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A Barycentric Coordinate Based Approach to Formation Control of Multi-agent Systems under Directed and Switching Topologies

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Abstract— **This paper studies the formation control problem for a leader-follower network in 3D. The objective is to control the agents to form a globally rigid formation, for which the sensing graph is directed and switching while the communication graph is undirected and switching. Under such a setup, a barycentric coordinate based approach is proposed for the design of formation control laws ensuring global convergence. A necessary and sufficient graphical condition is obtained to guarantee that the followers converge to form a globally rigid formation together with the leaders. By this approach, the formation of the whole group, namely, the orientation, translation and formation scale, can be reconfigured by the leaders.**

I. Introduction

Formation control is one of the most actively studied topics in multi-agent systems, whose objective is to control a group of autonomous robots to achieve prescribed constraints on their states. Based on different types of sensed and controlled variables, [1] categorizes the existing literature on formation control into three kinds:

- Position-based control: Each agent can sense its own position with respect to a global coordinate system, and control its position to achieve the desired formation, which is prescribed by the desired position with respect to the global coordinate system.
- Displacement-based control: Each agent can sense relative positions of its neighboring agents with respect to the global coordinate system, and is controlled to achieve the desired formation, which is specified by the desired displacements with respect to the global coordinate system.
- Distance-based control: Each agent can sense relative positions of its neighboring agents with respect to its own local coordinate system, and is driven to achieve the desired formation, which is specified by the desired inter-agent distances.

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Recently, a new formation control approach based on barycentric coordinates is introduced in [2]–[4]. The barycentric coordinate is a geometric notion characterizing the relative position of a point with respect to other points [5] in absence of the global coordinate system. For barycentric-coordinate-based control, each agent can sense relative positions of its neighboring agents with respect to its own local coordinate system, and is controlled to achieve the desired formation, which is specified by the desired barycentric coordinates of every agent with respect to its neighboring agents. In comparison, barycentric-coordinate-based control requires less advanced sensing capability than position-based control and displacement-based control, and needs less interactions than distance-based control (Fig. 1).

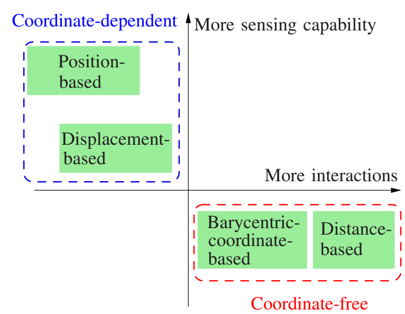

Fig. 1. Sensing capability vs. interactions.

Along the approach based on barycentric coordinates, [2] and [3] solve the formation control problem in 2D with complex barycentric coordinates, while [4] addresses the formation control problem in d-dimensional spaces with real barycentric coordinates. These works all consider fixed topologies. However, the information flow may be influenced by unpredictable changes, so considering switching interaction topologies is more attractive.

We initiate the study for formation control over switching topologies for a leader-follower network in our previous work [6], but an assumption is needed that the followers lie in the convex hull spanned by the leaders in the target configuration. In this paper, we aim to remove this convex assumption by allowing the interaction weights to be negative, which, however, leads to much more challenges in control synthesis and convergence analysis. To overcome the difficulties, a communication graph is introduced and an auxiliary state information is exchanged, with which a fully distributed control law is proposed. A necessary and sufficient graphical condition is then obtained to ensure that a leader-follower network globally converges to form a globally rigid formation.

Notation: $\mathbb{R}$ denotes the set of real numbers. $\mathbf{1}_n$ represents the n-dimensional vector of ones and I_n represents the identity matrix of order n. The symbol $\otimes$ denotes the Kronecker product.

II. Preliminaries and Problem Statement

A. Preliminaries

A *directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a non-empty finite set $\mathcal{V}$ of elements called *nodes* and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ of ordered pairs of nodes called *edges*. For each node $i \in \mathcal{V}$, let $\mathcal{N}_i^+ := \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ denote the set of its *in-neighbors*, and let $\mathcal{N}_i^- := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ denote the set of its *out-neighbors*.

For a directed graph, the Laplacian matrix $L \in \mathbb{R}^{n\times n}$ is defined as follows:

$$L(i,j) = \begin{cases} -w_{ij} & \text{if } i \neq j \text{ and } j \in \mathcal{N}_i^+ \\ 0 & \text{if } i \neq j \text{ and } j \notin \mathcal{N}_i^+ \\ \sum\limits_{k\in\mathcal{N}_i^+} w_{ik} & \text{if } i = j \end{cases}$$

where $w_{ij} \neq 0$ is called the weight on edge (j, i).

For a graph $\mathcal{G}$, a node v is said to be *reachable* from another node u if there exists a path from u to v. Moreover, a node v is said to be *4-reachable* from a set $\mathcal{R}$ if there exists a path from a node in $\mathcal{R}$ to v after removing any three nodes except v.

A *time-varying graph* $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ represents a graph whose edge set changes over time. For a time-varying graph $\mathcal{G}(t)$, a node v is called *uniformly jointly 4-reachable* from $\mathcal{R} \subset \mathcal{V}$ if there exists $T > 0$ such that for all t, v is 4-reachable from $\mathcal{R}$ in the union graph $\mathcal{G}([t, t + T))$, whose edge set is the union of the edge set of $\mathcal{G}(t)$ over the time interval $[t, t + T)$. An example is given in Fig. 2, for which node 5 is uniformly jointly 4-reachable from the set $\{1, 2, 3, 4\}$ since we can take $T = 2$ and for any t the union graph $\mathcal{G}([t, t + T)) = \mathcal{G}_1 \cup \mathcal{G}_2$.

