Quantized Output-feedback Control for Linear Systems with Multiplicative Noises in Measurement

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***Abstract*—In this paper, we consider the quantized quadratic performance control problem for a class of stochastic systems which are subject to multiplicative noises in the measurement, we look for a dynamic output feedback controller to guarantee certain level of performance. By using the sector bound approach to characterize the quantized error, we show that the existence of the solution of quantized quadratic guaranteed cost problem can be found by solving the so-called guaranteed cost control problem of the associated system with sector bound uncertainty. The main result of this paper show that this problem can be effectively solved using linear matrix inequalities (LMIs).**

I. Introduction

Research of the problem of quantized feedback control can be traced back to 1956 [1], where Kalman investigated the effects of quantization in a sampled data control system and pointed out that the quantized feedback system would exhibit limited cycles and chaotic behavior if finite-alphabet quantizer is used. For the early works of quantized feedback control problem [2] [3] [4] [5], the quantization are always considered as undesirable, either as noise or state uncertainty, so most of the works try to eliminate its influence.

The widespread use of network based control where the information between the controller and the the plants is exchanged through a network medium with limited capacities has further strengthened the importance of the study of the quantized feedback control problem. Different from the early views towards quantization, quantization is now considered to be useful instead of undesirable. As for the fundamental problem in network control system how much is the least data rate that has to be sent to stabilize the system, [6] shows that the coarsest quantizer is logarithmic for single input deterministic system, where the quantization density can be characterized by the unstable poles of the system matrix. Quantization density with respect to the feedback subject to the Bernoulli packets dropouts is considered in [7], which is related to both the unstable states and the statistical properties of Bernoulli noises. For the more general case with the input channel subject to an independent and identically distributed packet dropout process in [8], the minimum data rate for the mean square stabilization is explicitly given in terms of the unstable eigenvalues of the open loop matrix and the packet dropout probability. For the stochastic systems, not only system matrix but also the statistical properties of the noises are related [9]. Results on feedback control with dynamic quantizer can be found in [10] [11]. In works about quantized feedback control, most of them are only confined to the problem of quantized stabilization, control performance is usually not addressed. Quantized feedback control problem with a quadratic performance index for deterministic systems is studied in [12], where a sector bound approach is used to characterize the quantized error. Both quantized state feedback and dynamic output feedback control are considered in [12]. As for quantized stabilization and performance with finite levels quantizer, [13] shows that asymptotic stabilization for the system can be achieved with a moderate number of quantization levels by introducing a dynamic scaling method for logarithmic quantizer. The quantized feedback stabilization problem for system with bounded noises is also given in [13].

In this paper we consider a quantized quadratic performance problem for stochastic systems with multiplicative noise in the measurements. By using the sector bound approach to characterize the quantized error, we show that the existence of the solutions to the quadratic quantized performance control problem is guaranteed by the existence of the solution to the so-called guaranteed cost control problem. Using the Schur complement technique and using the elimination lemma to deal with the stochastic noises, we get the solutions to the problem in terms of linear matrices inequalities. The paper is organized as follows. In section II, the system is introduced, some fundamental knowledge about quantization is introduced, and the problem formulation is given. In section III, the relation of the existence of the quadratic quantized performance control problem and the guaranteed cost control problem is established, and the existence of the solution to the problem of guaranteed cost control problem is given in terms of linear matrices inequalities. Section IV draws some conclusions about this paper.

II. Problem Formulation

Consider the following linear discrete-time system with multiplicative noise in the measurement:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= (1+\gamma(t))Cx(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the system state vector with initial state x_0 assumed to be white noise with $Ex^T(0)x(0) = \mu^2 I$ for some $\mu > 0$; $u(t) \in \mathcal{R}$ is the control input, $y(t) \in \mathcal{R}$ is the measurement, $\gamma(t)$ is a white scalar noise with $E\gamma^2(t) = \sigma^2 > 0$ for some $\sigma > 0$, and uncorrelated with the initial state $x(0)$. The measurement is sent through a band-limited channel that has to be quantized by a logarithmic quantizer in the following form:

