Exploiting Structures of Nonlinear Parametric Perturbations for Robust Stability

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Abstract. The research on robust stability of uncertain systems has been undertaking for a number of years. We now seem to have a good understanding of the so-called linear (or affine) perturbations. The progress for tackling nonlinear perturbations has been relatively slow due to obvious difficulties. One approach to nonlinear perturbations is to convert the robust stability problem into a certain type of nonlinear programming problem. The deficiencies of this approach are the numerical inefficiency and the lack of analytical insights.

This paper is meant to emphasize the advantages of exploiting special structures of nonlinear perturbations for robust stability. This is motivated by the fact that in many practical situations nonlinear perturbations are parameterized in a simple manner, such as uncertainties in the zeros, poles and gains of a transfer function, serial interconnections of perturbed subsystems, and uncertain time-delay constants. We exam these commonly used structures of nonlinear perturbations and provide efficient tests for robust stability.

1 Introduction

Inspired by the seminal work of Kharitonov on Hurwitz invariance of the so-called interval polynomials [1], there has been significant progress in the parametric approach to robust stability analysis. In particular, the so-called *affine perturbation problem*, i.e., the robust stability problem of families of polynomials with coefficients subject to affine perturbations, has been deeply studied. On one side, many so-called *extreme-point results* have been developed. For example, Kharitonov's result for interval polynomials [1] and many Kharitonov-like results [2,3,4]. In recent papers by Rantzer [5,6], some elegant necessary and sufficient conditions are provided for a stability region to hold extreme-point results. On the other side, for affine perturbations and/or stability regions for which extreme point results do not hold, other numerically efficient results are available. The well-celebrated *edge theorem* by Bartlett, Hollot and Lin [7] and its variations [8,9,10] for polytopes of polynomials and quasipolynomials give solutions in the parameter space. The *frequency sweeping techniques*, initially introduced by Argoun [11] and Dasgupta [12], now have become very popular due to both its numerical efficiency, simple frequency domain interpretation, and, perhaps most importantly, its ability to compute the robust stability margin (i.e., the maximum size of perturbations, in certain sense, for preserving stability). For example, Barmish in [13] gave a frequency sweeping technique to check robust stability of a polytope of polynomials. A different frequency sweeping technique was given in [14] by the author to obtain the robust stability margin which is the maximum size of a polytope of polynomials for preserving robust stability. This technique was recently improved by Tsypkin and Polyak [15] who obtained an elegant closed-form expression for the robust stability margin.

Despite of the significant achievement in the affine perturbation problem, the progress in tackling nonlinear (and multilinear, in particular) perturbations has been much less successful. One approach to this problem is to treat it in its broadest generality, as is done in [16,17]. In [16], Vicino, Tesi and Milanese considered a very general class of nonlinear perturbations and proposed to compute the robust stability margin by using a nonlinear programming algorithm technique called *branch and bound*. The numerical features of the branch and bound technique was intensively studied by Balakrishnan, Boyd and Balemi [17]. This approach, although appealing due to its universality, has two limitations: First of all, it is often computationally intensive, especially when the number of parameters is large and/or the degree of polynomials is high; Secondly, it does not provide much analytic insight into the problem and its solution, i.e., the relationship between robust stability and the parameters cannot be revealed. An alternative approach is to consider particular parameterizations reflecting specific

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forms of structural information supplied by the modeling process. This allows verification of robust stability and computation of robust stability margin computationally less demanding. Examples of this approach include [18,19,20,21], which treat different types of multilinear perturbations by exploiting their specific multilinear dependence; and [22], which accounts for a special class of nonlinear perturbations.

This paper is meant to emphasize the importance of exploiting special structures of nonlinear perturbations for robustness analysis. Several special structures will be discussed in detail:

- **Uncertain real zeros, poles and gains [22].** For systems with independent parameter perturbations in real zeros, poles and gains, we explain that it is necessary and sufficient to check certain one-dimensional lines in the parameter space to determine the Hurwitz invariance of the family of systems. A striking phenomenon about these line segments is that their descriptions are independent of the parameter set and the multiplicities of the zeros and poles.