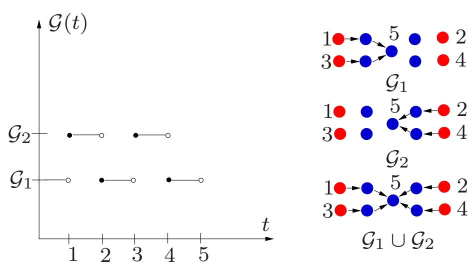

Fig. 2. A time-varying graph $\mathcal{G}(t)$.

A *configuration* in $\mathbb{R}^3$ of a set of n nodes is defined by their coordinates in the Euclidean space $\mathbb{R}^3$, denoted as $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal \in \mathbb{R}^{3n}$, where each $p_i \in \mathbb{R}^3$ for $1 \leq i \leq n$. A *framework* in $\mathbb{R}^3$ is a graph $\mathcal{G}$ equipped with a configuration p, denoted as $\mathcal{F} = (\mathcal{G}, p)$. A framework $(\mathcal{G}, p)$ is said to be *generic* if the coordinates $p_1, \ldots, p_n$ do not satisfy any nontrivial algebraic equation with integer coefficients [7].

For a square matrix $E \in \mathbb{R}^{n\times n}$, the *associated graph* $\mathcal{G}(E)$ consists of n nodes $v_1, \ldots, v_n$ where an edge leads from v_j to v_i $(i \neq j)$ if and only if the (i, j)-th entry of E is nonzero.

B. Problem statement

We consider a leader-follower network consisting of $N = m + n$ agents in 3D with m leaders labeled $1, \ldots, m$ and n followers labeled $m+1, \ldots, N$. Define a target configuration $p = [p_a^\intercal, p_b^\intercal]^\intercal$ where $p_a = [p_1^\intercal, \ldots, p_m^\intercal]^\intercal \in \mathbb{R}^{3m}$ for the leaders and $p_b = [p_{m+1}^\intercal, \ldots, p_N^\intercal]^\intercal \in \mathbb{R}^{3n}$ for the followers. Let $z_i \in \mathbb{R}^3$ be the position vector of agent i. The motion of each agent is governed by the following dynamics

$$\dot{z}_i(t) = u_i(t), \;\; i = 1, \ldots, N, \tag{1}$$

where $u_i(t) \in \mathbb{R}^3$ is the control input.

We assume that the leaders are already in a globally rigid formation and we only focus on controlling the followers. The leaders are governed by

$$\dot{z}_i(t) = v_r(t), \;\; i = 1, \ldots, m, \tag{2}$$

where $v_r(t)$ is the synchronized velocity known to all the followers. We say that the leaders are already in a globally rigid formation if the positions of the leaders satisfy

$$z_i(t) - \int_{t_0}^{t} v_r(\tau)d\tau = A(t_0)p_i + c(t_0), \text{ for } i = 1, \ldots, m,$$

where $A(t_0) \in \mathbb{R}^{3\times3}$ is a unitary matrix representing a rotation, and $c(t_0) \in \mathbb{R}^3$ is a constant vector representing a translation, which means that the formation shape of the leaders is obtained from p_a via a rigid-body transformation.

Remark 2.1: If $v_r(t)$ is only known to a subset of the followers, the ideas developed in [6] can be adopted and extended. For simplicity, we consider in this paper that $v_r(t)$ is known to all followers.

Next we give the precise definition for formation control.

Definition 2.1: A globally rigid formation $p = [p_a^\intercal, p_b^\intercal]^\intercal$ is said to be uniformly asymptotically reached if for any $\delta > 0$ and for any $\varepsilon > 0$ there exists $T > 0$ such that for any t_0 and for any $z_i(t_0)$ satisfying $\|z_i(t_0) - Ap_i - c\| \leq \delta$, $i = 1, \ldots, N$,

$$(\forall t \geq t_0 + T)(\forall i) \left\| z_i(t) - Ap_i - c - \int_{t_0}^{t} v_r(\tau)d\tau \right\| \leq \varepsilon, \tag{3}$$

where A and c are determined by the leaders.

Suppose that every agent can measure relative positions of its neighbors. We use a time-varying graph $\bar{\mathcal{G}}(t) = (\mathcal{V}, \bar{\mathcal{E}}(t))$ to represent the *sensing graph*, where $\mathcal{V} = \mathcal{V}_a \cup \mathcal{V}_b$ with $\mathcal{V}_a = \{1, \ldots, m\}$ and $\mathcal{V}_b = \{m+1, \ldots, N\}$. In other words, $(j, i) \in \bar{\mathcal{E}}(t)$ if agent i can measure the relative position of agent j at time t. Let $\bar{\mathcal{N}}_i^+(t)$ be the set of in-neighbors of agent i in $\bar{\mathcal{G}}(t)$ and $\bar{\mathcal{N}}_i^-(t)$ be the set of its out-neighbors.

