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An LMI Approach to Robust $\mathcal{H}_\infty$ Filtering for Linear Systems[1]

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Abstract

In this paper, we consider the robust $\mathcal{H}_\infty$ filtering problem for both continuous and discrete-time linear uncertain systems with energy bounded exogenous noise inputs and structural uncertainties satisfying some general integral quadratic constraints (IQCs). We apply the well-known $\mathcal{S}$-procedure and show that the robust $\mathcal{H}_\infty$ filtering problem can be effectively solved using linear matrix inequalities (LMIs).

1. Introduction

One of the very popular signal estimation algorithms in signal processing is the $\mathcal{H}_2$ filtering which minimizes the $\mathcal{L}_2$ norm of the corresponding estimation error. The well-known Kalman filtering is one of the celebrated $\mathcal{H}_2$ filtering approaches widely used in various fields of signal processing and control. A common feature of the $\mathcal{H}_2$ filtering algorithms is that they all assume the exogenous input signals have known statistics (typically the input signals are assumed as zero mean white noises) and the system under consideration has known dynamics described by certain well-posed model. These assumptions limit the application scope of the $\mathcal{H}_2$ filtering technique when there are uncertainties in either the exogenous input signals or the system model. It has been known that the standard Kalman filtering algorithms will generally not guarantee satisfactory performance when there exists uncertainty in the system model; see e.g. [1].

$\mathcal{H}_\infty$ filtering has different time and frequency domain properties to the $\mathcal{H}_2$ filtering. The exogenous input signals in $\mathcal{H}_\infty$ filtering are assumed belonging to $\mathcal{L}_2[0,\ \infty]$ signal space instead of the strict white noises. With $\mathcal{H}_\infty$ filtering, the $\mathcal{H}_\infty$ norm of the operator that relates the exogenous input signals with a desired output is minimized. Therefore, the magnitude of the power spectrum of the corresponding estimation error is lower than that of using $\mathcal{H}_2$ filtering. Although $\mathcal{H}_\infty$ filtering doesn't minimizes the variance of the corresponding estimation error, there are still application situation that the $\mathcal{H}_\infty$ filtering instead of the $\mathcal{H}_2$ filtering is more appropriate. One of such applications is reported in [2] for seismic signal deconvolution. There are further indications that $\mathcal{H}_\infty$ filtering may find applications in fault detection and radar systems. Meanwhile, minimizing $\mathcal{H}_\infty$ norm may guarantee better filtering performance when there exist uncertainties in the exogenous input signals or the system model; see e.g. [3] for example.

Several results have been obtained about robust $\mathcal{H}_\infty$ filtering for continuous and discrete time linear systems with exogenous noise input and parametric uncertainty; see [4, 5, 6, 7] for example. These results are obtained using the ARE approach. The problem of robust $\mathcal{H}_\infty$ filtering contains two aspects: $\mathcal{H}_\infty$ filtering analysis and $\mathcal{H}_\infty$ filtering synthesis. The analysis aspect is to determine the worst-case $\mathcal{H}_\infty$ performance when a filter is given while the synthesis aspect is to design a suitable filter such that the worst-case $\mathcal{H}_\infty$ performance is satisfactory. The ARE approach in [4, 5, 7] involves a conversion from the robust $\mathcal{H}_\infty$ filtering problem to a "scaled" $\mathcal{H}_\infty$ filtering one by converting the norm-bounded uncertainty into some scaling parameters. The conversion used there significantly simplifies the robust $\mathcal{H}_\infty$ filtering problem and makes it possible to use the standard $\mathcal{H}_\infty$ filtering results. However, besides the traditional computation problem with the ARE approach, the conversion above introduces the following disadvantages: 1) The scaling parameters enter the AREs nonlinearly; 2) the norm-bounded uncertainty assumption can only describe limited applications.

