Actuator Saturation Control

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Chapter 1

Regional $\mathcal{H}_2$ performance Synthesis

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1.1. Introduction

Actuator saturation is inevitable in feedback control systems. If it is ignored in the design, a controller may "wind up" the actuator, possibly resulting in degraded performance or even instability. A classical approach to avoiding such undesirable behaviors is to add an anti-windup compensator to the original controller [1–7]. This approach has an advantage of providing control engineers with insights, for the role of each control component is clear.

On the other hand, higher performance may be expected if a controller is designed *a priori* considering the saturation effect. Lin, Saberi and their coworkers (see [8–10] and the references therein) have developed control design methods along this line using Riccati equations as a basic tool. Other Riccati equation approaches include [11–13]. Recent results also include those developed using the circle and the Popov criteria within the framework of linear matrix inequalities (LMIs) [14–18]. The idea is based on Lyapunov functions that are valid in a certain domain of the state space, and is very close in spirit to that of [10] mentioned above. For more detail and overview of the recent development, we refer the reader to [19, 20].

This chapter presents some methods for designing controllers to achieve a certain $\mathcal{H}_2$ (or linear quadratic) performance. In the design, the troublesome saturation nonlinearity is captured in a specific state space region by a sector-bound condition and the circle criterion is applied to guarantee stability (i.e. convergence to the origin) and the $\mathcal{H}_2$ performance. This will be called the circle analysis. When the state space region is restricted to those states that do not activate the saturation nonlinearity (i.e. the linear region), the sector bound reduces to a single line, resulting in a simpler but seemingly more conservative performance bound. This will be called the linear analysis.

In [21], it is shown that (i) the circle analysis can give a better estimate of the domain of attraction than the linear analysis for a given system, but (ii) the former provides no better result than the latter when they are used to design a controller that maximizes the estimated domain of attraction. This chapter first shows a result analogous to this for the case where our main concern is the domain of $\mathcal{H}_2$ performance rather than the domain of attraction. Thus, the "optimal" controller within the framework of circle analysis can be designed using simple linear analysis conditions. However, the second half of this chapter shows by numerical examples that the "optimal" controller thus designed may not be the best in terms of the actual $\mathcal{H}_2$ performance (or others such as settling time and overshoot) due to inherent conservatism of the $\mathcal{H}_2$ performance bound. It is illustrated by an example that the circle criterion can indeed be useful to improve the actual performance over the controller designed via the linear analysis.

We use the following notation. The set of $n \times m$ real matrices is denoted by $\mathbb{R}^{n\times m}$. For a matrix M, M^{T} denotes the transpose. For a vector x, x_i is the i^{th} entry of x. For vectors x and y, $x > y$ means that $x_i > y_i$ for all i, and similarly for $x \geq y$. For a symmetric matrix X, $X > 0$ ($X \geq 0$) means that X is positive (semi)definite. For a square matrix Y, $\mathsf{He}(Y) := Y + Y^{\mathsf{T}}$. Finally, a transfer function is denoted by

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := C(sI - A)^{-1}B + D.$$

1.2. Analysis

1.2.1. A general framework

Consider the feedback system depicted in Fig. 1, where $H(s)$ is a linear time-invariant (LTI) system given by

$$\dot{x} = \mathcal{A}x + \mathcal{B}u, \quad z = \mathcal{K}x, \quad e = \mathcal{C}x + \mathcal{D}u \tag{1.1}$$

and $\phi : \mathbb{R}^m \to \mathbb{R}^m$ is a saturation nonlinearity, i.e.

$$u = \phi(z) \quad \Leftrightarrow \quad u_i = \begin{cases} \alpha_i & (u_i > \alpha_i) \\ z_i & (|u_i| \leq \alpha_i) \\ -\alpha_i & (u_i < -\alpha_i) \end{cases} \tag{1.2}$$

where $\alpha \in \mathbb{R}^m$ is a given vector with positive entries.

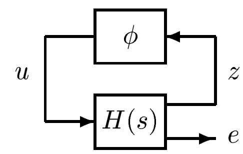

Figure 1: Feedback system with saturation nonlinearity

The set of state vectors $\mathbf{A}$ is called a *domain of attraction* if any state trajectory starting from a point in $\mathbf{A}$ converges to the origin as the time goes to infinity. Moreover, the set of state vectors $\mathbf{P}$ is called a *domain of performance* (with level γ) if it is a domain of attraction and any output e in response to $x(0) \in \mathbf{P}$ has its $\mathcal{L}_2$ norm squared less than or equal to γ. Our first objective is to characterize a domain of performance.

The following lemma is the basis for our analysis. A similar idea has been used in the literature on saturating control; see e.g. [15, 16].

Lemma 1.1. Consider the nonlinear system

$$\dot{x} = f(x), \quad e = g(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are continuous functions passing through the origin. Let $\mathbf{X}$ be a subset of $\mathbb{R}^n$ containing the origin. Assume existence and uniqueness of the solution to $\dot{x} = f(x)$ for any initial state $x(0) \in \mathbf{X}$. Suppose there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying, for some positive constants a, b, and c,

$$\begin{gathered} a\|x\|^2 \leq V(x) \leq b\|x\|^2, \quad {}^\forall x \in \mathbf{X}, \\ \frac{\partial V}{\partial x} f(x) + \frac{1}{\gamma} g(x)^\mathsf{T} g(x) \leq -c\|x\|^2, \quad {}^\forall x \in \mathbf{X} \\ \mathbf{P} \subset \mathbf{X} \end{gathered} \tag{1.3}$$

where

$$\mathbf{P} := \{\ x \in \mathbb{R}^n :\ V(x) \leq 1\ \}. \tag{1.4}$$

Then for each nonzero $x(0) \in \mathbf{P}$, the resulting response satisfies

$$x(t) \in \mathbf{P}, \quad \forall\, t \geq 0, \qquad \lim_{t\to\infty} x(t) = 0,$$

$$\int_0^\infty \|e(t)\|^2 dt < \gamma V(x(0)).$$

Proof. The fact that $\mathbf{P}$ is an invariant set and that $x(t)$ approaches the origin directly follows from Lemma 2 of [21]. From (1.3) and the system equations, we have

$$\|e(t)\|^2 \leq -c\gamma\|x(t)\|^2 - \gamma\frac{\partial V}{\partial x}\dot{x}(t).$$

Integrating from $t = 0$ to ∞ and noting the stability property, we have

$$\int_0^\infty \|e(t)\|^2 \leq -c\gamma \int_0^\infty \|x(t)\|^2 dt - \gamma\left(V(0) - V(x(0))\right) < \gamma V(x(0)).$$

■

1.2.2. Applications — linear and circle analyses

Applying Lemma 1.1 to our system

$$f(x) := \mathcal{A}x + \mathcal{B}\phi(\mathcal{K}x), \qquad g(x) := \mathcal{C}x + \mathcal{D}\phi(\mathcal{K}x),$$

one can obtain characterizations of the domain of performance $\mathbf{P}$. The characterization will depend on the choices of $V(x)$ and $\mathbf{X}$. We consider quadratic storage function $V(x) := x^{\mathsf{T}}Px$, elliptic domain of performance $\mathbf{P}$, and polytopic outer region $\mathbf{X}$ as follows:

$$\mathbf{P} := \{\ x \in \mathbb{R}^n :\ x^{\mathsf{T}}Px \leq 1\ \} \tag{1.5}$$

$$\mathbf{X} := \{\ x \in \mathbb{R}^n :\ |\mathcal{K}_i x| \leq \rho_i \quad (i = 1, \ldots, m)\ \}$$

where $\mathcal{K}_i$ is the ith row of matrix $\mathcal{K}$ and ρ_i are real scalars to be specified in the analysis.

If ρ_i are chosen as $\rho_i = \alpha_i$, then $\phi(\mathcal{K}x) = \mathcal{K}x$ for all $x \in \mathbf{X}^n$ and thus the above analysis becomes very simple. In this special case, we have the following linear analysis result.

Lemma 1.2. Let a symmetric matrix P and a scalar $\gamma > 0$ be such that

$$\mathsf{He}\begin{bmatrix} P(\mathcal{A} + \mathcal{B}\mathcal{K}) & 0 \\ \mathcal{C} + \mathcal{D}\mathcal{K} & -\gamma I \end{bmatrix} < 0 \tag{1.6}$$

$$\mathcal{K}_i^{\mathsf{T}}\mathcal{K}_i < \alpha_i^2 P \quad (i = 1, \ldots, m). \tag{1.7}$$

Then $\mathbf{P}$ in (1.5) is a domain of performance with level 2γ.