Moreover, each agent is assumed to communicate with its communication neighbors that are not necessarily its sensing neighbors. We use another time-varying graph $\mathcal{H}(t)$ to represent the *communication graph*, for which an edge (j, i) indicates that agent j can communicate to agent i. We make the following assumption for the communication graph.

Assumption 2.1: The communication graph $\mathcal{H}(t)$ is bidirectional. Moreover, the communication graph $\mathcal{H}(t)$ contains the sensing graph $\bar{\mathcal{G}}(t)$ as a subgraph at any time t.

To make the problem solvable, we also assume the following.

Assumption 2.2: The target configuration $p = [p_a^\intercal, p_b^\intercal]^\intercal$ is generic.

To avoid infinite switching within a finite time interval, we assume the following.

Assumption 2.3: The interval between any two switching instants satisfies a dwell time condition. That is to say, there exists $\tau_D > 0$ such that

$$t_{i+1} - t_i \geq \tau_D \text{ for all } i = 0, 1, \dots$$

if $\bar{\mathcal{G}}(t)$ switches at $t_0, t_1, t_2, \dots$.

Thus, the formation control problem is summarized as follows.

Suppose that the leaders are already in a globally rigid formation p_a. Given Assumptions 2.1-2.3 and relative position measurements $z_j - z_i$ for $j \in \bar{\mathcal{N}}_i^+$, design a fully distributed control law u_i for each follower i and find the corresponding necessary and sufficient graphical condition such that (3) is satisfied.

III. Distributed control law

In this section, we propose our fully distributed control law for the followers.

First we introduce a *neighbor-selecting rule* for the followers. That is to say, the followers may not interact with all the neighbors in the sensing graph $\bar{\mathcal{G}}(t)$. They choose their neighbors to interact with according to certain rules and then form the *interaction graph* $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$. Let $\mathcal{N}_i^+(t)$ be the set of in-neighbors of agent i in $\mathcal{G}(t)$ and $\mathcal{N}_i^-(t)$ be the set of out-neighbors.

Neighbor-selecting rule: If $|\bar{\mathcal{N}}_i^+(t)| < 4$, then $\mathcal{N}_i^+(t) = \emptyset$. Otherwise, $\mathcal{N}_i^+(t) = \bar{\mathcal{N}}_i^+(t)$.

Remark 3.1: If the number of follower i's neighbors is less than 4, then there is no barycentric coordinate representation in the three-dimensional space. This is why we introduce this neighbor-selecting rule.

An example is given in Fig. 3 to demonstrate the neighbor-selecting rule.

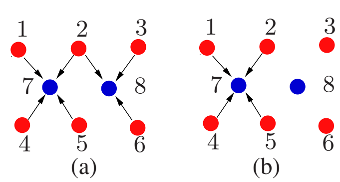

Fig. 3. (a) Sensing graph $\bar{\mathcal{G}}$. (b) Interaction graph $\mathcal{G}$.

We propose the following linear switching control law for each follower $i = m+1, \dots, N$,

$$\begin{cases} \dot{\zeta}_i = -\frac{1}{2}\zeta_i - \sum_{j \in \mathcal{N}_i^+(t)} k_{ij}(t)(z_j - z_i) \\ u_i = v_r(t) - \sum_{j \in \mathcal{N}_i^+(t)} k_{ij}(t)\zeta_i + \sum_{j \in \mathcal{N}_i^-(t)} k_{ji}(t)\zeta_j \end{cases} \tag{4}$$

where ζ_i is an auxiliary state ($\zeta_1 = \cdots = \zeta_m = 0$) and $k_{ij}(t) \in \mathbb{R}/\{0\}$ is a weight associated to edge (j, i) in the interaction graph $\mathcal{G}(t)$, which will be designed later.

In (4), for each follower i, $z_j - z_i$ ($j \in \mathcal{N}_i^+$) is acquired by sensors and $k_{ji}\zeta_j$ ($j \in \mathcal{N}_i^-$) is transmitted through communication. To apply ζ_j properly in the control law, it is required that all the followers share a common sense of orientation in their frames.

According to Assumption 2.3, $\mathcal{G}(t)$ is piecewise constant and thus $k_{ij}(t)$'s are piecewise constant. Now we give the principle for each follower i to design $k_{ij}(t)$ based on the target configuration p. That is, for agent i, $k_{ij}(t)$'s are chosen to be the barycentric coordinates about its neighbors in the target configuration, which means that each follower i selects $k_{ij}(t)$ satisfying

$$\sum_{j \in \mathcal{N}_i^+(t)} k_{ij}(t)(p_j - p_i) = 0. \tag{5}$$

Note that computing $k_{ij}(t)$'s from (5) is distributed for each agent i with p_i and $p_j, j \in \mathcal{N}_i^+(t)$. We denote by $L(t)$ the Laplacian of $\mathcal{G}(t)$ with these weights $k_{ij}(t)$'s.