$$Q(y)=\begin{cases} u_i, & if\, \frac{1}{1+\delta}u_i < y \leq \frac{1}{1-\delta}u_i, y > 0 \\ 0 & if\ y = 0 \\ -Q(-y) & if\ y < 0 \end{cases} \tag{2}$$

with quantization levels as

$$U = \{\pm u_i : u_i = \rho_i u_0, i = 1, 2, \cdots\} \bigcup \{\pm u_0\} \bigcup \{0\}, \\ 0 < \rho < 1, u_0 > 0, \tag{3}$$

where ρ is the quantized density of the logarithmic quantizer. As illustrated in [12], using the sector bound approach we have

$$|y - Q(y)| = |\Delta(t)y| \leq \delta |y|, \tag{4}$$

where

$$\delta = \frac{1-\rho}{1+\rho},\ \Delta(t) = \frac{y(t) - Q(y(t))}{y(t)}. \tag{5}$$

Consider the following quadratic performance cost control function:

$$J(x(0)) = \sum_{t=0}^{\infty} (x^T(t)Sx(t) + u^T(t)Ru(t)), \\ S \geq 0,\ R > 0. \tag{6}$$

Suppose quantized output feedback control is used for the system to be quadratically mean-square stabilized, i.e., a controller in the following form is to be designed:

$$x_c(t+1) = A_c x_c(t) + B_c Q(y(t)),\ x_c(0) = 0, \\ u(t) = C_c x_c(t) + D_c Q(y(t)), \tag{7}$$

such that the closed loop system is quadratically and that the performance index is minimized in the sense of

$$\min EJ(x(0)). \tag{8}$$

III. SOLUTIONS

From the system (1) and the controller (7), we can write the closed-loop system as

$$\xi(t+1) = [\bar{A} + \bar{B}(\Delta(t)) + \gamma(t)\hat{B}(\Delta(t))]\xi(t), \tag{9}$$

with the system state as

$$\xi(t) = \begin{bmatrix} x_c(t) \\ x(t) \end{bmatrix}, \tag{10}$$

and parameters are defined as

$$\bar{A} = \begin{bmatrix} A_c & B_c C \\ BC_c & A + BD_c C \end{bmatrix}, \bar{B}(\Delta(t)) = \begin{bmatrix} 0 & \Delta(t)B_c C \\ 0 & \Delta(t)BD_c C \end{bmatrix}, \\ \hat{B}(\Delta(t)) = \begin{bmatrix} 0 & B_c C(1+\Delta(t)) \\ 0 & BD_c C(1+\Delta(t)) \end{bmatrix}. \tag{11}$$

Using the system state $\xi(t)$ of the closed-loop system (9), the performance cost (6) can be rewritten as

$$J(\xi(0)) = \sum_{t=0}^{\infty} (\xi^T(t)\bar{S}\xi(t) + u^T(t)Ru(t)) \tag{12}$$

with

$$\xi(0)) = \begin{bmatrix} 0 \\ x(0) \end{bmatrix}, \bar{S} = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}. \tag{13}$$

For the closed-loop system (9) to be quadratical mean-square stable, there exists an associated Lyapunov function $V(\xi) = \xi^T \bar{P} \xi$ with $\bar{P} = \bar{P}^T$ such that

$$E\nabla V(\xi(t)) = E\{V(\xi(t+1)) - V(\xi(t))\} < 0, \tag{14}$$

for all $t \geq 0$. The performance index (12) can be given as

$$\begin{aligned} EJ(\xi(0)) \;&=\; E\sum_{t=0}^{\infty} \nabla V(\xi(t)) + \xi^T(t)\bar{S}\xi(t) + u^T(t)Ru(t) \\ &+\; E\xi^T(0)\bar{P}\xi(0) \\ &= E\xi^T(0)\bar{P}\xi(0) + E\sum_{t=0}^{\infty} \xi^T(t)\bar{\Omega}(\Delta(t))\xi(t), \end{aligned} \tag{15}$$

