H_∞ Control and Quadratic Stabilization of Systems with Parameter Uncertainty Via Output Feedback

Lihua Xie, Minyue Fu, and Carlos E. de Souza

Abstract—**This note focuses on linear systems which are subject to both *time-varying norm-bounded* parameter uncertainty and exogenous disturbance, and addresses the following robust H_∞ control problem: designing a *linear dynamic output feedback controller* such that the closed-loop system is quadratically stable and achieves a prescribed level of disturbance attenuation for all admissible parameter uncertainties. It is shown that such a problem is equivalent to a scaled H_∞ control problem.**

I. Introduction

The last decade has witnessed significant advances in the H_∞ control theory; see [5]–[8], [13], [15], and the references therein. It is a well-known fact that H_∞ control is closely associated with many robustness problems such as sensitivity minimization [20] and stabilization of uncertain systems [8], [9], [12]. However, when there is parameter uncertainty in plant modelings, no robust behavior on H_∞ performance along with stability can be guaranteed by the standard H_∞ control method. To date, little attention has been paid to designing H_∞ controllers which would be robust against *time-varying real parameter uncertainty* in the controlled system.

Stabilization of systems with parameter uncertainty is one of the most vital subjects in control research. Among various techniques for robust stabilization (e.g., [9], [12], [17]), the so-called *quadratic stabilization theory* [1], [8], [10], [12], [14] seems to be most effective in dealing with time-varying parameter uncertainty. The problem of quadratic stabilization is to find a feedback controller such that the closed-loop system is stable with a fixed (uncertainty-independent) Lyapunov function. This problem was initially proposed in [10] to study the control of uncertain systems satisfying the so-called "matching conditions." Since then, numerous results have been reported in the literature, including a necessary and sufficient condition given in [1] for quadratic stabilizability via nonlinear state feedback control and the Riccati equation approach proposed in [14]. Recently, it was remarkably shown in [8] that a certain type of quadratic stabilization problem is essentially an H_∞ control problem. Similar results for discrete-time systems can be found in [12]. Despite the richness of the existing results on quadratic stabilization, no guarantee on system performance, such as disturbance attenuation, can be offered along with stability, even for the "nominal system."

In attempting to further bridge the gap between H_∞ control and robust stabilization, we in this note consider linear systems subject to both time-varying parameter uncertainty of a certain type (as in the robust stabilization case) and input disturbance (as in the H_∞ control case). The problem addressed is to design a *linear dynamic output feedback controller* such that the closed-loop system is quadratically stable and achieves a prescribed level of disturbance attenuation. Since this problem involves the notions of both robust stabilization and H_∞ control, we refer to it as *robust H_∞ control*; see Section II for its precise definition. Such a problem in the linear state feedback setting has been completely solved in [18], [19] by using a Riccati equation approach. In the case of output feedback, sufficient conditions for designing robust H_∞ controllers with a fixed order have been derived in [11], while [16] has developed H_∞ controllers for systems with uncertainty in the state matrix only. The focal point of this note is the linear dynamic output feedback and we consider uncertain systems with parameter uncertainties appearing in all the state, input, and output matrices. It should be noted that a similar robust H_∞ control problem for complex uncertainties has been tackled in [3], [4] and the so-called μ-synthesis method has been developed which employs the structured singular value and the H_∞ control techniques to search for a suitable controller. The result in this note can be viewed as an analogy of those in [4] for time-varying parameter uncertainties. However, due to the fact that time-varying uncertainty is considered in this note, a different machinery based on Riccati equations is used rather than a transfer function approach.

Our first main result establishes an interconnection between: i) a robust H_∞ control problem; and ii) a scaled H_∞ control for a system without parameter uncertainty, thus allowing us to solve the robust H_∞ control problem via existing H_∞ control techniques. As a special case, the problem of *quadratic stabilization via linear dynamic output feedback control* is shown to be equivalent to a standard H_∞ control problem, which slightly generalizes a result in [8] to systems with more general uncertainty structure.

II. Problem Statement and Definitions

Consider the class of uncertain linear systems described by state-space models of the form

(Σ_1):

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + B_1 w(t) + [B_2 + \Delta B(t)]u(t) \tag{2.1a}$$

$$z(t) = C_1 x(t) + D_{12} u(t) \tag{2.1b}$$

$$y(t) = [C_2 + \Delta C(t)]x(t) + D_{21} w(t) + [D_{22} + \Delta D(t)]u(t) \tag{2.1c}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^q$ is the disturbance input, $y(t) \in \mathbb{R}^r$ is the measured output, $z(t) \in \mathbb{R}^p$ is the controlled output, A, B_1, B_2, C_1, C_2, D_{12}, D_{21}, and D_{22} are real constant matrices of appropriate dimensions that describe the nominal system, and $\Delta A(\cdot)$, $\Delta B(\cdot)$, $\Delta C(\cdot)$, and $\Delta D(\cdot)$ are real-valued matrix functions representing time-varying parameter uncertainties. The parameter uncertainties considered here are norm-bounded and of the form