IV. Stability analysis

Define z and ζ the aggregated vectors of all z_i's and ζ_i's, respectively. By (2) and (4), the overall closed-loop system can be described as

$$\begin{bmatrix} \dot{z} \\ \dot{\zeta} \end{bmatrix} = \left(\begin{bmatrix} 0 & -H(t) \\ L(t) & -U \end{bmatrix} \otimes I_3 \right) \begin{bmatrix} z \\ \zeta \end{bmatrix} + \begin{bmatrix} \mathbf{1}_N \\ 0 \end{bmatrix} \otimes v_r, \tag{6}$$

where

$$H(t) = \begin{bmatrix} 0 & 0 \\ 0 & L_{ff}^\intercal(t) \end{bmatrix}, \; U = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2}I_n \end{bmatrix}.$$

$L_{ff}(t)$ is the sub-matrix of $L(t)$ with the following form

$$L(t) = \left[\begin{array}{c|c} 0_{m\times m} & 0_{m\times n} \\ \hline L_{lf}(t) & L_{ff}(t) \end{array} \right]. \tag{7}$$

According to the design of $k_{ij}(t)$'s, $L(t)$ satisfies

$$(L(t) \otimes I_3)p = 0 \text{ and } L(t)\mathbf{1}_N = 0. \tag{8}$$

Let $\mathcal{R}$ be a subset of $\mathcal{V}$ and let $L_{\mathcal{R}}$ be the sub-matrix of L with the rows and columns corresponding to nodes in $\mathcal{V} - \mathcal{R}$ crossed out. The following lemma provides the relationship between the determinant of $L_{\mathcal{R}}$ and the connectivity of $\mathcal{G}$.

Lemma 4.1 ([4]): Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \mathcal{V}_a \cup \mathcal{V}_b$ and a generic configuration $p = [p_a^\intercal, p_b^\intercal]^\intercal$. For almost all[1] L satisfying $(L \otimes I_3)p = 0$, $\det(L_{\mathcal{V}_b}) \neq 0$ if and only if every node in $\mathcal{V}_b$ is 4-reachable from $\mathcal{V}_a$.

In what follows, we present our main theorem.

Theorem 4.1: Suppose the leaders are in a globally rigid formation p_a. Then under control law (4), a globally rigid formation $[p_a^\intercal, p_b^\intercal]^\intercal$ can be uniformly asymptotically reached

[1]Here "for almost all L" means "for almost all weights used to construct L". And "for almost all" parameter values is to be understood as "for all parameter values except for those in some proper algebraic variety in the parameter space". The proper algebraic variety for which a property is not true is the zero set of some nontrivial polynomial with real coefficients in the parameters. A proper algebraic variety has Lebesgue measure zero [8].

if and only if every follower is uniformly jointly 4-reachable from $\mathcal{V}_a$ in the interaction graph $\mathcal{G}(t)$.

The proof of Theorem 4.1 needs two lemmas. Before introducing the lemmas, we let C_Δ be a set $\{t_i\}$ of points in $[0,\infty)$ for which there exists a Δ such that for any $t_i, t_j \in C_\Delta$ with $t_i \neq t_j$, one has $|t_i - t_j| \geq \Delta$. Thus, C_Δ comprises points spaced at least Δ apart.

Denote by Γ the set of real functions $v(\cdot)$ on $[0,\infty)$ such that for each $v(\cdot) \in \Gamma$ there corresponds some Δ and some C_Δ such that

(1) $v(t)$ and $\dot{v}(t)$ are continuous and bounded on $[0,\infty)/C_\Delta$.

(2) $v(t)$ and $\dot{v}(t)$ have finite limits as $t \downarrow t_i$ and $t \uparrow t_i, t_i \in C_\Delta$.

The following lemma is taken from [9].

Lemma 4.2 ([9]): Let $V(\cdot): \mathbb{R}_+ \to \mathbb{R}^{n\times r}$ be a regulated matrix function (i.e., one-sided limits exist for all $t \in \mathbb{R}_+$), and satisfy for some positive δ and α_1, and all $t \in \mathbb{R}_+$,

$$\int_t^{t+\delta} \|V(\tau)\|^2 d\tau < \alpha_1. \tag{9}$$

Suppose also that the entries of $V(\cdot)$ lie in Γ. Let M be a real constant $n \times n$ matrix with $M + M^\intercal = -I_n$ and B a real constant $n \times r$ matrix with rank r. Then

$$\dot{x} = \begin{bmatrix} 0 & -VB^\intercal \\ BV^\intercal & M \end{bmatrix} x \tag{10}$$

is exponentially stable if and only if for some positive δ and α_3, and all $t \in \mathbb{R}_+$,

$$\int_t^{t+\delta} V(\tau)V(\tau)^\intercal d\tau \geq \alpha_3 I. \tag{11}$$

The next result shows the relationship between $\mathcal{G}(L^\intercal(t)L(t))$ and $\mathcal{G}(L(t))$.

Lemma 4.3: For almost all $L(t)$ satisfying $(L(t) \otimes I_3)p = 0$, the associated graph $\mathcal{G}(L(t))$ is a subgraph of $\mathcal{G}(L^\intercal(t)L(t))$ at any time t.

Proof: We omit t for $L(t)$ and $k_{ij}(t)$ for simplicity unless it is necessary.

First, for each i we show that if $|\mathcal{N}_i^+| \neq 0$, then for almost all L satisfying $(L \otimes I_3)p = 0$, it holds that $k_{ii} := -\sum_{j\in\mathcal{N}_i^+} k_{ij} \neq 0$.

Notice that $k_{ii} \neq 0 \Leftrightarrow \sum_{j\in\mathcal{N}_i^+} k_{ij} \neq 0$. Without loss of generality, we say follower i has $|\mathcal{N}_i^+| = l$ neighbors ($l \neq 0$) labelled $i_1, i_2, \ldots, i_l$. Then there are two cases: namely, $|\mathcal{N}_i^+| = 4$ and $|\mathcal{N}_i^+| > 4$.