In this paper, we consider a new approach to the robust $\mathcal{H}_\infty$ filtering problem for continuous and discrete time uncertain systems. The systems considered are subject to an energy bounded exogenous noise input and several uncertainties described by the integral quadratic constraints (IQCs) which are more general than the norm bounded structure. Similar to [4, 5, 7], the robust $\mathcal{H}_\infty$ filtering problem also involves two aspects, namely, the $\mathcal{H}_\infty$ filtering analysis and the $\mathcal{H}_\infty$ filtering synthesis. We apply the $\mathcal{S}$-procedure to show that the robust $\mathcal{H}_\infty$ filtering problem can be solved using several linear matrix inequalities (LMIs). It is interesting to see that the analysis problem can be solved using a single LMI which is jointly linear in terms of a positive-definite matrix for $\mathcal{H}_\infty$ filtering performance, the scaling parameters and the $\mathcal{H}_\infty$

filtering performance bound; the synthesis problem is more complicated which involves two LMIs with one jointly linear in a positive-definite matrix, the inverses of the scaling parameters as well as the $\mathcal{H}_\infty$ filtering performance bound and the other jointly linear in a positive-definite matrix, the scaling parameters as well as the $\mathcal{H}_\infty$ filtering performance bound. However, we show that the two LMIs are jointly linear in all of the three set of variables in special cases. Our results naturally reduce to those in [4, 5, 7] when norm-bounded uncertainty assumption is enforced.

This paper is organized as follows: Section 2 states the the robust $\mathcal{H}_\infty$ filtering problem, section 3, the robust $\mathcal{H}_\infty$ filtering analysis; section 4, the robust $\mathcal{H}_\infty$ filtering synthesis; and the concluding remarks are given in section 5.

Due to space limitation, we delete all of the proofs from this paper. For details of these proofs and an example, see [8].

We use the following notational table in this paper:

Notation	Continuous	Discrete
$\partial x(t)$	$\dot{x}(t)$	$x(t+1)$
$\mathbf{S}_0^T \, \|\|\xi_i(t)\|\|^2$	$\int_0^T \|\|\xi_i(t)\|\|^2 dt$	$\sum_{t=0}^T \|\|\xi_i(t)\|\|^2$

2. Problem Statement

Consider the following uncertain linear system:

$$\partial x(t) = Ax(t) + Bw(t) + \sum_{i=1}^{p} H_{1i}\xi_i(t) \qquad \text{(1a)}$$

$$z(t) = C_1 x(t) + D_1 w(t) + \sum_{i=1}^{p} H_{2i}\xi_i(t) \qquad \text{(1b)}$$

$$y(t) = C_2 x(t) + D_2 w(t) + \sum_{i=1}^{p} H_{3i}\xi_i(t) \qquad \text{(1c)}$$

where $x(t) \in \mathbf{R}^n$ is the state, $w(t) \in \mathbf{R}^q$ the exogenous noise input which belongs to $\mathcal{L}_2[0,\infty)$, $z(t) \in \mathbf{R}^r$ the output to be estimated, $y(t) \in \mathbf{R}^{r_y}$ the measured output, and $\xi_i(t) \in \mathbf{R}^{k_i}$ the uncertain variables satisfying the following IQCs:

$$\mathbf{S}_0^T \, ||\xi_i(t)||^2 \le \mathbf{S}_0^T \, ||E_{1i}x(t) + E_{2i}w(t) + E_{3i}\xi(t)||^2, \quad \text{as } T \to \infty, \; i = 1, 2, \ldots, p \qquad (2)$$

with

$$\xi(t) = [\xi_1^T(t) \; \ldots \; \xi_p^T(t)]^T.$$

Also, A, B, C_1, C_2, D_1 and D_2 are constant matrices of appropriate dimension. H_{1i}, H_{2i}, H_{3i}, E_{1i}, E_{2i} and E_{3i} are constant matrices. Without loss of generality, we assume that $H_{1i} \in \mathbf{R}^{n\times k_i}$, $H_{2i} \in \mathbf{R}^{r\times k_i}$, $H_{3i} \in \mathbf{R}^{r_y\times k_i}$, $E_{1i} \in \mathbf{R}^{k_i\times n}$, $E_{2i} \in \mathbf{R}^{k_i\times q}$ and $E_{3i} \in \mathbf{R}^{k_i\times \sum_{i=1}^{p} k_i}$.