- **Value set characterization for cascaded plants [22].** Lots of physical plants consist of a cascade of subplants which may involve independent parameter perturbations. Furthermore, parameter perturbations in each subplant are usually in simple forms while their compound effects might be highly complex. Suppose each subplant admits affine perturbations, then the value set of the plant transfer function will be shown to be determined also by certain one-dimensional line segments in the parameter space. In this case, however, these line segments are "frequency dependent," i.e., they move in the parameter space when the plant transfer function is evaluated at different frequencies, or complex points in general. Despite of this frequency dependence feature, these line segments are simply characterized, and very useful in plotting value sets and, consequently, in determining robust stability.

- **Time-delay systems with an uncertain time-delay constant [23].** Uncertain time-delay constants are another type of nonlinear perturbations. Robust stability analysis of time-delay systems with uncertain time-delay constants is known to be difficult. However, for the case when the plant transfer function consists of a single uncertain time-delay constant, we show that the maximum time-delay for preserving robust stability can be simply determined, each in the case when the plant has additional affine perturbations and/or some additional known time-delay constants.

The results above are to be detailed in the sections to follow.

2 Uncertain Real Zeros, Poles and Gains

For illustrative purposes, let us consider a simple example of uncertain plant described by the following transfer function:

$$G(s,q) = K\frac{s+z}{(s+p_1)^2(s+p_2)} \tag{1}$$

with $q := (K, z, p_1, p_2) \in Q$, and

$$Q = [1,2] \times [1,2] \times [2,4] \times [3,4]\ . \tag{2}$$

When a unity feedback is applied to the above plant, the closed-loop stability is determined by the following characteristic polynomial:

$$p(s,q) = s^3 + (p_2 + 2p_1)s^2 + (p_1^2 + 2p_1p_2 + k)s + p_1^2p_2 + zk. \tag{3}$$

Nonlinear perturbations are obviously observed in the above polynomial. Thus, checking robust stability seems a difficult task. However, the technique we are going to present here, which was derived in [22], will tell us that the family of polynomials

$$p(s,Q) := \{p(s,q), q \in Q\} \tag{4}$$

is robustly stable (i.e., Hurwitz invariance) if and only if $p(s,q)$ is stable for all q on the edges of Q and on the following four line segments:

$$\{(K^*, z^*, p_1, p_2) : 3 \le p_1 = p_2 \le 4\},\ K^*, z^* = 1, 2. \tag{5}$$

Note that checking robust stability on a single line segment is a simple numerical task.

The class of parameter perturbations to be considered are generalized to allow the following transfer function:

$$G(s,q) = K\,H(s)\frac{(s+z_1)^{\mu_1}\cdots(s+z_m)^{\mu_m}}{(s+p_1)^{\nu_1}\cdots(s+p_n)^{\nu_n}} \tag{6}$$

where

$$q := (K, z_1, \cdots, z_m, p_1, \cdots, p_n) \tag{7}$$

is the parameter vector which belongs to the following hyperrectangle

$$Q = [\underline{K}, \bar{K}] \times [\underline{z}_1, \bar{z}_1] \times \cdots \times [\underline{p}_n, \bar{p}_n]\ . \tag{8}$$

Considering the above uncertain plant with feedback controller $C(s)$, our objective is to determine whether the closed-loop system is robustly stable, i.e., whether the characteristic equation

$$1 + C(s)G(s,q) = 0 \tag{9}$$

has zeros within the open left half plane for all $q \in Q$.