For the case that $|\mathcal{N}_i^+| = 4$, suppose on the contrary that $k_{ii} = 0$. From $\sum_{j\in\mathcal{N}_i^+} k_{ij}(p_j - p_i) = 0$ and $\sum_{j\in\mathcal{N}_i^+} k_{ij} = 0$ we obtain that $k_{ii_1}(p_{i_1} - p_{i_4}) + k_{ii_2}(p_{i_2} - p_{i_4}) + k_{ii_3}(p_{i_3} - p_{i_4}) = 0$, which means $p_{i_1}, p_{i_2}, p_{i_3}$ and p_{i_4} stay on the same plane, a contradiction to generic p. Thus, $k_{ii} \neq 0$.

For the case that $|\mathcal{N}_i^+| > 4$, we consider the following neighbor sets and each of them only contains 4 neighbors of i, i.e., $\{i_a, i_{a+1}, i_{a+2}, i_{a+3}\}$, $a = 1, 2, \ldots, l-3$. For each a, we choose $k_{ii_j}^a$ to satisfy $\sum_{j=a}^{a+3} k_{ii_j}^a (p_{i_j} - p_i) = 0$ with $k_{ii}^a := -\sum_{j=a}^{a+3} k_{ii_j}^a$. From previous analysis it is obtained that $k_{ii}^a \neq 0$. Then we consider a liner combination for all a, i.e., $k_{ii_j} = \alpha_1 k_{ii_j}^1 + \cdots + \alpha_{l-3} k_{ii_j}^{l-3}$, $j = 1, \ldots, l$, which leads to $k_{ii} = \alpha_1 k_{ii}^1 + \cdots + \alpha_{l-3} k_{ii}^{l-3}$. So we can find $\alpha_1, \ldots, \alpha_{l-3}$ such that $k_{ii} \neq 0$ and $k_{ii_j} \neq 0$, $j = 1, \ldots, l$. Thus, we find a L satisfying $(L \otimes I_3)p = 0$ and $k_{ii} \neq 0$ if $|\mathcal{N}_i^+| \neq 0$. Applying the fact that either a polynomial is zero or it is not zero almost everywhere, it follows that for almost all L satisfying $(L \otimes I_3)p = 0$, $k_{ii} \neq 0$ if $|\mathcal{N}_i^+| \neq 0$.

Second, we prove that if $(j,i) \in \mathcal{E}$, then for almost all L satisfying $(L \otimes I_3)p = 0$, $L^\intercal L(i,j) \neq 0$.

If $(j,i) \in \mathcal{E}$, we know $k_{ii} \neq 0$ and $k_{ij} \neq 0$. Notice that

$$L^\intercal L(i,j) = L(:,i)^\intercal L(:,j) = k_{1i}k_{1j} + \cdots + k_{ii}k_{ij} + \cdots + k_{Ni}k_{Nj}.$$

It is known that $k_{ii}k_{ij} \neq 0$. Note that $k_{ii}k_{ij}$ is picked by agent i and thus we can infer that if $(j,i) \in \mathcal{E}$, for almost all L satisfying $(L \otimes I_3)p = 0$, $L^\intercal L(i,j) \neq 0$.

Third, we have shown that for almost all L satisfying $(L \otimes I_3)p = 0$, $(j,i) \in \mathcal{E} \Rightarrow L^\intercal L(i,j) \neq 0$. With the fact that $(j,i) \in \mathcal{E} \Leftrightarrow L(i,j) \neq 0$, it can be directly obtained that $L(i,j) \neq 0 \Rightarrow L^\intercal L(i,j) \neq 0$, which implies that for almost all L satisfying $(L \otimes I_3)p = 0$, the associated graph $\mathcal{G}(L)$ is a subgraph of $\mathcal{G}(L^\intercal L)$. ■

Proof of Theorem 4.1: ($\Leftarrow$) Let $y = z - \mathbf{1}_N \otimes \int_{t_0}^t v_r(\tau)d\tau$ and (6) changes to

$$\begin{bmatrix} \dot{y} \\ \dot{\zeta} \end{bmatrix} = \left(\begin{bmatrix} 0 & -H(t) \\ L(t) & -U \end{bmatrix} \otimes I_3 \right) \begin{bmatrix} y \\ \zeta \end{bmatrix}, \tag{12}$$

or equivalently

$$\begin{bmatrix} \dot{y}_a \\ \dot{y}_b \\ \dot{\zeta}_a \\ \dot{\zeta}_b \end{bmatrix} = \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -L_{ff}^\intercal(t) \\ 0 & 0 & 0 & 0 \\ L_{lf}(t) & L_{ff}(t) & 0 & -\frac{1}{2}I_n \end{bmatrix} \otimes I_3 \right) \begin{bmatrix} y_a \\ y_b \\ \zeta_a \\ \zeta_b \end{bmatrix}$$

where the subscript a represents the states for the leaders and b for the followers, which are used throughout the paper.