$$\begin{bmatrix} \Delta A(\cdot) & \Delta B(\cdot) \\ \Delta C(\cdot) & \Delta D(\cdot) \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F(\cdot)[E_1 \quad E_2] \tag{2.2}$$

where $H_1 \in \mathbb{R}^{n \times i}$, $H_2 \in \mathbb{R}^{r \times i}$, $E_1 \in \mathbb{R}^{j \times n}$, and $E_2 \in \mathbb{R}^{j \times m}$ are known constant matrices and $F(t) \in \mathbb{R}^{i \times j}$ is an unknown matrix function satisfying

$$F^T(t)F(t) \le \varrho^2 I \tag{2.3}$$

with the elements of $F(\cdot)$ being Lebesgue measurable and $\varrho > 0$ a given constant. In the above, the superscript "T" denotes the transpose.

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This note addresses the problem of designing a *linear dynamic output feedback controller* for the system (2.1) such that the closed-loop system is quadratically stable and achieves a prescribed level of disturbance attenuation in the H_∞-norm sense for all admissible uncertainties which satisfy (2.3). We will also discuss the problem of quadratic stabilization of system (2.1) via linear dynamic output feedback.

To motivate the technique used in this note, we will first recall the notions of quadratic stability and disturbance attenuation and some related results.

Let us consider the following system simplified from (2.1):

$$(\Sigma_2): \quad \dot{x}(t) = Ax(t) + B_1 w(t) \quad (2.4a)$$

$$z(t) = C_1 x(t). \quad (2.4b)$$

Definition 2.1 [13]: Given a scalar $\gamma > 0$, the system (2.4) is said to be *stable with disturbance attenuation* γ if it satisfies the following conditions:

i) A is a stable matrix; and

ii) the transfer function from disturbance w to controlled output z satisfies

$$\|C_1(sI - A)^{-1}B_1\|_\infty < \gamma. \quad \square$$

Lemma 2.1 (See [8] for Proof): Let $\gamma > 0$ be given. The system (2.4) is stable with disturbance attenuation γ if and only if there exists a symmetric matrix $P > 0$ such that

$$A^T P + PA + \gamma^{-2} PB_1 B_1^T P + C_1^T C_1 < 0. \quad (2.5)$$

▽ ▽ ▽

When there is parameter uncertainty $\Delta A(t)$ in the state matrix of (2.4), the system reads

$$(\Sigma_3): \quad \dot{x}(t) = [A + \Delta A(t)]x(t) + B_1 w(t) \quad (2.6a)$$

$$z(t) = C_1 x(t). \quad (2.6b)$$

Definition 2.2 [1]: The system (2.6) is said to be *quadratically stable* if there exists a positive definite symmetric matrix P such that for all admissible uncertainty $\Delta A(\cdot)$

$$[A + \Delta A(t)]^T P + P[A + \Delta A(t)] < 0. \quad (2.7)$$

Similarly, the uncertain system (2.1) is said to be *quadratically stabilizable via linear dynamic output feedback* if there exists a linear dynamic output feedback compensator $K(s)$ such that with $u = K(s)y$ the resulting closed-loop system is quadratically stable. □

Remark 2.1: Note that quadratic stability implies uniformly asymptotic stability for all admissible $\Delta A(\cdot)$. However, it is a conservative notion for robust stability since a fixed P matrix is required which results in a fixed Lyapunov function $V(x) = x^T P x$ for all uncertainties $\Delta A(\cdot)$. Nevertheless, the simplicity of this notion has been proven to be an effective means to deal with time-varying uncertainty [1], [8], [10], [12], [14]. □

In order to guarantee an H_∞ performance of (2.6) for all admissible parameter uncertainties, we adopt the technique used in quadratic stability, i.e., we incorporate $\Delta A(\cdot)$ in (2.5). This leads to the notion of *quadratic stability with disturbance attenuation*.

Definition 2.3 [19]: Given a scalar $\gamma > 0$, the system (2.6) is said to be *quadratically stable with disturbance attenuation* γ if there exists a symmetric positive-definite matrix P such that for all admissible uncertainty $\Delta A(\cdot)$

$$[A + \Delta A(t)]^T P + P[A + \Delta A(t)] + \gamma^{-2} PB_1 B_1^T P + C_1^T C_1 < 0. \quad (2.8)$$

Similarly, given a scalar $\gamma > 0$, the uncertain system (2.1) is said to be *quadratically stabilizable with disturbance attenuation* γ *via linear dynamic output feedback* if there exists a linear dynamic output feedback compensator $K(s)$ such that with $u = K(s)y$ the resulting closed-loop system is quadratically stable with disturbance attenuation γ. □

The notion of quadratic stability with disturbance attenuation implies the following result. (The proof is similar to that of [19, lemma 2.1] and thus is omitted.)