Remark: Several special cases of the system (1) have been treated in the literature. For example, [4, 5, 6, 9, 7, 10, 11, 12, 3] considered the following system:

$$\sigma x(t) = (A + \Delta A)x(t) + Bw(t) \qquad (3)$$

$$z(t) = C_1 x(t) \qquad (4)$$

$$y(t) = (C_2 + \Delta C)x(t) + D_2 w(t) \qquad (5)$$

with norm-bounded uncertainty

$$\begin{bmatrix} \Delta A \\ \Delta C \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F(t)E \qquad (6)$$

where $F^T(t)F(t) \le I, \; \forall t \ge 0$. Another widely used system uncertainty description in $\mathcal{H}_\infty$ analysis involves the so-called linear fractional uncertainty (see, e.g., [13]) where the following system is considered

$$\sigma x(t) = (A + \Delta A)x(t) + Bw(t) \qquad (7)$$

$$z(t) = C_1 x(t) \qquad (8)$$

$$y(t) = C_2 x(t) + D_2 w(t) \qquad (9)$$

with one-block type uncertainty

$$\Delta A = HF(t)(I - E_3F(t))^{-1}E_1,$$

$$F^T(t)F(t) \le I, \; \forall t \ge 0 \qquad (10)$$

However, to our knowledge, there is no existing $\mathcal{H}_\infty$ filtering result available for linear systems with linear fractional uncertainty.

It is straightforward to see that the aforementioned norm-bounded and one-block type uncertainties are special cases of the IQCs (2) with $p = 1$.

Consider the following filter:

$$\partial x_f(t) = A_f x_f(t) + B_f y(t) \qquad \text{(11a)}$$

$$z_f(t) = C_f x_f(t) + D_f y(t) \qquad \text{(11b)}$$

where $x_f(t) \in \mathbf{R}^{n_f}$ is the estimated state, $z_f(t) \in \mathbf{R}^r$ the estimated output, $y(t)$ is the measured output of (1) and A_f, B_f, C_f and D_f are constant matrices of appropriate dimension to be chosen.

Define the output filtering error as

$$e(t) = z(t) - z_f(t), \qquad (12)$$

then the filtering error dynamics is given by

$$\partial x_e(t) = A_e x_e(t) + B_e w(t) + \sum_{i=1}^{p} H_{1ie}\xi_i(t) \qquad \text{(13a)}$$

$$e(t) = C_e x_e(t) + D_e w(t) + \sum_{i=1}^{p} H_{2ie}\xi_i(t) \qquad \text{(13b)}$$

where

$$x_e(t) = \begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix} \qquad (14)$$

$$A_e = \begin{bmatrix} A & 0 \\ B_f C_2 & A_f \end{bmatrix}; \quad B_e = \begin{bmatrix} B \\ B_f D_2 \end{bmatrix} \quad (15)$$

$$C_e = [C_1 - D_f C_2 \quad -C_f]; \; D_e = D_1 - D_f D_2 (16)$$

$$H_{1e} = \begin{bmatrix} H_1 \\ B_f H_3 \end{bmatrix}; \quad H_{2e} = H_2 - D_f H_3 \quad (17)$$

with simplified notation

$$H_1 = [H_{11} \; \dots \; H_{1p}]; \quad H_2 = [H_{21} \; \dots \; H_{2p}];$$

$$H_3 = [H_{31} \; \dots \; H_{3p}] \quad (18)$$

Remark: It is obvious that robust $\mathcal{H}_\infty$ state estimation problem is a special case of the above robust $\mathcal{H}_\infty$ filtering problem with $D_1 = 0$, $H_2 = 0$, $D_f = 0$, $C_1 = I$ and $C_f = I$.