Next we show that

$$\begin{cases} z^*(t) = (I_N \otimes A)p + \mathbf{1}_N \otimes (c + \int_{t_0}^t v_r(\tau)d\tau) \\ \zeta^* = 0 \end{cases}$$

is an equilibrium solution of system (6), which is equal to show that

$$\begin{cases} y^* = (I_N \otimes A)p + \mathbf{1}_N \otimes c \\ \zeta^* = 0 \end{cases} \tag{13}$$

is an equilibrium point of system (12), and this can be seen from $(L(t) \otimes I_3)[(I_N \otimes A)p + \mathbf{1}_N \otimes c] = (L(t) \otimes A)p = (I_N \otimes A)(L(t) \otimes I_3)p = 0$.

Notice that the leaders already move in a globally rigid formation. Considering the followers and applying the coordinate transformation $e_b = y_b - y_b^*$, we obtain

$$\begin{bmatrix} \dot{e}_b \\ \dot{\zeta}_b \end{bmatrix} = \left(\begin{bmatrix} 0 & -L_{ff}^\intercal(t) \\ L_{ff}(t) & -\frac{1}{2}I_n \end{bmatrix} \otimes I_3 \right) \begin{bmatrix} e_b \\ \zeta_b \end{bmatrix}. \tag{14}$$

In addition, we see that (14) is derived by substituting

$$x = \begin{bmatrix} e_b \\ \zeta_b \end{bmatrix}, M = -\frac{1}{2}I_n \otimes I_3, B = I_n \otimes I_3 \text{ and } V = L_{ff}^\intercal \otimes I_3$$

into (10) in Lemma 4.2.

Suppose $\mathcal{G}(t)$ switches at $t = t_0, t_1, t_2, \ldots$. Recall that by our dwell time assumption, $t_{i+1} - t_i \geq \tau_D$ for all $i = 0, 1, 2 \ldots$. Moreover, we are always able to find a $\tau_m > \tau_D$ large enough such that $t_{i+1} - t_i \leq \tau_m$ for all $i = 0, 1, 2, \ldots$. If for some interval $[t_i, t_{i+1})$ there is no such a τ_m, we can partition $[t_i, t_{i+1})$ artificially.

Suppose now every follower is uniformly jointly 4-reachable from $\mathcal{V}_a$. Then by definition there exits $T > 0$ such that for all t in the union graph $\mathcal{G}([t, t+T))$ every follower is 4-reachable from $\mathcal{V}_a$. It is known that $L_{ff}(t)$ is piecewise constant, so $L_{ff}(t)$ is regulated. Let

$$\delta = T + 2\tau_m.$$

Notice that $k_{ij}(t)$'s are finite due to finite nodes and edges, then we derive $\|L_{ff}(t)\|^2$ is uniform upper-bounded, which implies that there must exist some positive α_1 and for all t, $\int_t^{t+\delta} \|L_{ff}^\intercal(\tau)\|^2 d\tau < \alpha_1$. Hence, according to Lemma 4.2, to prove the sufficiency of Theorem 4.1, it remains to prove that for some positive α_3, and for all t,

$$E := \int_t^{t+\delta} L_{ff}(\tau)^\intercal L_{ff}(\tau) d\tau \geq \alpha_3 I. \tag{15}$$

For any t, without loss of generality, let $t \in (t_m, t_{m+1}]$, and $t + \delta \in [t_n, t_{n+1})$, and define

$$D := L^\intercal(t_{m+1})L(t_{m+1}) + \cdots + L^\intercal(t_{n-1})L(t_{n-1}).$$

Recall the fact that either a polynomial is zero or it is not zero almost everywhere. Then we can infer that for almost all $L(t)$,

$$L^\intercal(t_k)L(t_k)(i,j) \neq 0,\ k = m+1, \ldots, n-1 \Rightarrow D(i,j) \neq 0,$$

which means that the associated graph

$$\mathcal{G}(D) = \mathcal{G}(L^\intercal(t_{m+1})L(t_{m+1})) \cup \cdots \cup \mathcal{G}(L^\intercal(t_{n-1})L(t_{n-1})).$$

Since $t_{m+1} - t \leq \tau_m$ and $t + \delta - t_n \leq \tau_m$, we know $t_n - t_{m+1} \geq T$. That is to say, in $\mathcal{G}([t_{m+1}, t_n))$ every follower is 4-reachable from $\mathcal{V}_a$. Notice that

$$\mathcal{G}([t_{m+1}, t_n)) = \mathcal{G}(L(t_{m+1})) \cup \cdots \cup \mathcal{G}(L(t_{n-1})).$$

By Lemma 4.3, the associated graph $\mathcal{G}(L(t))$ is a subgraph of $\mathcal{G}(L^\intercal(t)L(t))$. This indicates that in $\mathcal{G}(D)$ every follower is 4-reachable from $\mathcal{V}_a$. Moreover, note that $(D \otimes I_3)p = 0$ and $D\mathbf{1}_N = 0$. So D can be regarded as a Laplacian matrix which has the following structure

$$\left[\begin{array}{c|c} * & * \\ \hline * & D_{ff} \end{array}\right]$$

where

$$D_{ff} = L_{ff}^\intercal(t_{m+1})L_{ff}(t_{m+1}) + \cdots + L_{ff}^\intercal(t_{n-1})L_{ff}(t_{n-1})$$

due to the special structure of $L(t)$. Then it follows from Lemma 4.1 that $\det(D_{ff}) \neq 0$. Also with the fact that D_{ff} is positive semi-definite, we know that D_{ff} is positive definite.