Proof. The result follows from Lemma 1.1 by noting that (1.3) and $\mathbf{P} \subset \mathbf{X}$ reduces to (1.6) and (1.7), respectively, where we redefine $\gamma/2$ to be γ. ∎

We now consider the case where $\rho_i \geq \alpha_i$. In this case, condition (1.3) reduces to the following:

$$2x^{\mathsf{T}} P(\mathcal{A}x + \mathcal{B}u) + \frac{1}{\gamma}(\mathcal{C}x + \mathcal{D}u)^{\mathsf{T}}(\mathcal{C}x + \mathcal{D}u) < 0 \tag{1.8}$$

holds for all $x \neq 0$ and u such that

$$u = \phi(\mathcal{K}x), \quad |\mathcal{K}_i x| \leq \rho_i, \quad \forall\, i = 1, \ldots, m. \tag{1.9}$$

Note that, if x and u satisfy (1.9), then

$$(u_i - \mathcal{K}_i x)(u_i - s_i \mathcal{K}_i x) \leq 0, \quad \forall\, i = 1, \ldots, m \tag{1.10}$$

holds where $s_i := \alpha_i/\rho_i$. This is easy to see once we notice the fact that the saturation nonlinearity ϕ will lie in the sector $[s_i, 1]$ when its input z_i is restricted by $|z_i| \leq \rho_i$ (see Fig. 2). Applying the circle criterion to the sector-bounded nonlinearity, we have the following:

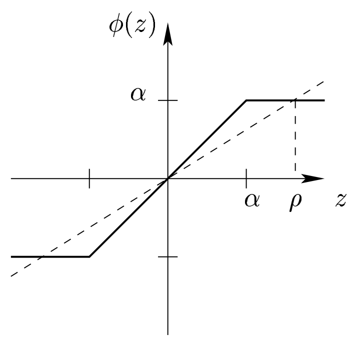

Figure 2: Sector bound for saturation nonlinearity

Lemma 1.3. Let a symmetric matrix P and a scalar $\gamma > 0$ be given. Suppose there exist diagonal matrices $0 \leq R < I$ and $T > 0$ such that

$$\mathsf{He} \begin{bmatrix} P(\mathcal{A} + \mathcal{B}\mathcal{K}) & P\mathcal{B} & 0 \\ -RT\mathcal{K} & -T & 0 \\ \mathcal{C} + \mathcal{D}\mathcal{K} & \mathcal{D} & -\gamma I \end{bmatrix} < 0 \tag{1.11}$$

$$\mathcal{K}_i^{\mathsf{T}} \mathcal{K}_i < \rho_i^2 P \quad (i = 1, \ldots, m) \tag{1.12}$$

where $\rho_i := \alpha_i/(1 - r_i)$ with r_i being the i^{th} diagonal entry of R. Then $\mathbf{P}$ in (1.5) is a domain of performance with level 2γ.

Proof. The result follows from Lemma 1.1 as discussed above. In particular, applying the S-procedure, a sufficient condition for (1.8) to hold for all $x \neq 0$ and w such that (1.9) is given by the existence of $t_i > 0$ $(i = 1, \ldots, m)$ satisfying

$$\begin{aligned} &2x^{\mathsf{T}} P(\mathcal{A}x + \mathcal{B}u) + \tfrac{1}{\gamma}(\mathcal{C}x + \mathcal{D}u)^{\mathsf{T}}(\mathcal{C}x + \mathcal{D}u) \\ &\qquad -\sum_{i=1}^{m} t_i(u_i - \mathcal{K}_i x)(u_i - s_i\mathcal{K}_i x) < 0, \end{aligned} \qquad \forall \begin{bmatrix} x \\ u \end{bmatrix} \neq 0.$$

It is straightforward to verify that this condition is equivalent to

$$\mathsf{He}\begin{bmatrix} P\mathcal{A} - \mathcal{K}^{\mathsf{T}} TS\mathcal{K} & P\mathcal{B} + \mathcal{K}^{\mathsf{T}} T & 0 \\ TS\mathcal{K} & -T & 0 \\ \mathcal{C} & \mathcal{D} & -(\gamma/2)I \end{bmatrix} < 0$$

where S and T are the diagonal matrices with $s_i := \alpha_i/\rho_i$ and t_i on the diagonal, respectively. A congruence transformation of this inequality leads to (1.11) by defining $R := I - S$ and redefining $\gamma/2$ to be γ. Finally, it can be verified that (1.12) implies $\mathbf{P} \subset \mathbf{X}$. ∎

It should be noted that the circle analysis in Lemma 1.3 reduces exactly to the linear analysis in Lemma 1.2 when $\rho = \alpha$. This can be seen once we notice that there exists a (sufficiently large) $T > 0$ satisfying (1.11) if and only if (1.6) holds, because $\rho = \alpha$ implies $R = 0$.

By the congruence transformation with $\mathsf{diag}(P^{-1}, T^{-1}, I)$ and by the Schur complement, it can be shown that the conditions in (1.11) and (1.12) are equivalent to

$$\begin{bmatrix} (\mathcal{A} + \mathcal{B}\mathcal{K})Q & \mathcal{B}V & 0 \\ -R\mathcal{K}Q & -V & 0 \\ (\mathcal{C} + \mathcal{D}\mathcal{K})Q & \mathcal{D}V & -\gamma I \end{bmatrix} < 0, \quad \rho_i^2 > \mathcal{K}_i Q \mathcal{K}_i^{\mathsf{T}} \tag{1.13}$$

where $Q := P^{-1}$ and $V := T^{-1}$. Thus, the largest estimate of the domain of performance is obtained by maximizing $\det(Q)$ subject to (1.13) over symmetric Q, diagonal V, and diagonal $0 \leq R < I$. This problem is difficult in general due to the product term $R\mathcal{K}Q$ which destroys the linearity of (1.13). However, if R is fixed, then the problem becomes a quasi-concave maximization [22] subject to LMI constraints and thus can be solved efficiently. This property is particularly appealing for the single input case, for the parameter R becomes scalar and its appropriate value can be found by a line search.

1.3. Synthesis

1.3.1. Problem formulation and a critical observation

Consider a linear time invariant system

$$\dot{\mathsf{x}} = A\mathsf{x} + Bu, \quad e = C\mathsf{x} + Du, \quad y = M\mathsf{x} \tag{1.14}$$

where $\mathsf{x}(t) \in \mathbb{R}^{\mathsf{n}}$ is the sate, $u(t) \in \mathbb{R}^m$ is the control input, $e(t) \in \mathbb{R}^\ell$ is the performance output (e.g., error signal), and $y(t) \in \mathbb{R}^k$ is the measured output. Suppose the actuator has a limited power and we have the following constraint on the magnitude of admissible control input:

$$|u_i(t)| \leq \alpha_i, \quad \forall\, t \geq 0, \quad i = 1, \ldots, m. \tag{1.15}$$

Our objective is to develop methods for designing a feedback controller that uses $y(t)$ to generate $u(t)$ satisfying the saturation constraint (1.15) such that the closed-loop system sustains a high $\mathcal{H}_2$ performance in a large region in the state space.

Let us formulate the following:

Synthesis Problem: Let the plant (1.14), the controller order $\mathsf{n}_c \geq 0$, and a desired domain of performance $\mathbf{P} \subset \mathbb{R}^n$ ($n := \mathsf{n} + \mathsf{n}_c$) be given. Design a controller such that:

(a) The saturation constraint (1.15) is satisfied;

(b) All the closed-loop states converge to the origin as the time goes to infinity whenever the initial state belongs to $\mathbf{P}$;

(c) The worst case $\mathcal{H}_2$ performance measure

$$J := \sup_{x(0) \in \mathbf{P}} \int_0^\infty \|e(t)\|^2 dt \tag{1.16}$$

is minimized.

This problem is difficult, and no exact solution is available till date. We will address this problem conservatively, using the analysis results developed in the previous section. Consequently, the desired domain of performance is restricted to the class of ellipsoids specified by (1.5) where $P = P^{\mathsf{T}} > 0$ and n is the dimension of the closed-loop state space, and the controller structure will be some combination of the saturation nonlinearity ϕ in (1.2) and a linear time invariant system.

We consider two classes of nonlinear controllers that meet the specification (a) of the Synthesis Problem, i.e. the saturation constraint (1.15). One is given by

$$u = \phi(z), \quad z = K_s(s)y \tag{1.17}$$

and the other is given by

$$u = \phi(z), \quad z = K_a(s) \begin{bmatrix} y \\ z - u \end{bmatrix}. \tag{1.18}$$

The closed-loop system with these controllers are depicted in Fig. 3 where the dashed part is absent and $K(s) := K_s(s)$ for (1.17) while the dashed part is present and $K(s) := K_a(s)$ for (1.18). The additional dashed feedback loop has been suggested in the literature to account for the actuator saturation effect and used as a basis for anti-windup compensation. Hereafter, we shall refer to (1.17) as the directly-saturating (DS) controller and to (1.18) as the anti-windup (AW) controller.