3. Analysis of Robust $\mathcal{H}_\infty$ Filtering

The main purpose for designing the robust $\mathcal{H}_\infty$ filter is to minimize the induced $\mathcal{L}_2$ norm from $w(t)$ to $e(t)$. However, we are also concerned about the stability of the augmented system (13). In this paper, we adopt the following stability notion:

Definition 1 *The filtering error dynamics (13) is called bounded-state stable (BS stable) if for any $x_e(0) \in \mathbf{R}^{n+n_f}$ and any $w(t) \in \mathcal{L}_2[0,\infty)$, there exists $M \geq 0$ such that*

$$\|x_e(t)\| \leq M, \qquad t \geq 0 \quad (19)$$

The robust $\mathcal{H}_\infty$ filtering analysis problem associated with the uncertain system (1) is as follows: *Given $\gamma > 0$ and a filter of the form (11), determine if the error dynamics (13) is BS stable for $w(t) = 0$ and satisfies the following condition:*

$$\mathbf{S}_0^T \; \|e(t)\|^2 < \gamma^2 \mathbf{S}_0^T \; \|w(t)\|^2, \quad \text{as } T \to \infty, x_e(0) = 0 \quad (20)$$

for all admissible uncertainty satisfying the IQCs (2).

For notational convenience, we define:

$$E_1^T = [E_{11}^T \; \dots \; E_{1p}^T]; \quad E_2^T = [E_{21}^T \; \dots \; E_{2p}^T]; \; (21)$$

$$E_3^T = [E_{31}^T \; \dots \; E_{3p}^T] \quad (22)$$

$$J = \text{diag}\{\tau_1 I_{k_1}, \dots, \tau_p I_{k_p}\} \quad (23)$$

where $\tau_1, \dots, \tau_p$ are scalars. $J > 0$ if and only if $\tau_i > 0$, $i = 1, 2, \dots, p$.

Applying the $\mathcal{S}$-procedure, we have the following result:

Lemma 1 *Given the system (13), condition (20) holds for all admissible uncertainties satisfying the IQCs (2) if there exist a positive definite matrix $P = P^T \in \mathbf{R}^{n \times n}$ and scaling parameters $\tau_1, \dots, \tau_p > 0$ such that the following condition holds:*

For the continuous-time case:

$$2x_e^T P(A_e x_e + B_e w + \sum_{i=1}^p H_{1ie}\xi_i) + \sum_{i=1}^p \tau_i (\|[E_{1i} \; 0]x_e + E_{2i} w + E_{3i}\xi\|^2 - \|\xi_i\|^2) + \|C_e x_e + D_e w + \sum_{i=1}^p H_{2ie}\xi_i\|^2 - \gamma^2 \|w\|^2 < 0,$$
$$\forall x_e \in \mathbf{R}^{n+n_f}, w \in \mathbf{R}^q, \xi_i \in \mathbf{R}^{k_i}, i = 1, 2 \dots, p, \text{ with } [x^T \; w^T \; \xi^T]^T \neq 0 \quad (24)$$

For the discrete-time case:

$$(A_e x_e + B_e w + \sum_{i=1}^p H_{1ie}\xi_i)^T P (A_e x_e + B_e w + \sum_{i=1}^p H_{1ie}\xi_i) - x_e^T P x_e + \sum_{i=1}^p \tau_i (\|[E_{1i} \; 0]x_e + E_{2i} w + E_{3i}\xi\|^2 - \|\xi_i\|^2) + \|C_e x_e + D_e w + \sum_{i=1}^p H_{2ie}\xi_i\|^2 - \gamma^2 \|w\|^2 < 0,$$
$$\forall x_e \in \mathbf{R}^{n+n_f}, w \in \mathbf{R}^q, \xi_i \in \mathbf{R}^{k_i}, i = 1, 2 \dots, p, \text{ with } [x^T \; w^T \; \xi^T]^T \neq 0 \quad (25)$$

Remark: If the uncertainties are described by quadratic constraints:

$$\|\xi_i(t)\|^2 \leq \|E_{1i}x(t) + E_{2i}w(t) + E_{3i}\xi(t)\|^2, \quad i = 1, 2, \dots, p, \quad (26)$$

then Lemma 1 naturally leads to asymptotic stability.