Next step we consider

$$\begin{aligned} E = & L_{ff}^\intercal(t_m)L_{ff}(t_m)(t_{m+1} - t) \\ & + L_{ff}^\intercal(t_{m+1})L_{ff}(t_{m+1})(t_{m+2} - t_{m+1}) + \cdots \\ & + L_{ff}^\intercal(t_{n-1})L_{ff}(t_{n-1})(t_n - t_{n-1}) \\ & + L_{ff}^\intercal(t_n)L_{ff}(t_n)(t + \delta - t_n). \end{aligned}$$

Since D_{ff} is symmetric and positive definite, for any vector $x \neq 0$ it holds that $x^\intercal D_{ff} x > 0$, from which we can get that for any vector $x \neq 0$, $x^\intercal E x > 0$. Thus, E is also symmetric and positive definite, from which we can infer that the smallest eigenvalue of E is positive for any t. Thus, to prove (15) holds, it is required to show that the smallest eigenvalue of E is uniform lower bounded.

Note that $L_{ff}(t)$ is finite. And the number of switches during $[t, t+\delta)$ is not more than $\lceil \frac{\delta}{\tau_D} \rceil$. This means that D is finite. And notice that for E, $\tau_D \leq t_{i+1} - t_i \leq \tau_m$ for all $i = 0, 1, 2, \ldots$, and $0 \leq t_{m+1} - t \leq \tau_m,\ 0 \leq t + \delta - t_n \leq \tau_m$. Then we know that there exists $\alpha_3 > 0$ such that for all t,

$$\int_t^{t+\delta} L_{ff}^\intercal(\tau)L_{ff}(\tau) d\tau \geq \alpha_3 I.$$

Hence, a globally rigid formation $[p_a^\intercal, p_b^\intercal]^\intercal$ can be uniformly asymptotically reached.

($\Rightarrow$) We prove the necessity in a contrapositive way. Assume that there exists a follower, say i, that is not uniformly jointly 4-reachable from $\mathcal{V}_a$. That is, for any $T > 0$ there exists $t^* \geq 0$ such that in the union graph $\mathcal{G}([t^*, t^* + T))$, i is not 4-reachable from $\mathcal{V}_a$.

Note that $\mathcal{G}(\int_{t^*}^{t^*+T} L(\tau)d\tau)$ is a subgraph of $\mathcal{G}([t^*, t^* + T))$, which means that in $\mathcal{G}(\int_{t^*}^{t^*+T} L(\tau)d\tau)$, i is not 4-reachable from $\mathcal{V}_a$. It is known that $\int_{t^*}^{t^*+T} L(\tau)d\tau$ is a Laplacian matrix corresponding to $\mathcal{G}(\int_{t^*}^{t^*+T} L(\tau)d\tau)$, then by Lemma 4.1 we know $\det\left(\int_{t^*}^{t^*+T} L_{ff}(\tau)d\tau\right) = 0$. Let

$$C(t) = \left(\begin{bmatrix} 0 & -L_{ff}^\intercal(t) \\ L_{ff}(t) & -\frac{1}{2}I_n \end{bmatrix} \otimes I_3\right)$$

and consider the interval $[t^*, t^* + T)$, then we attain from (14) that

$$\begin{bmatrix} e_b(t^* + T) \\ \zeta_b(t^* + T) \end{bmatrix} = \exp\left[\int_{t^*}^{t^*+T} C(\tau)d\tau\right] \begin{bmatrix} e_b(t^*) \\ \zeta_b(t^*) \end{bmatrix}.$$

Since we have shown that $\det\left(\int_{t^*}^{t^*+T} L_{ff}(\tau)d\tau\right) = 0$, there exists a $\tilde{e}_b \neq 0$ satisfying

$$\left(\int_{t^*}^{t^*+T} L_{ff}(\tau)d\tau \otimes I_3\right)\tilde{e}_b = 0$$

such that

$$\begin{bmatrix} e_b(t^*) \\ \zeta_b(t^*) \end{bmatrix} = \begin{bmatrix} \tilde{e}_b \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} e_b(t^* + T) \\ \zeta_b(t^* + T) \end{bmatrix} = \begin{bmatrix} \tilde{e}_b \\ 0 \end{bmatrix}.$$

So we choose δ and ε such that $\varepsilon = \frac{1}{2}\delta$. Then for all $T > 0$, there exists $t_0 = t^*$ and a follower i satisfying $\tilde{e}_i \neq 0$. Choose $z_i(t_0) = Ap_i + c + k\tilde{e}_i$ where $k > 0$ is a scale factor such that $\frac{1}{2}\delta < k\|\tilde{e}_i\| \leq \delta$. Then we have found $z_i(t_0)$ satisfying $\|z_i(t_0) - Ap_i - c\| \leq \delta$,

$$\begin{aligned} & (\exists t = t_0 + T)(\exists i) \left\| z_i(t) - Ap_i - c - \int_{t_0}^t v_r(\tau)d\tau \right\| \\ & > \frac{1}{2}\delta = \varepsilon. \end{aligned}$$

Therefore, the globally rigid formation p cannot be uniformly asymptotically reached. ■

Remark 4.1: The formation of the whole group, namely, the orientation, translation and formation scale, can be reconfigured by the leaders. To see this, denote a target configuration $\bar{p}$ such that for all $i, j \in \mathcal{V}$ and $i \neq j$,

$$\|\bar{p}_i - \bar{p}_j\| = \gamma \|p_i - p_j\|,$$

where γ is a scale factor. The barycentric coordinates remain unchanged, i.e.,

$$\sum_{j \in \mathcal{N}_i^+(t)} k_{ij}(t)(p_j - p_i) = 0 \Rightarrow \sum_{j \in \mathcal{N}_i^+(t)} k_{ij}(t)(\bar{p}_j - \bar{p}_i) = 0.$$

Thus, by the same process of proof we can get that the desired formation is reached.