where

$$\begin{aligned} \bar{\Omega}(\Delta(t)) \;=\;& [A_0 + B_0 K I_1 + B_0 K I_{20}^T + (1+\Delta(t))(1+\gamma(t))\bar{E}] \\ &\times \bar{P}[A_0 + B_0 K I_1 + B_0 K I_{20}^T + (1+\Delta(t))(1+\gamma(t))\bar{E}] \\ &+ [I_{20} K I_1^T + I_{20} K I_{20}^T (1+\gamma(t))\bar{E}]^T R \\ &\times\ [I_{20} K I_1^T + I_{20} K I_{20}^T (1+\gamma(t))\bar{E}] - \bar{P} + \bar{S} \\ =&\quad \Omega_0 + \Omega_1 \bar{E}(1+\gamma(t)) + [\bar{E}(1+\gamma(t))]^T \Omega_1^T \end{aligned} \tag{16}$$

for some constant matrices Ω_0 and Ω_1. In the above,

$$K = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}, A_0 = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}, B_0 = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}, \\ I_{10} = \begin{bmatrix} I & 0 \end{bmatrix}, I_{20} = \begin{bmatrix} 0 & I \end{bmatrix}, \\ \bar{E} = \begin{bmatrix} 0 & C \end{bmatrix}, I_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{17}$$

It follows that

$$\begin{aligned} E\bar{\Omega}(\Delta(t)) \;=\;& [\bar{A} + \bar{B}(\Delta(t))]^T \bar{P}[\bar{A} + \bar{B}(\Delta(t))] - \bar{P} + \bar{S} \\ &+\ \sigma^2[0\ D_c C(1+\Delta(t))]^T R[0\ D_c C(1+\Delta(t))] \\ &+\ [C_c\ D_c C(1+\Delta(t))]^T R[C_c\ D_c C(1+\Delta(t))] \\ &+\ \sigma^2 \hat{B}^T(\Delta(t))\bar{P}\hat{B}(\Delta(t)). \end{aligned} \tag{18}$$

In the presence of quantizer, the performance control problem can be formulated as follows: *Given a performance bound $\tau > 0$ and quantization density $\rho > 0$, find $\bar{P}$, K, if exist, such that*

$$tr(\bar{P}) < \tau \tag{19}$$

subject to

$$E\xi^T\bar{\Omega}(\Delta(t))\xi \leq 0,\ \forall\ \xi \neq 0,\ t \geq 0, \tag{20}$$

we call the problem quantized quadratic performance control (QQGC) problem. Define the following auxiliary system:

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\
y(t) &= (1+\gamma(t))Cx(t) \\
v(t) &= (1+\Delta(t))y(t)
\end{aligned} \tag{21}
$$

where $v(t)$ is the output available for feedback. The solution to QQGC is related to the so-called guaranteed-cost control (GCC) problem for the auxiliary system (21), i.e. we want to find $\bar{P}$, K such that (19) holds subject to

$$E\bar{\Omega}(\Delta) < 0, \ \forall \ |\Delta| \le \delta. \tag{22}$$

Theorem 1: Consider the system (1) with performance cost in (6), controller (7), performance bound $\tau > 0$ and quantization density as ρ. Suppose the GCC problem has a solution, then there exists a solution to the QQPC problem. Conversely, if the QQPC problem has a solution, then for any arbitrarily small $\eta > 0$, the GCC problem for (22) has a solution with δ replaced with $\delta - \eta$.

Next lemma and theorem give necessary and sufficient conditions for the existence of the solution of the GCC problem.

Lemma 1: For system (1) and controller (7), (22) holds if and only if there exists a $\varepsilon > 0$, such that the following matrix inequality holds:

$$
\begin{bmatrix}
-\bar{P}^{-1} & \bar{A} & 0 & 0 & 0 \\
\bar{A}^T & -\bar{P} + \bar{S} + \varepsilon \bar{E}^T \bar{E} & \bar{C}^T & \sigma \hat{D}^T & \sigma \bar{B}^T \\
0 & \bar{C} & -R^{-1} & 0 & 0 \\
0 & \sigma \hat{D} & 0 & -R^{-1} & 0 \\
0 & \sigma \bar{B} & 0 & 0 & -\bar{P}^{-1}
\end{bmatrix}
+ \varepsilon^{-1} H H^T < 0, \tag{23}
$$

where $H = \begin{bmatrix} \bar{H} \\ 0 \\ \bar{D} \\ \sigma \bar{D} \\ \sigma \bar{H} \end{bmatrix}$, $\hat{D} = \begin{bmatrix} 0 & D_c C \end{bmatrix}$, $\bar{E} = \begin{bmatrix} 0 & C \end{bmatrix}$, $\bar{C} = \begin{bmatrix} C_c & D_c C \end{bmatrix}$, $\bar{B} = \begin{bmatrix} 0 & B_c C \\ 0 & B D_c C \end{bmatrix}$, $\bar{D} = \delta D_c$, $\bar{H} = \begin{bmatrix} \delta B_c \\ \delta B D_c \end{bmatrix}$.