Using lemma 1 and denoting

$$E_{1c} = [E_1 \; 0] \quad (27)$$

we obtain the following theorem on robust $\mathcal{H}_\infty$ filtering analysis for continuous-time systems:

Theorem 1 *(Continuous-time) The following conditions are all equivalent and they guarantee the solution to the robust $\mathcal{H}_\infty$ filtering analysis problem associated with the uncertain system (1) and the filter (11):*

- **(i)** *There exist $P = P^T > 0$ and $\tau_1 > 0, \dots, \tau_p > 0$ such that (24) holds;*
- **(ii)** *There exist $P = P^T > 0$ and $J > 0$ in (23) solving the following LMI:*

$$\begin{bmatrix} A_c^T P + P A_e + E_{1c}^T J E_{1c} + C_c^T C_c \\ B_c^T P + E_2^T J E_1 + D_c^T C_c \\ H_{1c}^T P + H_{2c}^T C + E_3^T J E_{1c} \end{bmatrix}$$

$$\begin{matrix} PB_c + E_{1c}^T JE_2 + C_c^T D_c \\ -\gamma^2 I + D_c^T D_e + E_2^T JE_2 \\ H_{2c}^T D + E_3^T JE_2 \end{matrix}$$

$$\left. \begin{matrix} PH_{1c} + C_c^T H_{2c} + E_{1c}^T JE_3 \\ D_e^T H_{2e} + E_2^T JE_3 \\ -J + H_{2c}^T H_{2e} + E_3^T JE_3 \end{matrix} \right] < 0 \tag{28}$$

(iii) *There exist $P = P^T > 0$ and $J > 0$ in (23) solving the following LMI:*

$$\begin{bmatrix} A_c^T P + PA_e & PB_e & PH_{1e} & C_c^T & E_{1e}^T J \\ B_c^T P & -\gamma^2 I & 0 & D_e^T & E_2^T J \\ H_{1c}^T P & 0 & -J & H_{2c}^T & E_3^T J \\ C_c & D_c & H_{2c} & -I & 0 \\ JE_{1c} & JE_2 & JE_3 & 0 & -J \end{bmatrix} < 0 \tag{29}$$

(iv) *There exists $J > 0$ in (23) such that the following auxiliary system is asymptotically stable and the $\mathcal{H}_\infty$-norm of the transfer function from $\hat{w}(\cdot)$ to $\hat{e}(\cdot)$ is less than 1:*

$$\dot{\hat{x}}_c(t) = A_c \hat{x}_e(t) + [\gamma^{-1} B_e \quad H_{1e} J^{-1/2}] \hat{w}(t) \tag{30}$$

$$\hat{e}(t) = \begin{bmatrix} C_e \\ J^{1/2} E_{1e} \end{bmatrix} \hat{x}_e(t) + \begin{bmatrix} \gamma^{-1} D_e & H_{2e} J^{-1/2} \\ \gamma^{-1} J^{1/2} E_2 & J^{1/2} E_3 J^{-1/2} \end{bmatrix} \hat{w}(t) \tag{31}$$

Moreover, the set of all J's satisfying (iv) is convex.

Similarly, we have the following theorem on robust $\mathcal{H}_\infty$ filtering analysis for discrete-time systems:

Theorem 2 *(Discrete-time) The following conditions are all equivalent and they guarantee the solution to the robust $\mathcal{H}_\infty$ filtering analysis problem associated with the uncertain system (1) and the filter (11):*

(i) *There exist $P = P^T > 0$ and $\tau_1 > 0, \ldots, \tau_p > 0$ such that (25) holds;*

(ii) *There exist $P = P^T > 0$ and $J > 0$ in (23) solving the following LMI:*

$$\left[\begin{matrix} A_c^T PA_c - P + E_{1c}^T JE_{1c} + C_c^T C_e \\ B_c^T PA_c + E_2^T JE_{1e} + D_c^T C_e \\ H_{1c}^T PA_c + H_{2c}^T C_e + E_3^T JE_{1c} \end{matrix} \right.$$

$$\begin{matrix} A_c^T PB_c + E_{1e}^T JE_2 + C_e^T D_e \\ -\gamma^2 I + B_c^T PB_e + D_e^T D_e + E_2^T JE_2 \\ H_{1c}^T PB_e + H_{2c}^T D_e + E_3^T JE_2 \end{matrix}$$

$$\left. \begin{matrix} A_c^T PH_{1e} + C_c^T H_{2e} + E_{1e}^T JE_3 \\ B_c^T PH_{1c} + D_e^T H_{2e} + E_2^T JE_3 \\ -J + H_{1e}^T PH_{1c} + H_{2e}^T H_{2e} + E_3^T JE_3 \end{matrix} \right] < 0 \tag{32}$$

