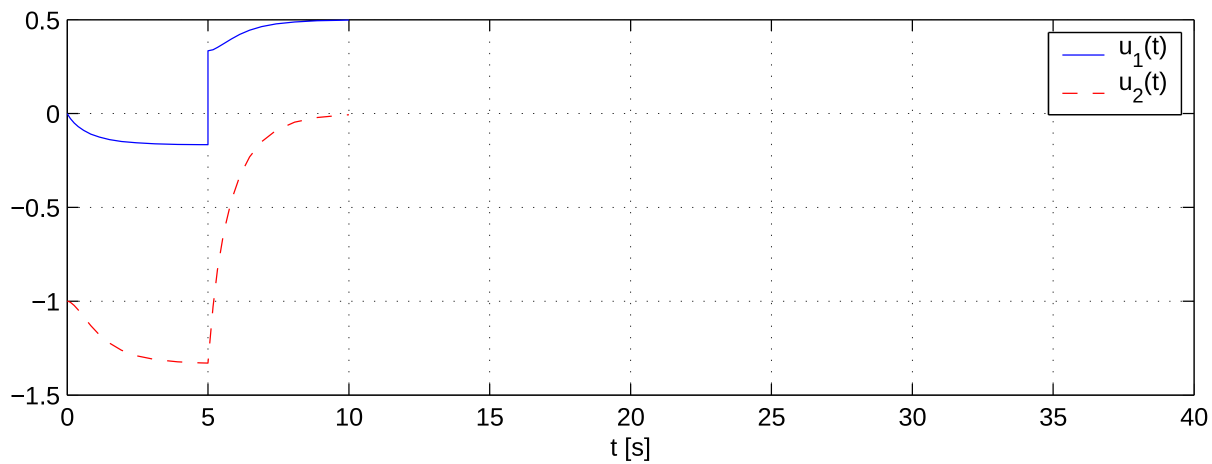

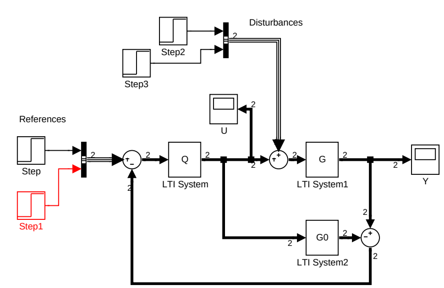

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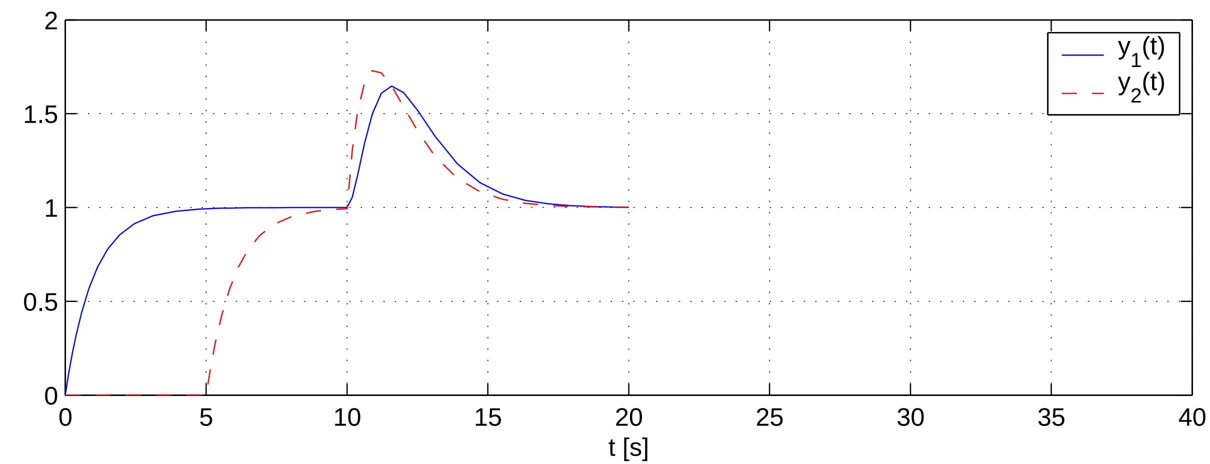