Remark 1: To simplify (23), we note that (23) is equivalent to the following inequality:

$$
\begin{bmatrix}
-\bar{P}^{-1} & \bar{A} & 0 & 0 & 0 & \bar{H} \\
\bar{A}^T & \maltese & \bar{C}^T & \sigma \hat{D}^T & \sigma \bar{B}^T & 0 \\
0 & \bar{C} & -R^{-1} & 0 & 0 & \bar{D} \\
0 & \sigma \hat{D} & 0 & -R^{-1} & 0 & \sigma \bar{D} \\
0 & \sigma \bar{B} & 0 & 0 & -\bar{P}^{-1} & \sigma \bar{H} \\
\bar{H}^T & 0 & \hat{D}^T & \sigma \hat{D}^T & \sigma \bar{H}^T & -\varepsilon I
\end{bmatrix} < 0, \tag{24}
$$

with $\maltese = -\bar{P} + \bar{S} + \varepsilon \bar{E}^T \bar{E}$.

Define $K_1 = \begin{bmatrix} A_c \\ C_c \end{bmatrix}$, $K_2 = \begin{bmatrix} B_c \\ D_c \end{bmatrix}$, then the inequality (24) can be written in the following form:

$$
\begin{aligned}
&\Omega_0 + F_1^T K_1 W_1 + W_1^T K_1^T F_1 + F_2^T K_2 W_2 + W_2^T K_2^T F_2 \\
&< \ 0,
\end{aligned} \tag{25}
$$

with

$$
\Omega_0 = \begin{bmatrix}
-\bar{P}^{-1} & A_0 & 0 & 0 & 0 & 0 \\
A_0^T & \maltese & 0 & 0 & 0 & 0 \\
0 & 0 & -R^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & -R^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & -\bar{P}^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & -\varepsilon I
\end{bmatrix},
$$

$$
F_1^T = \begin{bmatrix} B_0 \\ 0 \\ I_{20} \\ 0 \\ 0 \\ 0 \end{bmatrix}, W_1 = \begin{bmatrix} 0 & I_{10} & 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
F_2^T = \begin{bmatrix} B_0 \\ 0 \\ I_{20} \\ \sigma I_{20} \\ \sigma B_0 \\ 0 \end{bmatrix}, W_2 = \begin{bmatrix} 0 & \bar{E} & 0 & 0 & 0 & \delta I \end{bmatrix}. \tag{26}
$$

Theorem 2: (25) holds if and only if

$$F_{10}^T(\Omega_0 + F_2^T K_2 W_2 + W_2^T K_2^T F_2) F_{10} < 0 \tag{27}$$

and

$$W_{10}^T(\Omega_0 + F_2^T K_2 W_2 + W_2^T K_2^T F_2) W_{10}, < 0 \tag{28}$$

where F_{10} and W_{10} denote the kernels of F_1 and W_1, respectively.

By computation, we get

$$
F_{10} = \begin{bmatrix}
I_{20}^T & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 \\
-B^T & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I
\end{bmatrix}, \tag{29}
$$

$$
W_{10} = \begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 \\
0 & I_{20}^T & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & I
\end{bmatrix}. \tag{30}
$$