Given the bounding set Q as in (8), we denote by $\partial^k Q$ a k-dimensional boundary of Q. Note that a k-dimensional boundary of Q is a k-dimensional hyperrectangle obtained by setting all but k components of q at their extreme values, and that there are $2^k C^k_{n+m+1}$ k-dimensional boundaries. We define the following affine line in $\mathbf{R}^{n+m+1}$:

$$L = \{q : q \in R^{n+m+1}, q_1 = q_2 = \cdots = q_{n+m+1}\}. \quad (10)$$

In the result below the term *open projection* is used. Consider a k-dimensional boundary $\partial^k Q$ of Q. Note that there are $(n+m+1-k)$ elements of $\mathbf{q}$ fixed at extreme values on this boundary. Modify L by fixing these $(n+m+1-k)$ parameters in q to the extreme values as they assume in $\partial^k Q$. Then the intersection of the *interior* of $\partial^k Q$ and this modified affine line is called the *open projection* of L on $\partial^k Q$.

We then have the following result:

Theorem 1. *[22] Consider the family of transfer functions described in (6)-(8) and a controller $C(s)$, then the closed-loop system is stable for all $q \in Q$* if and only if *it is stable on the following line segments of Q:*

(a) *all the edges of Q and*

(b) *the open projections of the affine line L in (10) on all k-dimensional boundaries $\partial^k Q$ of Q with $K = \underline{K}$ or $\bar{K}$, $2 \le k \le n+m$, except those for which the following condition holds:*

$$\sum_{i \in \Psi_\mu} \mu_i = \sum_{i \in \Psi_\nu} \nu_i \quad (11)$$

where Ψ_μ (resp. Ψ_ν) is the subset of $\{1, \cdots m\}$ (resp. $\{1, \cdots n\}$) associated with $\partial^k Q$ such that z_i (resp. p_i) is a variable for all $i \in \Psi_\mu$ (resp. $i \in \Psi_\nu$).

We now return to the example in (1)-(2). Notice that in this case the critical subset of Q consists of all the edges of Q plus the line segments in (5). Therefore, according to Theorem 1, the robust stability of the closed-loop system is achieved if and only if it is stable for all q on the edges of Q and the line segments in (5).

The following result [22], derived from Theorem 1, gives an edge result for real zero-pole-gain perturbations.

Corollary 2. *Consider the family of transfer functions described in (6)-(8) and a controller $C(s)$. Suppose that the zeros and poles are non-overlapping, i.e., the open intervals $(\underline{z}_i, \bar{z}_i), (\underline{p}_j, \bar{p}_j)$, $1 \le i \le m$, $1 \le j \le n$, do not overlap. Then, the closed-loop system is robustly stable* if and only if *it is stable on every edge of Q.*

3 Value Sets of Cascaded Uncertain Plants

A general description for the transfer function of a cascaded uncertain plant is given by

$$G(s,q) = G_0(s) \prod_{i=1}^{\ell} p_i(s,q_i)^{k_i} \quad (12)$$

where $G_0(s)$ represents the unperturbed part of $G(s,q)$, $p_i(s,q_i)$ is the ith perturbed polynomial, k_i denotes the multiplicities of $p_i(s,q_i)$ and is allowed to be negative for capturing perturbations in the denominator of $G(s,q)$, q_i is the parameter vector of $p_i(s,q_i)$, and $q = [q_1, \cdots, q_\ell]$, which belongs to the following bounding set

$$Q = Q_1 \times \cdots \times Q_\ell , \quad (13)$$

is the parameter vector of the cascaded plant. The dependence of the polynomials $p_i(s,q_i)$ on q_i is assumed to be affine, i.e.,

$$p_i(s,q_i) = p_{i0}(s) + P_i(s) q_i \quad (14)$$

where $p_{i0}(s)$ is the nominal part and $P_i(s)$ is a polynomial matrix.