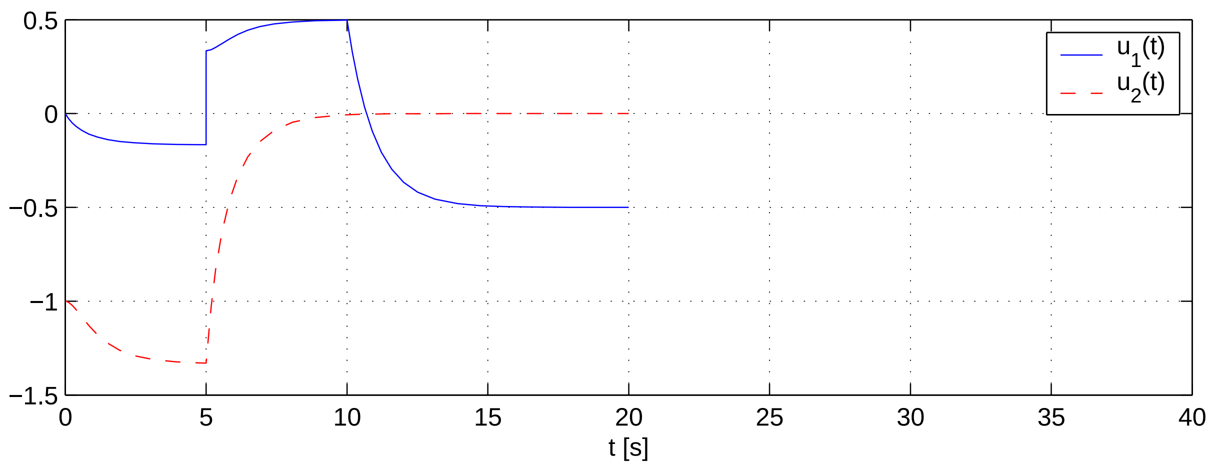

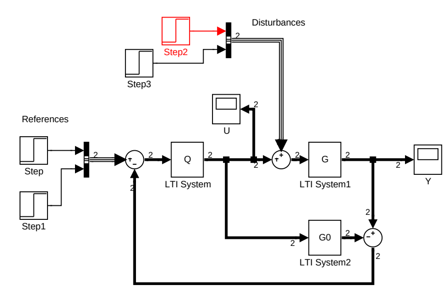

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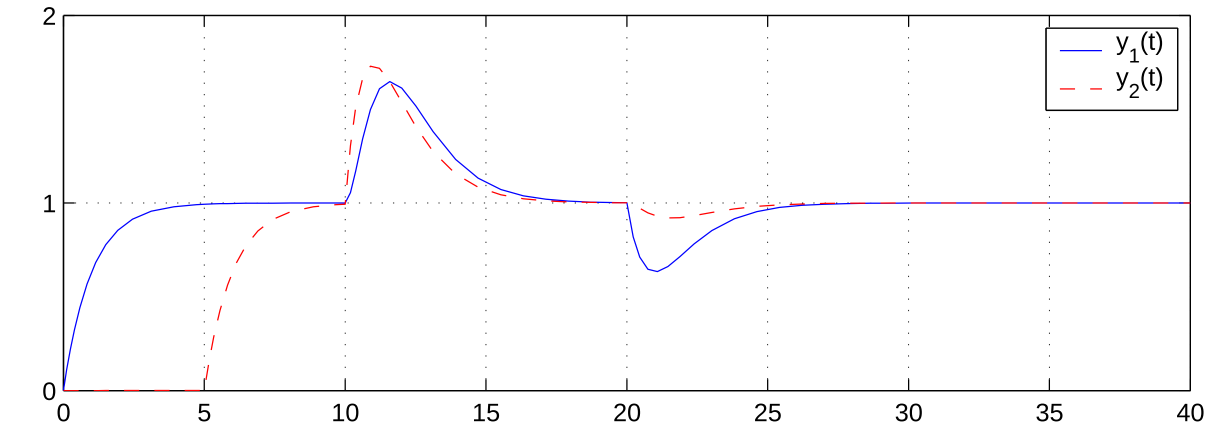

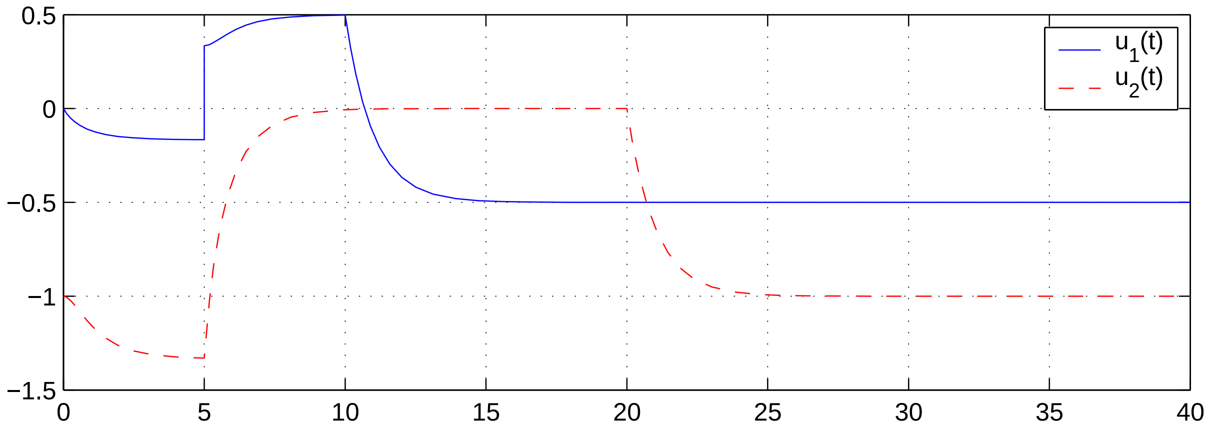

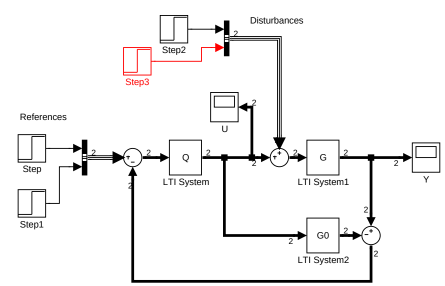

Difficulties of MIMO IMC Design

However, in general the choice of a suitable matrix $\mathbf{Q}(s)$ for a MIMO IMC design becomes much more complicated than in SISO systems.

- In particular, the key attributes in the synthesis of $\mathbf{Q}(s)$
  - relative-degree (i.e., zeros at infinity)
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- **How can we deal with possibly difficult, nonsquare, large scale MIMO systems?**

Use state space control design!

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- Decentralised control might be an option when the plant is diagonally dominant. A practical rule to pair inputs and outputs is based on the **Relative Gain Array**.

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Summary

- The concepts of stability, sensitivity functions, performance and robustness, generalise directly to MIMO systems.
- IMC design for MIMO systems is essentially the same as for SISO systems. Yet, the synthesis process is more subtle, and the required computations may get much more involved.
- We will come back to MIMO systems with **State Space System Theory and Control Design**.