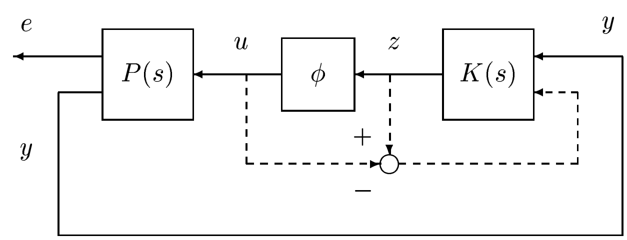

Figure 3: Feedback control system

Recall that Lemmas 1.2 and 1.3 provide sufficient conditions for a closed-loop system to meet the specifications (a) and (b) in the Synthesis Problem and to satisfy $J < 2\gamma$ where J is the worst case performance in (1.16) and γ is a given scalar. Hence, an approximation of the original Synthesis Problem would be to minimize γ over the class of DS or AW controllers subject to the conditions given by either Lemma 1.2 or Lemma 1.3. Thus we have four possible problem formulations.

Table 1.1 shows the four formulations, where each of γ's denotes the optimal performance bound achievable by the corresponding problem formulation. Clearly, the class of AW controllers is larger than the class of DS controllers, and the sufficient condition in Lemma 1.3 is no more conservative than that in Lemma 1.2. Hence we immediately see the following

	Directly-saturating Control (1.17)	Anti-windup Control (1.18)
Linear Analysis Lemma 1.2	$\gamma_{\ell s}$	$\gamma_{\ell a}$
Circle Criterion Lemma 1.3	γ_{cs}	γ_{ca}

Table 1.1: Possible problem formulations

relations:

$$\begin{cases} \gamma_{\ell s} \geq \gamma_{\ell a} \geq \gamma_{ca} \\ \gamma_{\ell s} \geq \gamma_{cs} \geq \gamma_{ca} \end{cases} .$$

It is tempting to expect that strict inequalities hold in the above relations. Surprisingly, however, it turns out that the achievable performance bounds are all equal:

$$\gamma_{\ell s} = \gamma_{\ell a} = \gamma_{cs} = \gamma_{ca}$$

and hence neither the circle criterion nor the anti-windup structure improves the achievable guaranteed performance within the control synthesis framework discussed here. The following formally states this result.

Theorem 1.1. Consider the plant $P(s)$ in (1.14) and the control system in Fig. 3. Fix the controller order n_c to be any nonnegative integer. Let a desired domain of performance $\mathbf{P}$ in (1.5) and a desired performance bound $\gamma > 0$ be given. Suppose there exists an AW controller (1.18) such that the corresponding closed-loop transfer function $H(s)$ in Fig. 1 is partially strictly proper from u to z and satisfies the condition in Lemma 1.3 (the circle criterion). Then there exists a DS controller (1.17) such that the corresponding closed-loop system satisfies the condition in Lemma 1.2 (the linear analysis). Consequently, we have $\gamma_{\ell s} = \gamma_{ca}$. Moreover, one such DS controller is given by (1.17) with $K_s(s)$ being the transfer function from y to u in Fig. 4.

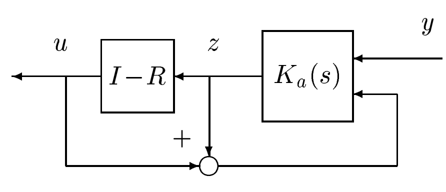

Figure 4: $K_s(s)$ of the DS controller

1.3.2. Proof of Theorem 1.1

Let $P = P^{\mathsf{T}} \in \mathbb{R}^{n\times n}$, $\gamma \in \mathbb{R}$, and a state space realization of the transfer function $K_a(s)$

$$K_a(s) = \left(\begin{array}{c|c} A_a & B_a \\ \hline C_a & D_a \end{array}\right) = \left(\begin{array}{c|cc} A_a & B_{a1} & B_{a2} \\ \hline C_a & D_{a1} & D_{a2} \end{array}\right)$$

with state vector $\mathsf{x}_c(t) \in \mathbb{R}^{\mathsf{n}_c}$ be given, where B_a and D_a are partitioned compatibly with the dimensions of the two input vectors in (1.18).

If we view the closed-loop system in Fig. 3 with $K(s) := K_a(s)$ as a special case of the general feedback system in Fig. 1, we see that $H(s)$ is determined by $P(s)$ and $K_a(s)$. Denote this particular transfer function by $H_a(s)$. The analysis results in Section 1.2 require that the upper $m \times m$ block of $H_a(s)$ is strictly proper. It can be verified that

$$H_a(\infty) = \begin{bmatrix} 0 \\ D \end{bmatrix} - \begin{bmatrix} I \\ 0 \end{bmatrix} (I - D_{a2})^{-1} D_{a2} = \begin{bmatrix} (D_{a2} - I)^{-1} D_{a2} \\ D \end{bmatrix}$$

and hence D_{a2} must be zero. A state space realization of $H_a(s)$ is then given by

$$H_a(s) = \left(\begin{array}{cc|c} A & 0 & B \\ (B_{a1} + B_{a2}D_{a1})M & A_a + B_{a2}C_a & -B_{a2} \\ \hline D_{a1}M & C_a & 0 \\ C & 0 & D \end{array}\right) =: \left(\begin{array}{c|c} \mathcal{A}_a & \mathcal{B}_a \\ \hline \mathcal{K}_a & 0 \\ \mathcal{C}_a & \mathcal{D}_a \end{array}\right),$$

where the closed-loop state vector is $x := [\ \mathsf{x}^{\mathsf{T}} \quad \mathsf{x}_c^{\mathsf{T}}\ ]^{\mathsf{T}}$.

Suppose there exist diagonal matrices $0 \le R < I$ and $T > 0$ such that (1.11) and (1.12) hold. Let $K_s(s)$ be defined as the mapping from y to u in Fig. 4 with its state space realization

$$K_s(s) = \left(\begin{array}{c|c} A_a + B_{a2}RC_a & B_{a1} + B_{a2}RD_{a1} \\ \hline (I-R)C_a & (I-R)D_{a1} \end{array}\right) =: \left(\begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array}\right)$$

where the state vector is x_c. Denote the corresponding closed-loop transfer function by $H_s(s)$. Its state space realization is given by

$$H_s(s) = \left(\begin{array}{cc|c} A & 0 & B \\ B_sM & A_s & 0 \\ \hline D_sM & C_s & 0 \\ C & 0 & D \end{array}\right) =: \left(\begin{array}{c|c} \mathcal{A}_s & \mathcal{B}_s \\ \hline \mathcal{K}_s & 0 \\ \mathcal{C}_s & \mathcal{D}_s \end{array}\right), \quad x = \begin{bmatrix} \mathsf{x} \\ \mathsf{x}_c \end{bmatrix} \qquad (1.19)$$

We show that this $H_s(s)$ satisfies linear analysis conditions (1.6) and (1.7).

Noting that

$$\mathcal{K}_s = (I - R)\mathcal{K}_a,$$

condition (1.12) with $\mathcal{K} := \mathcal{K}_a$ implies (in fact is equivalent to) (1.7) with $\mathcal{K} := \mathcal{K}_s$. Also note that satisfaction of (1.11) by $H_a(s)$ implies

$$\mathsf{He}\; N^{\mathsf{T}} \begin{bmatrix} P(\mathcal{A}_a + \mathcal{B}_a\mathcal{K}_a) & P\mathcal{B}_a & 0 \\ -RT\mathcal{K}_a & -T & 0 \\ \mathcal{C}_a + \mathcal{D}_a\mathcal{K}_a & \mathcal{D}_a & -\gamma I \end{bmatrix} N < 0$$

$$\Rightarrow \quad \mathsf{He} \begin{bmatrix} P(\mathcal{A}_a + \mathcal{B}_a(I - R)\mathcal{K}_a) & 0 \\ \mathcal{C}_a + \mathcal{D}_a(I - R)\mathcal{K}_a & -\gamma I \end{bmatrix} < 0$$

where

$$N := \begin{bmatrix} I & 0 \\ T^{-1}\mathcal{B}_a^{\mathsf{T}}P & T^{-1}\mathcal{D}_a \\ 0 & I \end{bmatrix}.$$

It is tedious but straightforward to verify that

$$\mathcal{A}_a + \mathcal{B}_a(I - R)\mathcal{K}_a = \mathcal{A}_s + \mathcal{B}_s\mathcal{K}_s, \quad \mathcal{C}_a + \mathcal{D}_a(I - R)\mathcal{K}_a = \mathcal{C}_s + \mathcal{D}_s\mathcal{K}_s.$$

Hence we see that $H_s(s)$ satisfies (1.6) as well. This completes the proof.

1.3.3. Fixed-gain control

In view of Theorem 1.1, the circle criterion (Lemma 1.3) does not help to improve the $\mathcal{H}_2$ performance bound for the closed-loop system when designing a controller with actuator saturation. Hence we use the linear analysis result (Lemma 1.2) as a basis for developing a control design method. The following provides a necessary and sufficient condition for existence of an output feedback controller that yields a closed-loop system satisfying the linear analysis condition.