For example, the cascade of

$$G_1(s,q_1) = \frac{s+q_1}{s+1}$$

and

$$G_2(s,q_2,q_3) = \frac{1}{(s^2+3s+q_2)(s+q_3)}$$

gives

$$G(s,q) = \frac{1}{s+1}(s+q_1)(s^2+3s+q_2)^{-1}(s+q_3)^{-1} . \quad (15)$$

The value set of $G(s,q), q \in Q$, evaluated at $s = s_0$, is defined to be

$$G(s_0,Q) = \{G(s_0,q) : q \in Q\} . \quad (16)$$

Note that the value set of each $p_i(s,q_i), q_i \in Q_i$ is a convex polygon with edges corresponding to the edges of Q_i.

We first consider a special case where every q_i is a scalar, i.e., $\underline{q}_i \le q_i \le \bar{q}_i$, $i = 1, \cdots, \ell$. In this case, a general description for $G(s_0,q)$ is given by

$$G(s_0,q) = \Delta \prod_{i=1}^{\ell} (q_i + \alpha_i + j\beta_i)^{k_i} \quad (17)$$

where Δ is a complex constant, α_i and β_i are real constants. It is assumed that if $\beta_i = 0$ and $k_i < 0$, then the interval $[\underline{q}_i + \alpha_i, \bar{q}_i + \alpha_i]$ does not contain zero, i.e., $G(s_0,Q)$ is bounded. It is also assumed in the above that q_i does not vanish in the evaluation, which is just for simplicity.

Similar to the previous section, the following affine line in $\mathbf{R}^\ell$ plays an important role:

$$L_G := \{q = \rho[\beta_1 \cdots \beta_\ell]^T \dot{-} [\alpha_1 \cdots \alpha_\ell]^T : -\infty < \rho < \infty\}. \quad (18)$$

Theorem 3. *[22] Consider the hyperrectangle Q in $\mathbf{R}^\ell$ and the complex function $G(s_0, q)$ in (17). Suppose $G(s_0, Q)$ is bounded, then there exists a collection of line segments in Q from which $G(s_0, (Q))$ is mapped. These line segments consist of*

(a) *all the edges of Q and*

(b) *the open projections of the affine line L_G in (18) on all k-dimensional boundaries $\partial^k Q$ of $Q, 2 \leq k \leq \ell$, except those for which the following condition holds: Let Ψ_k be the subset of $\{1, 2, \cdots \ell\}$ associated with $\partial^k Q$ such that q_i is a variable for all $i \in \Psi_k$, then, either*

(i) *$\beta_i = 0$, for some $i \in \Psi_k$; or*

(ii) $\sum_{i \in \Psi_k} k_i = 0$.

Remark: In order to illustrate this theorem, especially conditions $(b - i)$ and $(b - ii)$, we consider the following example:

$$G(s_0, q) = \frac{(q_1 + 1 + j)}{(q_2 + 2 + 3j)(q_3 + 4)}.$$

Then projections on the interior of Q and the faces $q_1 = \underline{q}_1$ or $\bar{q}_1$ and the faces $q_2 = \underline{q}_2$ or $\bar{q}_2$ are excluded by the restriction (i) above; likewise the projections on the faces at which $q_3 = \underline{q}_3$ or $\bar{q}_3$ are excluded by (ii). Thus for this example the edges comprise the critical set.

We now return to the general case where q_i are not restricted to scalars. In this case, a result very similar to Theorem 3 also holds. This is due to the fact that the boundary of each value set $p_i^{k_i}(s_0, Q_i)$ is determined by the edges of Q_i which are obtained by fixing all but one parameters in q_i. Denote by $\mathcal{Q}^\ell$ the collection of all those ℓ-dimensional boundaries of Q obtained by allowing only one free parameter in each Q_i. Then, we have the following result:

Theorem 4. *[22] Given the cascaded uncertain plant in (12). Then, the boundary of the value set $G(s_0, Q)$ is contained in the union of the boundaries of $G(s_0, \partial^\ell Q), \partial^\ell Q \in \mathcal{Q}^\ell$. The latter can be determined by Theorem 3.*