Denote $\Omega_{01} = F_{10}^T \Omega_0 F_{10}$, $\Omega_{02} = W_{10}^T \Omega_0 W_{10}$, $F_{11}^T = F_{10}^T F_2^T$, $W_{11} = W_2 F_{10}$, $F_{12}^T = W_{10}^T F_2^T$, $W_{12} = W_2 W_{10}$, so we get that

$$
\Omega_{01} = \begin{bmatrix}
-I_{20}\bar{P}^{-1} I_{20}^T - B R^{-1} B^T & I_{20} A_0 & 0 & 0 & 0 \\
A_0^T I_{20}^T & \maltese & 0 & 0 & 0 \\
0 & 0 & -R^{-1} & 0 & 0 \\
0 & 0 & 0 & -\bar{P}^{-1} & 0 \\
0 & 0 & 0 & 0 & -\varepsilon I
\end{bmatrix}, \tag{31}
$$

$$\Omega_{02}=\begin{bmatrix} -\bar{P}^{-1} & A_0 I_{20}^T & 0 & 0 & 0 & 0 \\ I_{20}A_0^T & I_{20}\maltese I_{20}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & -R^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -R^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\bar{P}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon I \end{bmatrix}, \quad (32)$$

with

$$F_{11}^T = \begin{bmatrix} 0 & 0 & \sigma I_{20}^T & \sigma B_0^T & 0 \end{bmatrix}^T, \quad (33)$$
$$W_{11} = \begin{bmatrix} 0 & \bar{E} & 0 & 0 & \delta I \end{bmatrix}, \quad (34)$$
$$F_{12}^T = \begin{bmatrix} B_0^T & 0 & I_{20}^T & \sigma I_{20}^T & \sigma B_0^T & 0 \end{bmatrix}^T, \quad (35)$$
$$W_{12} = \begin{bmatrix} 0 & \bar{E}\begin{bmatrix} 0 \\ I \end{bmatrix} & 0 & 0 & 0 & \delta I \end{bmatrix}. \quad (36)$$

Then the inequalities of (25) are equivalent to

$$\Omega_{01} + F_{11}^T K_2 W_{11} + W_{11}^T K_2 F_{11} < 0$$
$$\Omega_{02} + F_{12}^T K_2 W_{12} + W_{12}^T K_2 F_{12} < 0 \quad (37)$$

Define $F_{21} = kerF_{11}$, $W_{21} = kerW_{11}$, $W_{21} = kerW_{11}$, $F_{22} = kerF_{12}$, $W_{22} = kerW_{12}$. By computation, we get that

$$F_{21} = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ -B^T & 0 & 0 & 0 \\ I_{20}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \quad (38)$$

,

$$W_{21} = kerW_{11} = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ \delta I_{20}^T & 0 & I_{10}^T & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ -C & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (39)$$

$$F_{22} = \begin{bmatrix} -\sigma I_{10}^T & I_{20}^T & -\sigma I_{20}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & -B^T & 0 & -\sigma I & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ I_{10}^T & 0 & I_{20}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}, \quad (40)$$

$$W_{22} = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ -\delta I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ C & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (41)$$

Denote

$$\bar{P} = \begin{bmatrix} X & X_1 \\ X_1^T & X_2 \end{bmatrix}, \bar{P}^{-1} = \begin{bmatrix} Y & Y_1 \\ Y_1^T & Y_2 \end{bmatrix}, \quad (42)$$

then the following theorem holds:

Theorem 3: The guaranteed cost control problem has a solution if and only if (19) holds subject to

$$\begin{bmatrix} -Y_2 - BR^{-1}B^T & 0 & A \\ 0 & -X & -X_1 \\ A^T & -X_1^T & -X_2 + S + \varepsilon C^T C \end{bmatrix} < 0, \quad (43)$$

$$\begin{bmatrix} -(1+\sigma^2)Y & \sigma Y_1 & -(1+\sigma^2)Y_1 & 0 & 0 \\ \sigma Y_1^T & -Y_2 - BR^{-1}B^T & \sigma Y_2 & -\sigma BR^{-1} & A \\ -(1+\sigma^2)Y_1^T & \sigma Y_2^T & -(1+\sigma^2)Y_2 & 0 & -\sigma A \\ 0 & -\sigma R^{-1}B^T & 0 & -(1+\sigma^2)R^{-1} & 0 \\ 0 & A^T & -\sigma A^T & 0 & -X_2 + S + \varepsilon C^T C \end{bmatrix} < 0, \quad (44)$$

$$\begin{bmatrix} \delta^2(-X_2 + S + \varepsilon C^T C) - \varepsilon C^T C & 0 & -\delta A^T \\ 0 & -Y & -Y_1 \\ -\delta A & -Y_1^T & -Y_2 \end{bmatrix} < 0. \quad (45)$$

For $S + \varepsilon C^T C$ is semi-positive, we can assume it can be decomposed in the form of $S + \varepsilon C^T C = M^T M$ for some M, under this assumption we get the following theorem.