Theorem 1.2. Consider the feedback system given by the plant (1.14) and the DS controller (1.17). Suppose there exist symmetric matrices X and Y and matrices F, K and L such that

$$\mathsf{He} \begin{bmatrix} XA + FM & 0 \\ C + DLM & -\gamma I \end{bmatrix} < 0 \tag{1.20}$$

$$\mathsf{He} \begin{bmatrix} AY + BK & 0 \\ CY + DK & -\gamma I \end{bmatrix} < 0 \tag{1.21}$$

$$\begin{bmatrix} X & I & M^{\mathsf{T}}L_i^{\mathsf{T}} \\ I & Y & K_i^{\mathsf{T}} \\ L_iM & K_i & \alpha_i^2 \end{bmatrix} > 0, \quad (i = 1, \ldots, m) \tag{1.22}$$

where L_i and K_i are the ith row of L and K, respectively. Then the controller (1.17) with

$$K_s(s) := \left(\begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array} \right) \tag{1.23}$$

$$\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} := \begin{bmatrix} Z & XB \\ 0 & I \end{bmatrix}^{-1} \left(\begin{bmatrix} N & F \\ K & L \end{bmatrix} - \begin{bmatrix} XAY & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} -Y & 0 \\ MY & I \end{bmatrix}^{-1}$$

$$N := -(A + BLM)^{\mathsf{T}} - (C + DLM)^{\mathsf{T}}(CY + DK)/(2\gamma), \quad Z := X - Y^{-1}$$

has the following properties: For any initial closed-loop state vector $x := [\,\mathsf{x}^{\mathsf{T}} \;\; \mathsf{x}_c^{\mathsf{T}}\,]^{\mathsf{T}}$ (where x and x_c are the plant and the controller state, respectively) satisfying

$$\begin{bmatrix} \mathsf{x}(0) \\ \mathsf{x}_c(0) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} X & Z \\ Z & Z \end{bmatrix} \begin{bmatrix} \mathsf{x}(0) \\ \mathsf{x}_c(0) \end{bmatrix} \le 1, \tag{1.24}$$

the control input never saturates (i.e. $u = z$ in (1.17)), the state trajectory converges to the origin, and

$$\int_0^\infty \|e(t)\|^2 dt < 2\gamma.$$

Proof. With the controller in (1.17), the closed-loop system in Fig. 3 can be described by Fig. 1 with $H(s)$ given by $H_s(s)$ in (1.19) where A_s, B_s, C_s and D_s are the state space matrices of $K_s(s)$. We show that the conditions in Lemma 1.2 with this $H_s(s)$ can be equivalently transformed to (1.20)–(1.22) by standard change of variables and parameter elimination.

It can be shown (see e.g. [23]) that if there exists a controller (1.17) of some order satisfying conditions (1.6) and (1.7) for some P, then there also exists a controller of the same order as the plant satisfying the same conditions for another P. Moreover, the state coordinates of such a controller can always be chosen (see e.g. [24,25]) such that the conditions are satisfied with P of the following structure:

$$P = \begin{bmatrix} X & Z \\ Z & Z \end{bmatrix}.$$

Let

$$\Gamma := \begin{bmatrix} I & 0 \\ Y & -Y \end{bmatrix}, \quad Y := (X - Z)^{-1}.$$

Then we have the following identity:

$$\begin{bmatrix} \Gamma & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} P(\mathcal{A}_s + \mathcal{B}_s\mathcal{K}_s) \\ \mathcal{C}_s + \mathcal{D}_s\mathcal{K}_s \\ \mathcal{K}_s \end{bmatrix} \Gamma^{\mathsf{T}} = \begin{bmatrix} XA + FM & N \\ A + BLM & AY + BK \\ C + DLM & CY + DK \\ LM & K \end{bmatrix}$$

where we used the following change of variables [23, 25]

$$\begin{bmatrix} N & F \\ K & L \end{bmatrix} := \begin{bmatrix} XAY & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} Z & XB \\ 0 & I \end{bmatrix} \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} \begin{bmatrix} -Y & 0 \\ MY & I \end{bmatrix}.$$

It is then easy to verify that (1.7) and (1.22) are related by the congruence transformation by Γ and the Schur complement. Similarly, by congruence transformation with $\mathsf{diag}(\Gamma, I)$, (1.6) can be equivalently transformed to

$$\mathsf{He} \begin{bmatrix} XA + FM & N & 0 \\ A + BLM & AY + BK & 0 \\ C + DLM & CY + DK & -\gamma I \end{bmatrix} < 0.$$

It can further be shown that this condition is equivalent to (1.20) and (1.21) by eliminating N using the projection lemma [26, 27]. ■

In Theorem 1.2, the domain of performance is characterized by (1.24). Note that the region of plant initial state $\mathsf{x}(0)$ to yield the desired $\mathcal{H}_2$ performance is dependent upon the choice of the initial controller state $\mathsf{x}_c(0)$. Rewriting (1.24) as

$$\mathsf{x}(0)^{\mathsf{T}}(X - Z)\mathsf{x}(0) + (\mathsf{x}(0) + \mathsf{x}_c(0))^{\mathsf{T}} Z(\mathsf{x}(0) + \mathsf{x}_c(0)) \leq 1,$$

we see that the best choice of $\mathsf{x}_c(0)$ to maximize the plant state domain of performance is given by $\mathsf{x}_c(0) = -\mathsf{x}(0)$, in which case the $\mathcal{H}_2$ performance is guaranteed whenever

$$\mathsf{x}(0)^{\mathsf{T}} Y^{-1} \mathsf{x}(0) \leq 1.$$

Thus the plant state domain of performance can be maximized within our framework by solving the following quasi-convex optimization problem:

$$\max_{X,Y,F,K,L} \det(Y) \quad \text{subject to (1.20)–(1.22)}$$

It can be shown that optimal choices of X and F are given by $X := \sigma X_o$ and $F := \sigma F_o$ for sufficiently large $\sigma > 0$ where X_o and F_o are any matrices satisfying $\mathsf{He}(X_o A + F_o M) < 0$ and $X_o > 0$. With these choices of X and F, condition (1.20) is always satisfied and condition (1.22) reduces to

$$\begin{bmatrix} Y & K_i^{\mathsf{T}} \\ K_i & \alpha_i^2 \end{bmatrix} > 0, \quad (i = 1, \ldots, m). \tag{1.25}$$

Hence the above problem becomes

$$\max_{Y,K} \det(Y) \quad \text{subject to (1.21) and (1.25).}$$

It turns out that this is exactly the same problem as that for the state feedback synthesis. This makes sense because it means that if the initial plant state is known then the output feedback with an observer achieves the same plant domain of performance as that achievable by state feedback.

When the initial plant state is not known, we may simply choose the zero initial controller state $\mathsf{x}_c(0) = 0$. In this case, the domain of performance is characterized by

$$\mathsf{x}(0)^\mathsf{T} X \mathsf{x}(0) \leq 1.$$

Therefore, the domain with the largest volume is obtained by minimizing $\det(X)$ subject to (1.20)–(1.22). This is not a quasi-convex optimization problem and is difficult to solve. On may replace $\det(X)$ by $\mathsf{tr}(X)$ to approximate the volume, in which case the problem becomes convex.

Letting γ be arbitrarily large, we have the following regional stabilization result that has been obtained in [21].

Corollary 1.1. Consider the feedback system given by the plant (1.14) and the DS controller (1.17). Suppose there exist symmetric matrices X and Y and matrices F, K and L such that

$$\mathsf{He}(XA + FM) < 0 \tag{1.26}$$

$$\mathsf{He}(AY + BK) < 0 \tag{1.27}$$

and (1.22) hold. Then the controller (1.17) with (1.23) has the following properties: For any initial closed-loop state vector satisfying (1.24), the control input never saturates, and the state trajectory converges to the origin.

We can give an alternative proof of the semi-global stabilization result in [19] using Corollary 1.1. That is, assuming that all the eigenvalues of A are in the closed left half plane and that those eigenvalues on the imaginary axis, if any, are simple[1], one can show that $\|X\|$ can be made arbitrarily small, indicating that the initial plant state region, with guaranteed state convergence to the origin, can be arbitrarily large.

Let $X_o > 0$, $Y_o > 0$, F_o and K_o be such that (1.26) and (1.27) hold. Such matrices exist if and only if (A, B, C) is a stabilizable and detectable triple. By the assumption on the purely imaginary eigenvalues, there exists $W > 0$ such that

$$AW + WA^\mathsf{T} \leq 0.$$

[1]The assumption on the purely imaginary eigenvalues is not required in the semi-global stabilization result of [19] but our proof below does.

We can assume that $W > X_o$ without loss of generality by scaling W properly. Then it can readily be verified that

$$X := \epsilon X_o, \quad F := \epsilon F_o, \quad Y := W/\epsilon + Y_o, \quad K := K_o, \quad L = 0$$

satisfy conditions (1.26), (1.27) and (1.22) for sufficiently small $\epsilon > 0$. Since $\|X\|$ can be arbitrarily close to zero, the domain of attraction for the plant state can be arbitrarily large, assuming zero initial controller state.