V. Simulation

In this section, we present some simulations to illustrate our result.

We consider 4 leaders and 6 followers in the simulation. Suppose the target configuration p is given in Fig. 4 with the leader set $\mathcal{V}_a = \{1, 2, 3, 4\}$ and the follower set $\mathcal{V}_b = \{5, \dots, 10\}$.

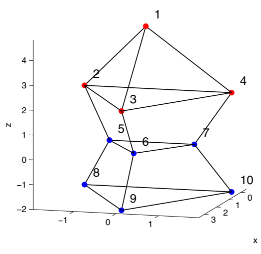

Fig. 4. The target configuration.

Suppose that the interaction graph $\mathcal{G}(t)$ is shown in Fig. 5, and it can be checked that every follower is uniformly jointly 4-reachable from $\mathcal{V}_a$ in $\mathcal{G}(t)$ by taking $T = 3$.

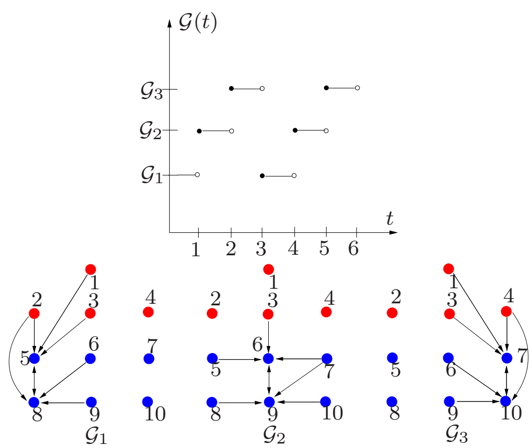

Fig. 5. A periodic switching graph $\mathcal{G}(t)$ that switches among three different topologies.

We carry out simulations using the control law (4) for the followers. A simulation result is shown in Fig. 6, which shows that a globally rigid formation is uniformly asymptotically reached. Another simulation result is presented in Fig. 7, from which we see that orientation, translation and formation scale can be reconfigured by the leaders.

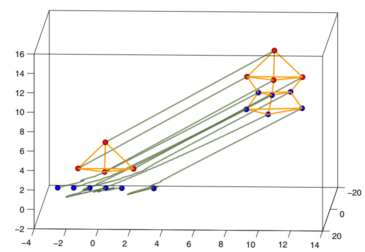

Fig. 6. The followers are uniformly asymptotically merged with the leaders.

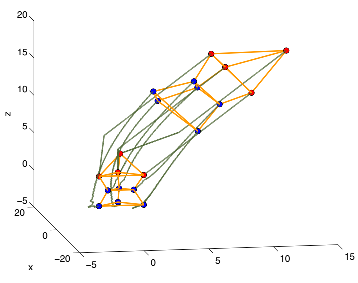

Fig. 7. The formation shape is guided by the leaders with the control law for the followers remaining unchanged.

VI. Conclusion

This paper studies the formation control problem for a leader-follower network and presents a barycentric coordinate based approach to solve the problem under directed and switching topologies. We introduce a communication graph to exchange an auxiliary state such that a fully distributed control law is available for the formation control purpose. A necessary and sufficient graphical condition is obtained to ensure global convergence.

References

[1] K. K. Oh, M. C. Park, and H. S. Ahn. A survey of multi-agent formation control. *Automatica*, 53:424–440, 2015.

[2] Z. Lin, L. Wang, Z. Han, and M. Fu. Distributed formation control of multi-agent systems using complex laplacian. *IEEE Transactions on Automatic Control*, 59(7):1765–1777, 2014.

[3] Z. Lin, L. Wang, Z. Han, and M. Fu. A graph laplacian approach to coordinate-free formation stabilization for directed networks. *IEEE Transactions on Automatic Control*, 2015.

[4] Z. Lin, L. Wang, Z. Chen, M. Fu, and Z. Han. Necessary and sufficient graphical conditions for affine formation control. *IEEE Transactions on Automatic Control*, 2015. accepted.

[5] H. Coxeter. Introduction to geometry. *John Wiley & Sons, Inc.*, 1969.

[6] T. Han, Z. Lin, and M. Fu. Three-dimensional formation merging control under directed and switching topologies. *Automatica*, 58:99–105, 2015.

[7] S. J. Gortler, A. D. Healy, and D. P. Thurston. Characterizing generic global rigidity. *American Journal of Mathematics*, 132(4):897–939, 2010.

[8] J. M. Dion, C. Commault, and J. van der Woude. Generic properties and control of linear structured systems: a survey. *Automatica*, 39(7):1125–1144, 2003.

[9] B. D. O. Anderson. Exponential stability of linear equations arising in adaptive identification. *IEEE Transactions on Automatic Control*, 22:83–88, 1977.