Theorem 4: The LMIs of (43) to (45) are equivalent to

$$\begin{bmatrix} -I + MY_2M^T & MY_2A^T \\ AY_2M^T & -Y_2 - BR^{-1}B^T + AY_2A^T \end{bmatrix} < 0, \quad (46)$$

$$\begin{bmatrix} -\frac{1}{1+\sigma^2}[Y_2 + BR^{-1}B^T] & \frac{1}{1+\sigma^2}A \\ \frac{1}{1+\sigma^2}A^T & -X_2 + S + \varepsilon C^T C + \frac{\sigma^2}{1+\sigma^2}A^T X_2 A \end{bmatrix} < 0, \quad (47)$$

$$\delta^2(-X_2 + S + \varepsilon C^T C) - \varepsilon C^T C + \delta^2 A^T X_2 A < 0. \quad (48)$$

Remark 2: For M depends on ε which is a scalar that has to be found, so strictly speaking (46) is not a linear matrix inequality, and the solution to the GCC problem requires sweeping ε. But since it is a scalar parameter, it is not a major problem.

Theorem 4 gives the equivalent conditions for (22), next we want to give a characterization of the existence of the solution of the guaranteed cost control problem. For $\bar{P}$ and $\bar{P}^{-1}$ are defined as in (42), then the constraint that $\bar{P}\bar{P}^{-1} = I$ can be characterized by the following lemma.

Lemma 2: The constraint $\bar{P}\bar{P}^{-1} = I$ implies that

$$\begin{bmatrix} X_2 & I \\ I & Y_2 \end{bmatrix} > 0. \quad (49)$$

It is also known that, given X_2 and Y_2, $\bar{P}$ can be fully constrained. We thus have the following main results.

Theorem 5: The QQGC problem for (19) (20) has a solution if there exists X_2 Y_2 and ε satisfying the inequalities (46)-(48) and (49), and such that the resulting $\bar{P}$ satisfying trace$\bar{P} < \tau$.

IV. Conclusion

This paper studies the quadratic quantized guaranteed cost control problem for stochastic systems with multiplicative noise in the measurements. Using the sector bound approach to characterize the quantized error, the existence of the solution to the problem can be solved by the existence of the solution to the guaranteed cost control problem. Using Schur complement and the elimination lemma, we get the sufficient conditions to the problem solutions in terms of linear matrices inequalities.

V. Appendix

Proof of Theorem 1 It is easy to know that (22) implies (20), so if the GCC problem has a solution, then, there exists a solution to the QQPC problem. Next we prove that if the QQPC problem has a solution, then, for any given arbitrarily small, the GCC problem for $\varepsilon > 0$ for (22) has a solution for $|\Delta| \leq \delta - \eta$. To see this we assume that (20) holds but (22) fails. Then there exist some ξ_0 and Δ_0 with $E\begin{bmatrix} 0 & C(1+\gamma(t)) \end{bmatrix}\xi_0 \neq 0$ and $|\Delta_0| \leq \delta$ such that

$$E\xi_0^T\bar{\Omega}(\Delta_0)\xi_0 \geq 0. \tag{50}$$

If Δ_0 is a boundary point, that is $\Delta_0 = \delta$, then the GCC problem has a solution for $|\Delta| \leq \delta - \eta$ for all $\eta > 0$. In the sequel, we assume that Δ_0 is an interior point.