We now consider the state feedback problem. The result basically follows from Theorem 1.2 by letting $M = I$.

Corollary 1.2. Let Q, K and γ be such that

$$\mathsf{He}\begin{bmatrix} AQ+BK & 0 \\ CQ+DK & -\gamma I \end{bmatrix} < 0, \qquad \begin{bmatrix} Q & K_i^{\mathsf{T}} \\ K_i & \alpha_i^2 \end{bmatrix} > 0. \tag{1.28}$$

Let $u = \mathcal{K}x$ with $\mathcal{K} := KQ^{-1}$ be the state feedback controller. Suppose the initial state satisfies

$$x(0)^{\mathsf{T}} Q^{-1} x(0) \leq 1. \tag{1.29}$$

Then we have $|u_i(t)| \leq \alpha_i$ for all $t \geq 0$ and

$$\int_0^\infty \|e(t)\|^2 dt < 2\gamma.$$

Proof. It is easy to verify that the closed-loop system with the indicated state feedback controller satisfy conditions (1.6) and (1.7) with

$$\mathcal{A} := A, \quad \mathcal{B} := B, \quad \mathcal{C} := C, \quad \mathcal{D} := D, \quad \mathcal{K} := KQ^{-1}, \quad P := Q^{-1}.$$

∎

In general, there exist Q and K satisfying (1.28) for any fixed value of $\gamma > 0$, provided (A, B) is stabilizable. This can be seen as follows. Stabilizability of (A, B) implies existence of Q_o and K_o such that $\mathsf{He}(AQ_o + BK_o) < 0$ and $Q_o > 0$. Since the second condition in (1.28) is equivalent to $\alpha_i^2 Q > K_i^{\mathsf{T}} K_i$, there exists a sufficiently small $\epsilon > 0$ such that $Q := \epsilon Q_o$ and $K := \epsilon K_o$ satisfy this condition. Now, for these Q and K, there exists a sufficiently large $\gamma > 0$ such that the first condition in (1.28) holds. Thus we have shown that there is a triple (Q, K, γ) satisfying (1.28) and $Q > 0$ whenever (A, B) is stabilizable. Let (Q_o, K_o, γ_o) be redefined to be one such triple. For $\gamma > \gamma_o$, it is clear that (Q_o, K_o, γ) satisfies (1.28). On the other hand, if $\gamma < \gamma_o$, then it can be verified that $(\sigma Q_o, \sigma K_o, \gamma)$ satisfies (1.28) where $\sigma := \gamma/\gamma_o$. In general, the more stringent the performance requirement (i.e. smaller γ), the smaller the domain of performance (i.e. smaller Q).

A state feedback controller that maximizes the domain of performance for a given level γ can be designed as follows: Fix $\gamma > 0$ and solve the following:

$$\max_{Q,K} \ \det(Q) \quad \text{subject to} \quad (1.28).$$

This is a quasi-concave maximization problem which can be solved efficiently. Once we find a solution (Q, K), a control gain is calculated as $\mathcal{K} = KQ^{-1}$.

1.3.4. Switching control

The performance of a fixed-gain state feedback controller can be improved by introducing a switching logic structure into the controller. The basic idea is as follows [13, 28]: Prepare a set of feedback gains $\mathcal{K}_k$ $(k = 0, \ldots, q)$ such that a certain performance is guaranteed by the fixed-gain control law $u = \phi(\mathcal{K}_k x)$ in the state space domain $\mathbf{P}_k \subset \mathbb{R}^n$. The first gain $\mathcal{K}_0$ is designed to yield a sufficiently large domain of performance $\mathbf{P}_0$ to cover the possible region of initial states. The other gains are determined so that the resulting domains of performances are nested:

$$\mathbf{P}_{k+1} \subset \mathbf{P}_k \quad (k = 0, \ldots, q)$$

where $\mathbf{P}_{q+1} := \{ \ 0 \ \}$. The control gains are switched in accordance with the following logic:

$$u = \phi(\mathcal{K}_k x) \quad (\text{when } x \in \mathbf{P}_k \text{ and } x \notin \mathbf{P}_{k+1}) \quad (k = 0, \ldots, q). \tag{1.30}$$

This strategy improves performance by successively switching the control gain from a low gain to a high gain as the state gets closer to the origin.

An important question is: How much is the performance improved by switching? If the gains are designed within the framework of Lemma 1.1, each gain $\mathcal{K}_k$ satisfies (1.3) for some $c > 0$, V_k and γ_k, and guarantees (without switching) the performance bound

$$\int_0^\infty \|e(t)\|^2 dt < \gamma_k V_k(x(0))$$

whenever $x(0) \in \mathbf{P}_k$. Now, suppose that the gains are switched in accordance with (1.30). Let t_k $(k = 1, \ldots, q)$ be the time instants when the switchings occur and define $t_0 := 0$, $t_{q+1} := \infty$, and $x_k := x(t_k)$

$(k = 0, \ldots, q+1)$. Then, from (1.3),

$$\int_0^\infty \|e(t)\|^2 dt = \sum_{k=0}^{q} \int_{t_k}^{t_{k+1}} \|e(t)\|^2 dt < \sum_{k=0}^{q} \gamma_k (V_k(x_k) - V_k(x_{k+1})).$$

If in particular $V_k(x)$ is given by a quadratic function $V_k(x) := x^\mathsf{T} P_k x$ with $P_k = P_k^\mathsf{T} > 0$,

$$\begin{aligned}\int_0^\infty \|e(t)\|^2 dt \; &< \; \sum_{k=0}^{q} \gamma_k (x_k^\mathsf{T} P_k x_k - x_{k+1}^\mathsf{T} P_k x_{k+1}) \\ &= \; \gamma_0 x_0^\mathsf{T} P_0 x_0 + \sum_{k=1}^{q} (\gamma_k - \gamma_{k-1} x_k^\mathsf{T} P_{k-1} x_k).\end{aligned}$$

For each $k = 1, \ldots, q$, by definition $x_k^\mathsf{T} P_k x_k = 1$ holds and hence the value of $x_k^\mathsf{T} P_{k-1} x_k$ is bounded below by

$$\min_x \{ \; x^\mathsf{T} P_{k-1} x : \; x^\mathsf{T} P_k x = 1 \; \} = \lambda_{\min}(P_k^{-1} P_{k-1})$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue. Thus we have the following $\mathcal{H}_2$ performance bound for the switching controller:

$$\int_0^\infty \|e(t)\|^2 dt < \gamma_0 + \sum_{k=1}^{q} (\gamma_k - \gamma_{k-1} \lambda_{\min}(P_k^{-1} P_{k-1})) \tag{1.31}$$

whenever $x(0) \in \mathbf{P}_0$.

Specializing the general idea to the linear analysis case, we have the following result. We consider for brevity the single input case only.

Theorem 1.3. Consider the system in (1.14) with $M = I$ and $m = 1$. Suppose that matrices $Q_k = Q_k^\mathsf{T}$ and K_k, and scalars γ_k and μ_k satisfy

$$\mathsf{He} \begin{bmatrix} AQ_k + BK_k & 0 \\ CQ_k + DK_k & -\gamma_k I \end{bmatrix} < 0, \qquad \begin{bmatrix} Q_k & K_k^\mathsf{T} \\ K_k & \alpha^2 \end{bmatrix} > 0, \tag{1.32}$$

for $k = 0, \ldots, q$, and

$$\frac{\gamma_k + \mu_k}{\gamma_{k-1}} Q_{k-1} < Q_k < Q_{k-1} \tag{1.33}$$

for $k = 1, \ldots, q$. Then the switching controller

$$u = \mathcal{K}_k x \quad (\text{when } x^\mathsf{T} Q_k^{-1} x \le 1 < x^\mathsf{T} Q_{k+1}^{-1} x) \quad (k = 0, \ldots, q)$$

with $\mathcal{K}_k := K_k Q_k^{-1}$ and $Q_{q+1}^{-1} := \infty I$ yields $|u(t)| \leq \alpha$ for all $t \geq 0$ and

$$\int_0^\infty \|e(t)\|^2 dt < 2\left(\gamma_0 - \sum_{k=1}^{q} \mu_k\right) \tag{1.34}$$

whenever $x(0)^\top Q_0^{-1} x(0) \leq 1$.