(iii) *There exist $P = P^T > 0$ and $J > 0$ in (23) solving the following LMI:*

$$\left[\begin{matrix} A_e^T PA_e - P & A_c^T PB_e \\ B_e^T PA_e & -\gamma^2 I + B_e^T PB_e \\ H_{1e}^T PA_e & H_{1e}^T PB_e \\ C_c & D_e \\ JE_{1e} & JE_2 \end{matrix} \right.$$

$$\left. \begin{matrix} A_e^T PH_{1e} & C_e^T & E_{1e}^T J \\ B_e^T PH_{1e} & D_e^T & E_2^T J \\ -J + H_{1e}^T PH_{1e} & H_{2e}^T & E_3^T J \\ H_{2e} & -I & 0 \\ JE_3 & 0 & -J \end{matrix} \right] < 0 \tag{33}$$

(iv) *There exists $J > 0$ in (23) such that the following auxiliary system is asymptotically stable and the $\mathcal{H}_\infty$-norm of the transfer function from $\hat{w}(\cdot)$ to $\hat{e}(\cdot)$ is less than 1:*

$$\hat{x}_e(t+1) = A_e \hat{x}_e(t) + [\gamma^{-1} B_e \quad H_{1e} J^{-1/2}] \hat{w}(t) \tag{34}$$

$$\hat{e}(t) = \begin{bmatrix} C_e \\ J^{1/2} E_{1e} \end{bmatrix} \hat{x}_e(t) + \begin{bmatrix} \gamma^{-1} D_e & H_{2e} J^{-1/2} \\ \gamma^{-1} J^{1/2} E_2 & J^{1/2} E_3 J^{-1/2} \end{bmatrix} \hat{w}(t) \tag{35}$$

Moreover, the set of all J's satisfying (iv) is convex.

4. Synthesis of Robust $\mathcal{H}_\infty$ Filters

For the synthesis problem, we need the following assumptions:

(A1) The nominal system matrix A is quadratically stable;

(A2) (A, C_2) is detectable.

The $\mathcal{H}_\infty$ filtering synthesis problem associated with the uncertain system (1) is as follows: *Given $\gamma > 0$, find a filter of the form (11) such that the corresponding error dynamics (13) is BS stable and satisfies the following condition:*

$$\mathbf{S}_0^T \|e(t)\|^2 < \gamma^2 \mathbf{S}_0^T \|w(t)\|^2, \quad \text{as } T \to \infty, x_e(0) = 0 \tag{36}$$

for all admissible uncertainties satisfying the IQCs (2).

It is straightforward to see that the auxiliary system (30)-(31) or (34)-(35) is the closed-loop system of the following auxiliary open loop system:

$$\partial \tilde{x}(t) = A\tilde{x}(t) + [\gamma^{-1} B \quad H_1 J^{-1/2}] \hat{w}(t) \tag{37a}$$

$$\hat{e}(t) = \begin{bmatrix} C_1 \\ J^{1/2} E_1 \end{bmatrix} \tilde{x}(t)$$

$$+\begin{bmatrix} \gamma^{-1}D_1 & H_2J^{-1/2} \\ \gamma^{-1}J^{1/2}E_2 & J^{1/2}E_3J^{-1/2} \end{bmatrix}\hat{w}(t)$$

$$+\begin{bmatrix} -I \\ 0 \end{bmatrix}u(t) \tag{37b}$$

$$y(t) = C_2\tilde{x}(t) + [\gamma^{-1}D_2 \quad H_3J^{-1/2}]\hat{w}(t) \tag{37c}$$