We claim that $E\begin{bmatrix} 0 & C(1+\gamma(t)) \end{bmatrix}\xi_0 \neq 0$. Suppose that $E\begin{bmatrix} 0 & C(1+\gamma(t)) \end{bmatrix}\xi_0 = 0$, then from (16) and (50) we can get that

$$\begin{aligned} &E\xi_0^T\bar{\Omega}(\Delta\begin{bmatrix} 0 & C(1+\gamma(t)) \end{bmatrix}\xi_0)\xi_0 = E\xi_0^T\Omega_0\xi \\ &= E\xi_0^T\bar{\Omega}(\Delta_0)\xi_0 \geq 0. \end{aligned} \tag{51}$$

which contradicts with (20), so $E\begin{bmatrix} 0 & C(1+\gamma(t)) \end{bmatrix}\xi_0 \neq 0$. For the strict convexity of $E\bar{\Omega}(\Delta)$, there exists Δ^1 with $|\Delta^1| \leq \delta - \eta_1$, for some $\eta_1 > 0$ such that

$$E\xi_0^T\bar{\Omega}(\Delta^1)\xi_0 \geq 0. \tag{52}$$

For it is continuous in ξ_0, we can perturb ξ_0 slightly such that (52) holds and with every element of $E\begin{bmatrix} 0 & C(1+\gamma(t)) \end{bmatrix}\xi_0 \neq 0$. Now for $\Delta(\alpha\begin{bmatrix} 0 & C(1+\gamma(t)) \end{bmatrix}\xi_0)$ covers $[-\delta, \delta]$ densely as α varies from $-\infty$ to ∞. Hence that $\alpha \neq 0$ such that

$$E\xi_0^T\bar{\Omega}(\Delta(\alpha\begin{bmatrix} 0 & C(1+\gamma(t)) \end{bmatrix}\xi_0))\xi_0 > 0. \tag{53}$$

Define $\xi_1 = \alpha\xi_0$, we get that

$$E\xi_1^T\bar{\Omega}(\Delta(\alpha\begin{bmatrix} 0 & C(1+\gamma(t)) \end{bmatrix}\xi_0))\xi_1 > 0 \tag{54}$$

which contradicts (20), which means Δ^0 cannot be an interior point. Hence, (20) implies (22) has a solution for $|\Delta| \leq \delta - \eta$.

Proof of Theorem 3: Using the elimination lemma to the inequalities (37), we can get that (37) hold if and only if

$$\begin{aligned} F_{21}^T\Omega_{01}F_{21} < 0 \\ W_{21}^T\Omega_{01}W_{21} < 0, \end{aligned} \tag{55}$$

and

$$\begin{aligned} F_{22}^T\Omega_{02}F_{22} < 0 \\ W_{22}^T\Omega_{02}W_{22} < 0. \end{aligned} \tag{56}$$

Plugging (31), (38) and (39) into the inequalities of (55), we can get (43) and the following inequality:

$$\begin{bmatrix} \delta^2(-X_2 + S + \varepsilon C^TC) - \varepsilon C^TC & \delta A^T & -\delta X_1^T \\ \delta A & -Y_2 - BR^{-1}B^T & 0 \\ -\delta X_1 & 0 & -X \end{bmatrix} < 0, \tag{57}$$

Multiply the first row and first column of (57) by δ^{-1}, then permuting its rows and columns, (57) can be converted into

$$\begin{bmatrix} -Y_2 - BR^{-1}B^T & 0 & A \\ 0 & -X & -X_1 \\ A^T & -X_1^T & -X_2 + S + \varepsilon C^TC - \delta^{-2}\varepsilon C^TC \end{bmatrix} < 0, \tag{58}$$

it is clear that this inequality is implied by (43) and thus it is not necessarily required. Plugging (32), (40) and (41) into the inequalities of (56), we can get (44) and (45).■

Proof of Theorem 4: Using Schur complement, (43) is equivalent to

$$\begin{bmatrix} -I & 0 & 0 & M \\ 0 & -Y_2 - BR^{-1}B^T & 0 & A \\ 0 & 0 & -X & -X_1 \\ M^T & A^T & -X_1^T & -X_2 \end{bmatrix} < 0, \tag{59}$$

then use Schur complement again, (59) is equivalent to

$$\begin{aligned} &- \begin{bmatrix} 0 & M \\ 0 & A \end{bmatrix} \begin{bmatrix} -X & -X_1 \\ -X_1^T & -X_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ M^T & A^T \end{bmatrix} \\ &+ \begin{bmatrix} -I & 0 \\ 0 & -Y_2 - BR^{-1}B^T \end{bmatrix} < 0, \end{aligned} \tag{60}$$

which is equivalent to (46).