Proof. The result basically follows from the preceding argument and the fixed-gain state feedback synthesis result of Corollary 1.2 with the change of variables $Q_k := P_k^{-1}$ and $K_k := \mathcal{K}_k Q_k$. The additional constraint $Q_k < Q_{k-1}$ in (1.33) is imposed to guarantee the nesting property $\mathbf{P}_k \subset \mathbf{P}_{k-1}$. The performance bound can be shown as follows. The second term on the right hand side of (1.31) is bounded above by $-\mu_k$ if and only if

$$\gamma_k + \mu_k < \gamma_{k-1} \lambda_{\min}(Q_{k-1}^{-1/2} Q_k Q_{k-1}^{-1/2})$$

or equivalently,

$$(\gamma_k + \mu_k) Q_{k-1} < \gamma_{k-1} Q_k.$$

Redefining $\gamma_k/2$ and $\mu_k/2$ to be γ_k and μ_k, respectively, we have the first inequality in (1.33) and the performance level is bounded as in (1.34). ∎

In view of Theorem 1.3, the switching mechanism in the controller improves the closed-loop performance bound by $2\mu := 2\sum_{k=1}^{q} \mu_k$, for the performance bound with the fixed-gain controller $u = \mathcal{K}_0 x$ is $2\gamma_0$. The best switching controller within this framework results when $\gamma_0 - \mu$ is minimized over the variables Q_k, K_k, γ_k and μ_k $(k = 0, \ldots, q)$ subject to the constraints in Theorem 1.3. This problem is nonconvex and it is difficult to compute the globally optimal solution. Hence, we propose a successive convex optimization to find a reasonable switching controller as follows.

Switching Control Design Algorithm:

1. Design an initial feedback gain $\mathcal{K}_0$ as follows. Fix γ and maximize $\det(Q)$ over K, Q and γ subject to (1.28). Let the optimizers be K_0, Q_0, and γ_0 and define $\mathcal{K}_0 := K_0 Q_0^{-1}$. Initialize k to be $k = 1$.

2. Find $\mathcal{K}_k$ as follows. Fix K_{k-1}, Q_{k-1}, and γ_{k-1}, and maximize μ_k over K_k, Q_k, and γ_k subject to constraints (1.32) and (1.33).

3. If $k = q$ then stop where q is the number of switchings specified in advance. Otherwise let $k \leftarrow k + 1$ and go to 2.

The switching controller thus obtained does not optimize the overall performance. However, the gain $\mathcal{K}_k$ is chosen so that the performance bound of the switching controller consisting of $\mathcal{K}_0, \ldots, \mathcal{K}_k$ is optimized for given gains $\mathcal{K}_0, \ldots \mathcal{K}_{k-1}$. With this compromise, each gain $\mathcal{K}_k$ can be obtained by solving the convex optimization problem defined in Step 2.

One can develop a similar design algorithm using the circle criterion instead of the linear analysis as has been done above. From Theorem 1.1 and its proof, however, it readily follows that the use of circle criterion does not improve the performance bound in (1.34). It will be illustrated later, in contrast, that the circle criterion indeed improves the actual performance for some cases.

1.4. Design examples

1.4.1. Switching control with linear analysis

We use the design condition derived from the linear analysis (Theorem 1.3) to design a switching state feedback controller. The following example illustrates the design procedure and the benefit of the switching strategy.

Example 1.1. Consider the system given by (1.14) with

$$A := \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C := \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D := 0, \quad M = I.$$

We consider the Synthesis Problem with $\alpha = 1$ and design a switching state feedback controller based on the Switching Control Design Algorithm with $q = 4$. We have chosen $\gamma_0 = 50$ and designed the initial feedback gain $\mathcal{K}_0$ as described in Step 1 of the algorithm. We then successively computed the gains $\mathcal{K}_1$ through $\mathcal{K}_4$ following Step 2. The results are as follows:

$$\begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix} = \begin{bmatrix} 50.0000 \\ 16.2234 \\ 5.3730 \\ 1.7991 \\ 0.6063 \end{bmatrix}, \quad \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} = \begin{bmatrix} 1.9348 \\ 0.6954 \\ 0.2447 \\ 0.0850 \end{bmatrix},$$

$$\begin{bmatrix} \mathcal{K}_0 \\ \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \\ \mathcal{K}_4 \end{bmatrix} = \begin{bmatrix} -0.2467 & -0.1497 \\ -0.3615 & -0.1795 \\ -0.5405 & -0.2179 \\ -0.8170 & -0.2667 \\ -1.2433 & -0.3279 \end{bmatrix}.$$

The domain of performance for each value of γ_i is plotted as an ellipse in the x_1–x_2 plane in Fig. 5. Note that the ellipses are nested, as specified.

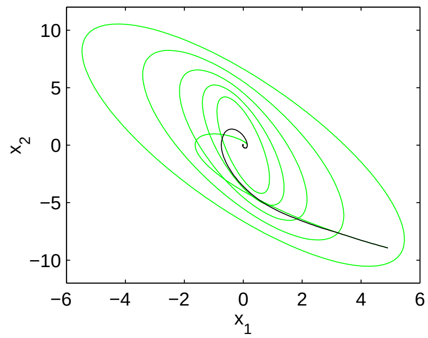

Figure 5: Domains of performance and state trajectories

We compare the performance of the switching controller (SWC) with that of the fixed-gain controller (FGC) $u = \mathcal{K}_0 x$. First of all, the guaranteed performance bound of the SWC is 94.08 as opposed to 100 guaranteed by the FGC, whenever the initial state is within the ellipse $x^\top Q_0^{-1} x \leq 1$. The state trajectories and the time responses are shown in Figs. 5 and 6, respectively, for the case $x(0) = [\ 5 \quad -9\ ]^\top$, where the dark curve is for the SWC and the lightly-colored curve is for the FGC. We see that the SWC performs much better than the FGC, although the improvement of the guaranteed performance is only 6%.

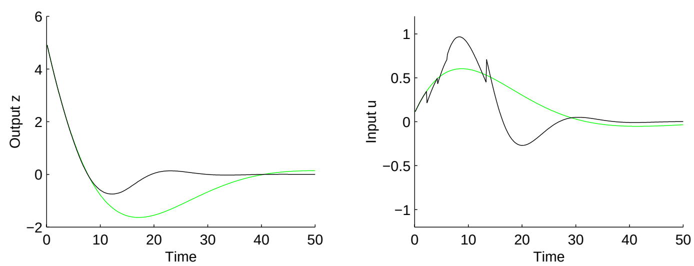

Figure 6: Initial state responses

1.4.2. Switching control with circle analysis

Recall that Theorem 1.1 states that the circle criterion does not help in the synthesis of saturating control in the sense that it does not improve (i.e. enlarge) the achievable domain of performance for a given performance level γ. Thus, the controller designed in Example 1.1 is the best within our framework in terms of the guaranteed $\mathcal{H}_2$ performance. However, the performance bound given by our analysis is conservative and hence Theorem 1.1 does not eliminate the possibility that the circle analysis may produce a controller with better actual performance. The purpose of this section is to show by an example that this is indeed the case and to illustrate how the circle criterion can be useful for improving the performance.

To this end, let us first present a switching control synthesis condition based on the circle analysis, which parallels the result in Theorem 1.3 derived from the linear analysis.

Theorem 1.4. Consider the system in (1.14) with $M = I$ and $m = 1$. Suppose that matrices K_k, symmetric matrices $Q_k > 0$, diagonal matrices $V_k > 0$ and scalars γ_k, μ_k and $0 \leq r < 1$ satisfy

$$\mathsf{He}\begin{bmatrix} AQ_k + BK_k & BV_k & 0 \\ -rK_k & -V_k & 0 \\ CQ_k + DK_k & DV_k & -\gamma_k I \end{bmatrix} < 0, \qquad \begin{bmatrix} Q_k & K_k^\mathsf{T} \\ K_k & \rho^2 \end{bmatrix} > 0 \tag{1.35}$$

for $k = 0, \ldots, q$, where $\rho := \alpha/(1-r)$, and

$$\frac{\gamma_k + \mu_k}{\gamma_{k-1}} Q_{k-1} < Q_k < Q_{k-1} \tag{1.36}$$

for $k = 1, \ldots, q$. Then the switching controller

$$u = \phi(z), \quad z = \mathcal{K}_k x \quad (\text{when } x^\mathsf{T} Q_k^{-1} x \leq 1 < x^\mathsf{T} Q_{k+1}^{-1} x) \quad (k = 0, \ldots, q)$$

with $\mathcal{K}_k := K_k Q_k^{-1}$ and $Q_{q+1}^{-1} := \infty I$ yields $|z(t)| \leq \rho$ for all $t \geq 0$, and

$$\int_0^\infty \|e(t)\|^2 dt < 2\left(\gamma_0 - \sum_{k=1}^{q} \mu_k\right)$$

whenever $x(0)^\mathsf{T} Q_0^{-1} x(0) \leq 1$.