with controller

$$\partial x_f(t) = A_f x_f(t) + B_f y(t) \tag{38a}$$

$$u(t) = C_f x_f(t) + D_f y(t) \tag{38b}$$

Notice that the robust $\mathcal{H}_\infty$ filtering synthesis problem of the original systems (1) with filter (11) is converted into the standard $\mathcal{H}_\infty$ control synthesis problem of the auxiliary open loop system (37) with controller (38). It is interesting to see that the auxiliary controller (38) is exactly the same form as the filter (11) which is to be designed. Therefore, instead of considering the original robust $\mathcal{H}_\infty$ filtering synthesis problem, we can tackle a standard $\mathcal{H}_\infty$ control synthesis problem which has readily applicable results; see [14] for example. Once the controller matrices are determined, we can directly use these matrices as desired filter matrices. This point suggests the following theorem which is the main result for the synthesis problem:

Theorem 3 *Given $\gamma > 0$. Denote by $\mathcal{N}_S$ any matrix whose columns form a basis of the null space of $[C_2 \;\; D_{21} \;\; H_3]$. Then the robust $\mathcal{H}_\infty$ filtering synthesis problem is solvable if there exist symmetric matrices $R, S \in \mathbf{R}^{n\times n}$ and $J > 0$ in (23) such that the following LMIs hold:*

For the continuous-time case:

$$\left[\begin{array}{c|c} \mathbf{A}_{RC} & \mathbf{B} \\ \hline \mathbf{B}^T & \mathbf{D}_1 \end{array}\right] < 0 \tag{39}$$

$$\left[\begin{array}{c|c} \mathcal{N}_S^T & 0 \\ \hline 0 & I \end{array}\right]\left[\begin{array}{c|c} \mathbf{A}_{SC} & \mathbf{C} \\ \hline \mathbf{C}^T & \mathbf{D}_2 \end{array}\right]\left[\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{40}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \tag{41}$$

For the discrete-time case:

$$\left[\begin{array}{c|c} \mathbf{A}_{RD} & \mathbf{B} \\ \hline \mathbf{B}^T & \mathbf{D}_1 \end{array}\right] < 0 \tag{42}$$

$$\left[\begin{array}{c|c} \mathcal{N}_S^T & 0 \\ \hline 0 & I \end{array}\right]\left[\begin{array}{c|c} \mathbf{A}_{SD} & \mathbf{C} \\ \hline \mathbf{C}^T & \mathbf{D}_2 \end{array}\right]\left[\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{43}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \tag{44}$$

where

$$\mathbf{A}_{RC} = \begin{bmatrix} AR + RA^T & RE_1^T \\ E_1R & -J^{-1} \end{bmatrix}$$

$$\mathbf{A}_{RD} = \begin{bmatrix} ARA^T - R & ARE_1^T \\ E_1RA^T & -J^{-1} + E_1RE_1^T \end{bmatrix}$$

$$\mathbf{A}_{SC} = \begin{bmatrix} A^TS + SA & SB & SH_1 \\ B^TS & -\gamma^2 I & 0 \\ H_1^TS & 0 & -J \end{bmatrix}$$

$$\mathbf{A}_{SD} = \begin{bmatrix} A^TSA - S & A^TSB & A^TSH_1 \\ B^TSA & -\gamma^2 I + B^TSB & B^TSH_1 \\ H_1^TSA & H_1^TSB & -J + H_1^TSH_1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} B & H_1J^{-1} \\ E_2 & E_3J^{-1} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} C_1^T & E_1^TJ \\ D_1^T & E_2^TJ \\ H_2^T & E_3^TJ \end{bmatrix}$$

$$\mathbf{D}_1 = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & -J^{-1} \end{bmatrix}, \quad \mathbf{D}_2 = \begin{bmatrix} -I & 0 \\ 0 & -J \end{bmatrix}.$$

Remark: LMI (39) is jointly linear in R, J^{-1} and γ^2 while LMI (40) is jointly linear in S, J and γ^2. Therefore, LMIs (39) and (40) are not jointly linear in J. However, in two special cases, they are jointly linear in J.