For (44), multiplying $\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & -\frac{\sigma}{1+\sigma^2}B & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & -\frac{\sigma}{1+\sigma^2}B^T & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$ to the left and right side of it, we can get it is equivalent to

$$\begin{bmatrix} -(1+\sigma^2)Y & \sigma Y_1 & -(1+\sigma^2)Y_1 & 0 & 0 \\ \sigma Y_1^T & -Y_2 - \frac{1}{1+\sigma^2}BR^{-1}B^T & \sigma Y_2 & 0 & A \\ -(1+\sigma^2)Y_1^T & \sigma Y_2^T & -(1+\sigma^2)Y_2 & 0 & -\sigma A \\ 0 & 0 & 0 & -(1+\sigma^2)R^{-1} & 0 \\ 0 & A^T & -\sigma A^T & 0 & -X_2 + S + \varepsilon C^TC \end{bmatrix} < 0. \tag{61}$$

For $-(1+\sigma^2)R^{-1}$ is unrelated with $\bar{P}$ or $\bar{P}^{-1}$, so the forth row and forth column can be deleted, so we get

$$\begin{bmatrix} -(1+\sigma^2)Y & \sigma Y_1 & -(1+\sigma^2)Y_1 & 0 \\ \sigma Y_1^T & -Y_2 - \frac{1}{1+\sigma^2}BR^{-1}B^T & \sigma Y_2 & A \\ -(1+\sigma^2)Y_1^T & \sigma Y_2^T & -(1+\sigma^2)Y_2 & -\sigma A \\ 0 & A^T & -\sigma A^T & -X_2 + S + \varepsilon C^TC \end{bmatrix} < 0, \tag{62}$$

swap the second and third rows, swap the second and third columns, it is equivalent to

$$\begin{bmatrix} -(1+\sigma^2)Y & -(1+\sigma^2)Y_1 & \sigma Y_1 & 0 \\ -(1+\sigma^2)Y_1^T & -(1+\sigma^2)Y_2 & \sigma Y_2^T & -\sigma A \\ \sigma Y_1^T & \sigma Y_2 & -Y_2 - \frac{1}{1+\sigma^2}BR^{-1}B^T & A \\ 0 & -\sigma A^T & A^T & -X_2 + S + \varepsilon C^T C \end{bmatrix} < 0, \quad (63)$$

using Schur complement we can get

$$\begin{aligned} &+ \frac{1}{1+\sigma^2}\begin{bmatrix} \sigma Y_1^T & \sigma Y_2 \\ 0 & -\sigma A^T \end{bmatrix}\begin{bmatrix} Y & Y_1 \\ Y_1^T & Y_2 \end{bmatrix}^{-1}\begin{bmatrix} \sigma Y_1 & 0 \\ \sigma Y_2^T & -\sigma A \end{bmatrix} \\ &+ \begin{bmatrix} -Y_2 - \frac{1}{1+\sigma^2}BR^{-1}B^T & A \\ A^T & -X_2 + S + \varepsilon C^T C \end{bmatrix} < 0, \quad (64) \end{aligned}$$

which is equivalent to (47).

For (45) using Schur complement we can get it is equivalent to

$$\begin{aligned} &- \begin{bmatrix} 0 & -\delta A^T \end{bmatrix}\begin{bmatrix} -Y & -Y_1 \\ -Y_1^T & -Y_2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -\delta A \end{bmatrix} \\ &+ \delta^2(-X_2 + S + \varepsilon C^T C) - \varepsilon C^T C < 0, \end{aligned} \quad (65)$$

which is equivalent to (48). ■

Proof of Lemma 2: Since that $X_1^T Y_1 + X_2 Y_2 = I$, $X_1^T Y + X_2 Y_1^T = 0$, solving X_1^T from the second equation and plugging it into the first equation gives

$$X_2 = (Y_2 - Y_1^T Y^{-1} Y_1)^{-1} > Y_2^{-1} \quad (66)$$

from which we can get the linear matrix inequality (49). ■

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