Proof. The result directly follows from condition (1.13) with a change of variable $K := \mathcal{K}Q$ and from the argument for switching control design presented in Section 1.3.4. ∎

Example 1.2. Consider the system treated in Example 1.1. We will design state feedback switching controllers using the circle criterion summarized in Theorem 1.4. The design steps are parallel to Switching Control Design Algorithm and are as follows. First fix r (and hence ρ) to be some value in the interval $0 \le r < 1$. Initialize k as $k = 0$. The numerical problem for designing the initial gain $\mathcal{K}_0$ is to maximize the domain of performance $\det(Q_k)$ for a fixed value of performance level γ_k, over the variables Q_k and K_k, subject to constraints (1.35). This problem is a quasi-concave maximization which can be solved efficiently. The initial gain is then obtained as $\mathcal{K}_0 := K_0 Q_0^{-1}$. We then increment k and go on to calculate additional gain $\mathcal{K}_k := K_k Q_k^{-1}$ by maximizing μ_k over K_k, Q_k, γ_k and μ_k subject to (1.35) and (1.36). Repeat this last step for $k = 1, \ldots, q$ where q is the number of switchings.

From Theorem 1.1 and its proof, we know that the optimal performance bound is obtained when $r = 0$ (i.e. $\rho = \alpha = 1$) so that the circle criterion (1.35) reduces to the linear analysis condition (1.32). Therefore, we should always let $\rho = 1$ to optimize the performance bound. However, as we show below, a "good" design may result when $\rho > 1$. In particular, it will be seen that increase in ρ does not substantially affect the domains of performance but yet the control gain is very much influenced so that the actual output response can be improved.

Fix ρ to be either $\rho = 2$ or $\rho = 10$, and follow the design procedure outlined above. The results are found to be

$$\begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix} = \begin{bmatrix} 50.0000 \\ 16.3991 \\ 5.4522 \\ 1.8300 \\ 0.6178 \end{bmatrix}, \quad \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} = \begin{bmatrix} 1.9014 \\ 0.6792 \\ 0.2388 \\ 0.0830 \end{bmatrix},$$

for the $\rho = 2$ case (the values of γ_k and μ_k for the $\rho = 10$ case are similar), and

$$\begin{bmatrix} \mathcal{K}_0 \\ \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \\ \mathcal{K}_4 \end{bmatrix} = \overset{(\text{Case: } \rho = 2)}{\begin{bmatrix} -0.4890 & -0.3018 \\ -0.7053 & -0.3635 \\ -1.0453 & -0.4420 \\ -1.5720 & -0.5412 \\ -2.3843 & -0.6656 \end{bmatrix}}, \quad \overset{(\text{Case: } \rho = 10)}{\begin{bmatrix} -2.3963 & -1.5302 \\ -3.4303 & -1.8405 \\ -5.0704 & -2.2372 \\ -7.6041 & -2.7386 \\ -11.5061 & -3.3676 \end{bmatrix}}$$

and the corresponding domains of performance are plotted as ellipses in Fig. 7. The ellipses for both $\rho = 2$ and $\rho = 10$ are almost identical to those obtained in Example 1.1 via linear analysis (i.e. $\rho = 1$), although the values of $\det(Q_k)$ are slightly smaller. Similarly, the values of γ_k and μ_k are found insensitive to ρ, and the resulting performance bounds are 94.20 when $\rho = 2$) and 94.40 when $\rho = 10$. The control gains, on the other hand, are heavily dependent upon the value of ρ. In particular, it seems that larger ρ yields higher gain in general.

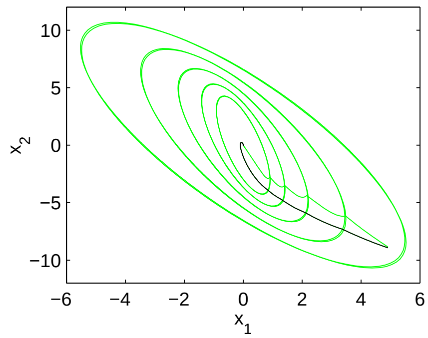

Figure 7: Domains of performance and state trajectories (circle criterion)

Using these gains we run simulations to obtain initial state responses with the same initial condition as in Example 1.1. The results are shown in Figs. 7 and 8 where the dark curves are the responses for $\rho = 2$ and the lightly-colored curves are for $\rho = 10$. We see that the response for the $\rho = 2$ case is actually better than the optimal $\mathcal{H}_2$ bound response obtained in Example 1.1 in the sense that it has no overshoot with shorter settling time. This indicates that circle criterion *can* improve actual performance although it does not help to improve theoretically guaranteed $\mathcal{H}_2$ performance bound (it is in fact slightly worse). Finally, when $\rho = 10$, the control gain is higher and the input u hits the saturation bound more often, but the output response is worse than the case $\rho = 2$. The purpose of showing this worse case is to illustrate that ρ can be used as a tuning parameter for "better" performance by adjusting the degree of saturation.

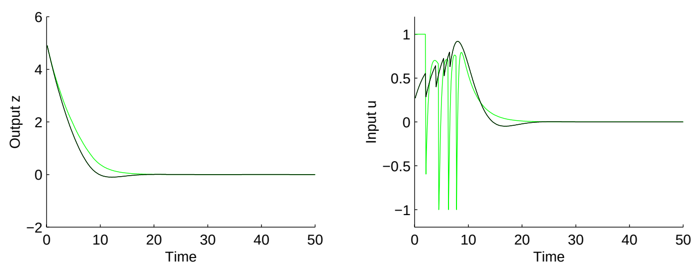

Figure 8: Initial state responses (circle criterion)

1.4.3. Fixed gain control with accelerated convergence

We have seen in Example 1.2 that the circle criterion can be used to heuristically improve the actual performance of the "theoretically optimal" controller designed in Example 1.1. In this section, we present another heuristic method to design a fixed-gain (without switching) state feedback controller that outperforms that in Example 1.1. To this end, we use the following synthesis condition obtained by modifying Corollary 1.2 to accelerate the convergence of the output signal.

Lemma 1.4. Fix $\beta \geq 0$ and let Q, K and γ be such that

$$\mathsf{He}\begin{bmatrix} (A+\beta I)Q+BK & 0 \\ CQ+DK & -\gamma I \end{bmatrix} < 0, \qquad \begin{bmatrix} Q & K_i^{\mathsf{T}} \\ K_i & \alpha_i^2 \end{bmatrix} > 0. \tag{1.37}$$

Let $u = KQ^{-1}x$ be the state feedback controller. Suppose the initial state satisfies

$$x(0)^{\mathsf{T}} Q^{-1} x(0) \leq 1. \tag{1.38}$$

Then we have $|u_i(t)| \leq \alpha_i$ for all $t \geq 0$ and $i = 1, \ldots, m$, and

$$\int_0^\infty \|\mathsf{e}^{\beta t} e(t)\|^2 dt < 2\gamma x(0)^{\mathsf{T}} Q^{-1} x(0).$$

Proof. Note that condition (1.37) is obtained by applying the condition in (1.28) to the new system obtained by replacing A by $A+\beta I$. The result simply follows from the well known fact: the initial state response of the modified system is given by

$$z_\beta(t) = (C+DK)\mathsf{e}^{(A+BK+\beta I)t}x(0) = \mathsf{e}^{\beta t}(C+DK)\mathsf{e}^{(A+BK)t}x(0) = \mathsf{e}^{\beta t}z(t)$$

where $z(t)$ is the response of the original system. ∎

Example 1.3. Consider the system given in Example 1.1. To accelerate the convergence, fix $\beta > 0$ and maximize $\det(Q)$ over Q and K subject to (1.37). In view of the previous examples, we choose $\gamma = 50$. For various values of $\beta > 0$, we solved the quasi-concave maximization problem and calculated the corresponding state feedback gains. For each gain, we estimate the domain of performance to guarantee the performance bound $\gamma = 50$ for the original system ($\beta = 0$) using the circle criterion (Lemma 1.3). With appropriate change of variables as in (1.13), the problem reduces to a quasi-concave maximization plus a line search over the "degree-of-saturation" parameter ρ. After some trial and error, we found that $\beta = 0.2$ gives the domain of performance whose size is nearly the same as the largest ellipse in Fig. 5. The feedback gain and the domain of performance for this case are given by

$$K = \begin{bmatrix} -0.6708 & -0.4171 \end{bmatrix}, \quad Q = \begin{bmatrix} 29.8150 & -44.6171 \\ -44.6171 & 108.8536 \end{bmatrix}$$

where the performance domain $x^{\mathsf{T}}Q^{-1}x \leq 1$ is plotted as the larger ellipse in Fig. 9. The optimal value of ρ that yielded this Q is $\rho = 2.7858$. Note that the two straight lines correspond to $\mathcal{K}x = \pm 1$ and thus the control input does not saturate if and only if the state is in the region between the lines. The smaller ellipse in Fig. 9 indicates the guaranteed domain of performance weighted by $\beta = 0.2$.