<u>Case 1</u>.

Suppose $p = 1$, i.e., $J = \tau_1 I$. This is called the "single uncertainty block" case. In this case, we left- and right-multiply (39) by

$$\mathrm{diag}\{J^{1/2}, J^{1/2}, I, I\},$$

then, the LMI (39) is equivalent to the following:

$$\left[\begin{array}{cc|cc} ARJ + RJA^T & RJE_1^T & J^{1/2}B & H_1J^{-1/2} \\ E_1RJ & -I & J^{1/2}E_2 & E_3J^{-1/2} \\ \hline J^{1/2}B^T & J^{1/2}E_2^T & -\gamma^2 I & 0 \\ J^{-1/2}H_1^T & J^{-1/2}E_3^T & 0 & -J^{-1} \end{array}\right] < 0. \tag{45}$$

Following Schur complement, the LMI (39) is further equivalent to the following:

$$\begin{bmatrix} A\tilde{R} + \tilde{R}A^T + \gamma^{-2}BB^TJ + H_1H_1^T & \tilde{R}E_1^T + \gamma^{-2}BE_2^TJ + H_1E_3^T \\ E_1\tilde{R} + \gamma^{-2}E_2B^TJ + E_3H_1^T & -I + \gamma^{-2}E_2E_2^TJ + E_3E_3^T \end{bmatrix} < 0 \tag{46}$$

where $\tilde{R} = RJ$. Similarly, the LMI (42) is equivalent to the following:

$$\begin{bmatrix} A\tilde{R}A^T - \tilde{R} + \gamma^{-2}BB^TJ + H_1H_1^T & A\tilde{R}E_1^T + \gamma^{-2}BE_2^TJ + H_1E_3^T \\ E_1^T\tilde{R}A + \gamma^{-2}E_2B^TJ + E_3H_1^T & -I + \gamma^{-2}E_2E_2^TJ + E_3E_3^T \end{bmatrix} < 0 \tag{47}$$

where $\tilde{R} = RJ$. Then, LMIs (46) and (40) for continuous-time systems as well as LMIs (47) and (43) for discrete-time systems are jointly linear in J.

<u>Case 2</u>.

Assume $E_1 = 0$. That is, the uncertainty variables $\xi_i(t)$ are independent of the state variables. Left- and right-multiplying (39) by

$$\mathrm{diag}\{I, J, I, J\},$$

the LMI (39) is equivalent to

$$\left[\begin{array}{cc|cc} AR+RA^T & 0 & B & H_1 \\ 0 & -J & JE_2 & E_3 \\ \hline B^T & E_2^T J & -\gamma^2 I & 0 \\ H_1^T & E_3^T J & 0 & -J \end{array}\right] < 0. \quad (48)$$

Similarly, the LMI (42) is equivalent to the following:

$$\left[\begin{array}{cc|cc} ARA^T - R & 0 & B & H_1 \\ 0 & -J & JE_2 & E_3 \\ \hline B^T & E_2^T J & -\gamma^2 I & 0 \\ H_1^T & E_3^T J & 0 & -J \end{array}\right] < 0. \quad (49)$$

Again, LMIs (48) and (40) for continuous-time systems as well as LMIs (49) and (43) for discrete-time systems are jointly linear in J.

Remark: Assume $p = 1, E_2 = 0, E_3 = 0, H_2 = 0, D_1 = 0, D_f = 0$, LMIs (39)-(41) for continuous-time systems and LMIs (42)-(44) for discrete-time systems naturally reduce to AREs which are exactly the same form as in [4, 5], see [4, 5] for details about the corresponding AREs.

5. Conclusion

In this paper, we have provided an LMI approach to the robust $\mathcal{H}_\infty$ filtering problem for linear systems with uncertainty described by general IQCs. Our approach has several advantages over the standard ARE approach. Namely, the LMI approach is computationally efficient due to the recent advancement in convex optimization [15]; The scaling parameters enter the LMIs linearly for the individual LMIs and joint linearly for all LMIs in the special cases; and the IQCs are very general in describing uncertainties.

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