4 Uncertain Time-delay Constant

Consider a single-input-single-output feedback system with open loop transfer function described by

$$W_\tau(s) = C(s)G(s)e^{-s\tau} \quad (19)$$

where $C(s)$ is the transfer function of the controller, $G(s)e^{-s\tau}$, the transfer function of the plant, and $e^{-s\tau}$, the uncertain time delay of the plant. In fact, $G(s)$ may include the known portion of the time delay. That is, the total time delay can be $T = T_0 + \tau$ with T_0 absorbed in $G(s)$. Moreover, $G(s)$ may consist of *known* time delays. The plant is subject to either parametric, unparametric uncertainties or both, and therefore, a general model for $G(s)$ is given by

$$G(s) = K[\bar{G}(s, q) + \Delta G(s)] \quad (20)$$

or

$$G(s) = K\bar{G}(s, q)[1 + L(s)] \quad (21)$$

where $K \in \mathbf{R}$ and $q = [q_1, \cdots, q_\ell] \in \mathbf{R}^\ell$ represent parametric uncertainties, and $\Delta G(s)$ and $L(s)$ characterize additive and multiplicative unparametric uncertainties, respectively. The parameter vector q usually lies in a bounding set Q similar to (13). On the other hand, $\Delta G(s)$ and $L(s)$ are bounded as follows:

$$|\Delta G(j\omega)| \leq \gamma\alpha(\omega) \quad (22)$$

and

$$|L(j\omega)| \leq \gamma\beta(\omega) \quad (23)$$

for some positive number γ which represents the size of the unparametric uncertainties, and $\alpha(\omega)$ and $\beta(\omega)$ for frequency weightings. It is further assumed that the closed-loop system is Hurwitz stable when $\tau = 0$.

The robust stability problem now is to determine the maximum time delay, τ_{max}, such that the stability of the closed-loop is preserved for all admissible uncertainties and all $0 \leq \tau < \tau_{max}$.

The result given below comes from the work [23] where various types of parametric and unparametric perturbations are considered and graphical methods are described for determining τ_{max}. To highlight the basic ideas for computing τ_{max}, we consider a special case and a general case.

Case 1: Uncertain Gain

We first consider a simple case where the open-loop plant is expressed by

$$G(s) = KG_0(s), \quad 0 < \underline{K} \leq K \leq \bar{K} \quad (24)$$

i.e., only the gain is uncertain.

Define Ω to be the set of critical radian frequencies at which the Nyquist plot of $W^0(s) = C(s)G_0(s)$ intersects the ring

$$R[1/\bar{K}, 1/\underline{K}] = \{c : c \in \mathbf{C}, 1/\bar{K} \leq |c| \leq 1/\underline{K}\}, \quad (25)$$

and denote by $\theta(\omega)$, for each $\omega \in \Omega$, the angle from $W^0(j\omega)$ to the negative real axis in the clockwise direction. Notice that the set Ω is often a simple interval. Then, we have

$$\tau_{max} = \min_{\omega \in \Omega} \frac{\theta(\omega)}{\omega}. \quad (26)$$

Case 2: General Case

Consider the general case where $G(s)$ is given by either (20) or (21) with $\bar{G}(s,q)$ by the following:

$$\bar{G}(s,q) = \frac{N(s,q)}{D(s,q)} = \frac{N_0(s) + \sum_{i=1}^{\ell} q_i N_i(s)}{D_0(s) + \sum_{i=0}^{\ell} q_i D_i(s)}, \quad q \in Q, \tag{27}$$

where $N_0(s)$ and $D_0(s)$ are nominal polynomials (or quasipolynomials if $G(s)$ include additional *known* time delays), and $N_i(s), D_i(s), i = 1, 2, \cdots, \ell$ are perturbation polynomials (or quasipolynomials).