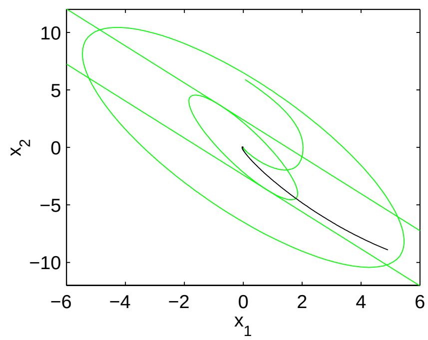

Figure 9: Domain of performance and state trajectories (Fixed gain)

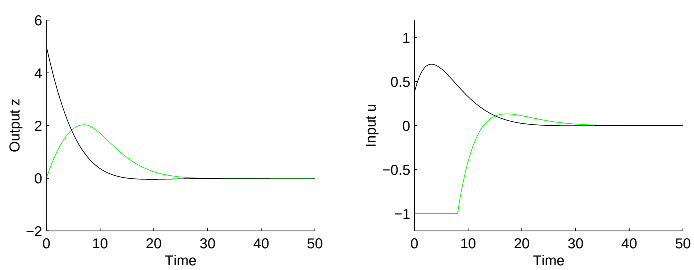

Figure 10: Initial state responses (Fixed gain)

Initial state responses are obtained for $x(0) = \xi_1$ or ξ_2 where

$$\xi_1 = \begin{bmatrix} 5 \\ -9 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

and plotted in Figs. 9 and 10. Note that the case $\mathsf{x}(0) = \xi_1$ has been treated in the previous examples. We see that the effective use of the circle criterion yielded a fixed gain controller that outperforms the switching controller designed in Example 1.1. Note, however, that the switching controller was systematically obtained without design iterations while the fixed-gain controller required a heuristic parameter tuning of β. Finally, we remark that the fixed gain controller does saturate for certain initial conditions as shown by the plots for the case $x(0) = \xi_2$, in contrast with the fact that any controller, fixed or switched, designed by using Corollary 1.2 or Theorem 1.3 would not saturate for all initial conditions within the domain of performance.

1.5. Further discussion

In the preceding analysis and synthesis, we assumed that the part of transfer function $H(s)$ from u to z in Fig. 1 is strictly proper to simplify the argument. In this section we show that this assumption can be made without loss of generality when the high frequency gain in question is diagonal. Below, we consider for simplicity the case where the saturation nonlinearity has the unity bound, i.e. $\alpha_i = 1$ in (1.2).

Consider the mapping from ξ to u in Fig. 11 (left), or

$$u = \phi(\xi + Gu) \tag{1.39}$$

where $\xi, u \in \mathbb{R}^m$, $G \in \mathbb{R}^{m\times m}$, and ϕ is the saturation nonlinearity defined in (1.2). This feedback loop is said to be well posed if, for each ξ, there exits a unique u satisfying (1.39).

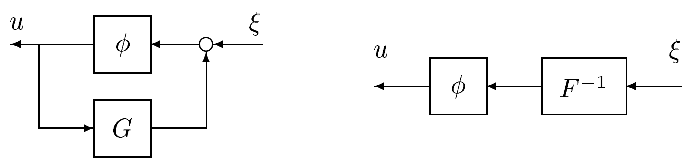

Figure 11: Saturation with algebraic loop

Lemma 1.5. Let $G \in \mathbb{R}^{m\times m}$ be a diagonal matrix and ϕ be the saturation nonlinearity defined in (1.2) with $\alpha_i = 1$. The feedback loop in Fig. 11 (left) is well posed if and only if $F := I - G > 0$, in which case, the mapping from ξ to u defined by Fig. 11 (left) is identical to that in Fig. 11 (right), that is,

$$u = \phi(\xi + Gu) \quad \Leftrightarrow \quad u = \phi(F^{-1}\xi).$$

Proof. We prove the result only for the case where u and ξ are scalars. The general case where u and ξ are vectors directly follows because the matrix G and the function ϕ are both diagonal.

If $G \geq 1$, then the equation $u = \phi(\xi + Gu)$ is satisfied by both $u = 1$ and $u = -1$ when ξ is zero. Thus the mapping from ξ to u is not uniquely defined at $\xi = 0$, and we conclude that $G < 1$ is a necessary condition for well-posedness. Below, we show that this condition is also sufficient, by explicitly constructing the mapping.

Suppose $G < 1$. We claim that if $u = \phi(\xi + Gu)$ holds then

$$\begin{aligned} \xi \geq 1 - G \quad &\Leftrightarrow \quad u = 1, \\ |\xi| \leq 1 - G \quad &\Leftrightarrow \quad u = (1-G)^{-1}\xi, \\ \xi \leq G - 1 \quad &\Leftrightarrow \quad u = -1, \end{aligned}$$

from which the result follows directly.

Consider the first equivalence. If $u = 1$, then

$$u = 1 \quad \Rightarrow \quad \xi + Gu \geq 1 \quad \Rightarrow \quad \xi \geq 1 - G.$$

To show the converse, suppose $\xi \geq 1 - G$ but $u \neq 1$. Then

$$\xi + Gu \geq 1 - G + Gu = (1-G)(1-u) + u > u \geq -1.$$

where we noted that $(1-G)(1-u) > 0$ due to $G < 1$, $u \neq 1$ and $u \leq 1$. Also note that $u < 1$ implies $\xi + Gu < 1$. Consequently,

$$-1 < \xi + Gu < 1 \quad \Rightarrow \quad u = \xi + Gu > u$$

which is a contradiction. Thus $u = 1$ must be true whenever $\xi \geq 1 - G$. The third equivalence can be shown similarly.

Finally, consider the second equivalence. If $u = (1-G)^{-1}\xi$, then

$$|(1-G)^{-1}\xi| \leq 1$$

since u is the output of the nonlinearity ϕ. Clearly, this condition is equivalent to $|\xi| \leq 1 - G$ and thus we have "$\Leftarrow$." To show the converse, suppose $|\xi| \leq 1 - G$ holds. If $\xi + Gu > 1$, then $u = 1$ and $1 - G < \xi$ which contradicts the supposition. Similarly, if $\xi + Gu < -1$, then $u = -1$ and $\xi < G - 1$, which is again a contradiction. Hence we must have $|\xi + Gu| \leq 1$, implying that $u = \xi + Gu$ or $u = (1-G)^{-1}\xi$ as claimed. ∎

Lemma 1.6. Consider the controller with anti-windup compensation depicted in Fig. 12 where ϕ is the saturation nonlinearity in (1.2) with $\alpha_i = 1$ and $K(s)$ is a transfer function with the following state space realization

$$K(s) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D_1 & D_2 \end{array} \right).$$

Suppose that D_2 is diagonal and the feedback loop in Fig. 12 is well posed, i.e. the output u is uniquely determined by the input y. Then $D_2 < I$ and the mapping from y to u in Fig. 12 with the above $K(s)$ is identical to the mapping from y to u in Fig. 12 with $K(s)$ replaced by

$$K_o(s) = \left(\begin{array}{c|cc} A & B_1 & B_2(I-D_2)^{-1} \\ \hline C & D_1 & 0 \end{array}\right).$$

Therefore, allowing for nonzero D_2 in the anti-windup controller does not enlarge the class of controllers unless D_2 has nonzero off-diagonal entries.

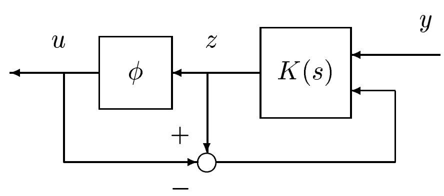

Figure 12: Controller with anti-windup compensation

Proof. Let us explicitly write down the equations describing the mapping from y to u in Fig. 12 as follows:

$$\dot{x} = Ax + B_1 y + B_2(z-u), \quad z = Cx + D_1 y + D_2(z-u), \quad u = \phi(z).$$

Well-posedness requires that $I - D_2$ is invertible, in which case the second equation can be solved for z as

$$z = (I-D_2)^{-1}(z_o - D_2 u), \quad z_o := Cx + D_1 y.$$

Substituting this expression for $u = \phi(z)$ we have

$$u = \phi((I-D_2)^{-1}(z_o - D_2 u)).$$

From Lemma 1.5, u is uniquely determined from z_o if and only if

$$(D_2 - I)^{-1} D_2 < I, \quad \text{or} \quad D_2 < I$$

holds. In this case, u is given by

$$u = \phi((I - (D_2 - I)^{-1} D_2)^{-1} (I - D_2)^{-1} z_o) = \phi(z_o).$$

Finally, noting that

$$\begin{aligned} \dot{x} &= Ax + B_1 y + B_2((I-D_2)^{-1}(z_o - D_2 u) - u) \\ &= Ax + B_1 y + B_2(I-D_2)^{-1}(z_o - u), \end{aligned}$$

we conclude the result. ■

When $D_2 = 0$ in the anti-windup controller in Fig. 12, well-posedness of the control law is automatically guaranteed. When D_2 is diagonal, the feedback loop is well posed if and only if $D_2 < I$. The above result shows that the feed-through term D_2 can be set to zero without loss of generality when specifying the class of controllers to be designed, provided that D_2 is diagonal. In other words, the nonzero diagonal D_2 term does not contribute to improve the achievable performance. Note, however, that this does not eliminate the possibility that a nonzero D_2 term of general structure may indeed improve the performance.

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