We define the following value sets:

$$\bar{W}(j\omega, Q) = C(j\omega)\bar{G}(j\omega, Q)\,, \tag{28}$$

and

$$W(j\omega, Q, \gamma) = \{c : c = C(j\omega)[\bar{G}(j\omega, q) + \Delta G(j\omega)], \; q \in Q, |\Delta G(j\omega)| \leq \gamma\alpha(\omega)\} \tag{29}$$

for additive perturbations, or

$$W(j\omega, Q, \gamma) = \{c : c = C(j\omega)\bar{G}(j\omega, q)[1 + L(j\omega)], \; q \in Q, |L(j\omega)| \leq \gamma\beta(\omega)\} \tag{30}$$

for multiplicative perturbations.

The computation of τ_{max} basically involves plotting the value sets $W(j\omega, Q, \gamma)$. Once these value sets are plotted, τ_{max} can be obtained by checking whether they intersect the ring $R[1/\bar{K}, 1/\underline{K}]$ and computing $\theta(\omega)$ which is the minimum angle from $W(j\omega, Q, \gamma)$ to the ring, similar to Case 2.

The determination of $W(j\omega, Q, \gamma)$ takes two steps. The first step is to plot $W(j\omega, Q)$. Notice that $\bar{G}(s,q)$ is not a simple affine function of q, brute force searching of the boundary of $W(j\omega, Q)$ could be a difficult numerical task. However, this difficulty can be avoided by using a recent result in [24] which proved an edge result for uncertain plants like $\bar{G}(s,q)$. More specifically, the result of [24] shows that the boundary of $W(j\omega, Q)$ is determined by the edges of Q. Therefore, the value set $W(j\omega, Q)$ can be simply plotted.

The second step is to "blur" the boundary of $W(j\omega, Q)$ to obtain the boundary of $W(j\omega, Q, \gamma)$ according to (22) or (23), which can be done easily.

To illustrate these ideas, we consider the following example which involves an uncertain time constant and multiplicative perturbations:

$$G(s) = K\left[\frac{1}{s(s+a)} + \Delta G(s)\right] \tag{31}$$

with

$$0.5 =\leq K \leq 1, \quad 0.5 \lesssim a \leq 1.5, \tag{32}$$

$$|\Delta(j\omega)| \leq \sqrt{\frac{0.02}{\omega^2 + 1}} \tag{33}$$

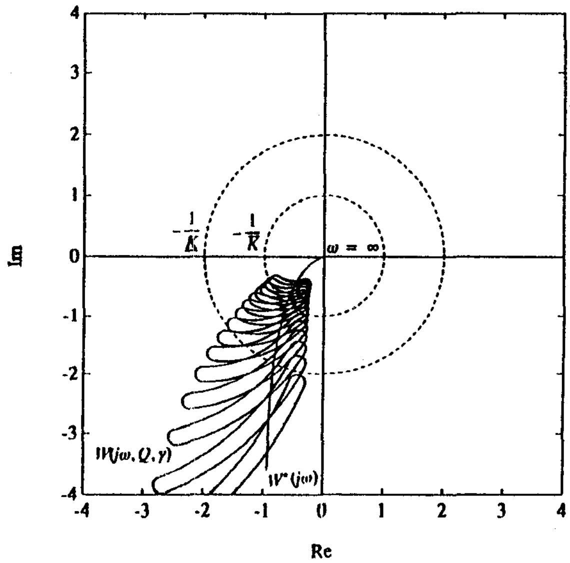

Figure 1: Plot for the example in Section 4

and $C(s) = 1$. The method described above is applied, and we obtain a sequence of value sets $W(j\omega, Q, \gamma)$, as shown in Figure 1. The corresponding τ_{max} is calculated to be 0.40 which occurs at $\omega = 0.96$.

5 Conclusion

In this paper, we have discussed several nonlinear perturbation structures for which robust stability can be relatively simply tested. These structures include uncertain real zeros, poles and gains; cascaded uncertain plants with each subplant admits affine perturbations in numerator and/or denominator; and uncertain time-delay systems with an uncertain time-delay constant. Further investigations are needed to discover new important structures of perturbations for robust stability analysis.

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