Lemma 2.2: Suppose the system (2.6) is quadratically stable with disturbance attenuation $\gamma > 0$. Then, this system is quadratically stable. Moreover, with zero-initial condition for $x(t)$, $\|z\|_2 < \gamma \|w\|_2$ for all admissible uncertainty $\Delta A(\cdot)$ and all nonzero $w \in L_2[0, \infty)$, where $\|\cdot\|_2$ denotes the usual $L_2[0, \infty)$-norm. ▽ ▽ ▽

Remark 2.2: The notion of quadratic stability with disturbance attenuation is a natural extension of quadratic stability to incorporate H_∞ performance and its conservativeness lies in the requirement of a fixed P matrix in (2.8) for all admissible parameter uncertainties as in the quadratic stability. Despite its conservativeness, we feel that this notion naturally combines both quadratic stability and disturbance attenuation, providing a feasible way of treating both parameter uncertainty and disturbance input.

III. Main Results

Our main results will develop the interconnections between both robust H_∞ control via linear dynamic output feedback and a scaled H_∞ control problem, and between quadratic stabilization via linear dynamic output feedback and a standard H_∞ control problem.

In connection with the system (2.1) we now introduce a system below that will allow us to establish the equivalence between robust H_∞ and a scaled H_∞ control problem.

(Σ_4):

$$\dot{x}(t) = Ax(t) + [\sqrt{\epsilon}\varrho H_1 \quad \gamma^{-1}B_1]\tilde{w}(t) + B_2 u(t) \quad (3.1a)$$

$$\tilde{z}(t) = \begin{bmatrix} \frac{1}{\sqrt{\epsilon}} E_1 \\ C_1 \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{\sqrt{\epsilon}} E_2 \\ D_{12} \end{bmatrix} u(t) \quad (3.1b)$$

$$y(t) = C_2 x(t) + [\sqrt{\epsilon}\varrho H_2 \quad \gamma^{-1}D_{21}]\tilde{w}(t) + D_{22}u(t) \quad (3.1c)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $\tilde{w}(t) \in \mathbb{R}^{q+i}$ is the disturbance input, $y(t) \in \mathbb{R}^r$ is the measured output, $\tilde{z}(t) \in \mathbb{R}^{p+j}$ is the controlled output, A, B_1, B_2, C_1, C_2, D_{12}, D_{21}, D_{22}, E_j, E_2, H_1, and H_2 are the same as in the system (2.1), $\epsilon > 0$ is a parameter to be chosen, and $\gamma > 0$ is the disturbance attenuation performance we wish to achieve for the system (2.1).

We first establish a key lemma which will lead to an interconnection between quadratic stability with disturbance attenuation of the unforced system of (2.1) (setting $u(t) \equiv 0$) and stability with disturbance attenuation of the unforced system (3.1) (setting $u(t) \equiv 0$).

Lemma 3.1: Let the constant $\gamma > 0$ be given. Then there exists a matrix $P > 0$ such that

$$[A + H_1 F(t) E_1]^T P + P[A + H_1 F(t) E_1] + \gamma^{-2} PB_1 B_1^T P + C_1^T C_1 < 0 \quad (3.2)$$

for all $F(t)$ satisfying (2.3) if and only if there exists a constant

$\epsilon > 0$ such that

$$A^TP + PA + \gamma^{-2}PB_1B_1^TP + \epsilon\varrho^2PH_1H_1^TP + \frac{1}{\epsilon}E_1^TE_1 + C_1^TC_1 < 0. \quad (3.3)$$

Proof: The sufficiency part of the proof follows immediately from the fact that for any $F(t)$ satisfying (2.3) and for any $\epsilon > 0$ we have that

$$E_1^TF^T(t)H_1^TP + PH_1F(t)E_1 \le \epsilon\varrho^2PH_1H_1^TP + \frac{1}{\epsilon}E_1^TE_1.$$

To show the necessity, we suppose there exists a matrix $P > 0$ such that (3.2) holds, i.e.,

$$Z \triangleq A^TP + PA + \gamma^{-2}PB_1B_1^TP + C_1^TC_1 < -E_1^TF^T(t)H_1^TP - PH_1F(t)E_1$$

for all $F(t)$ satisfying (2.3). By a technique similar to that used in [8, proof of theorem 3.2], it follows that there exists a constant $\epsilon > 0$ such that

$$\epsilon^2\varrho^2PH_1H_1^TP + \epsilon Z + E_1^TE_1 < 0,$$

i.e., (3.3) holds. ▽ ▽ ▽

In view of Definition 2.3 and Lemma 2.1, Lemma 3.1 leads to the following corollary.

Corollary 3.1: Given a constant $\gamma > 0$, the unforced system of (2.1) (setting $u(t) \equiv 0$) is quadratically stable with disturbance attenuation γ if and only if there exists a scaling parameter $\epsilon > 0$ such that the unforced system of (3.1) (setting $u(t) \equiv 0$) is stable with unitary disturbance attenuation.

Theorem 3.1: Let $\gamma > 0$ be a prescribed level of disturbance attenuation and $K(s)$ denote a given linear dynamic controller. Then, the system (2.1) is quadratically stabilizable with disturbance attenuation γ via the output feedback controller $K(s)$ if and only if there exists a constant $\epsilon > 0$ such that the closed-loop system corresponding to (3.1) and $K(s)$ is stable with unitary disturbance attenuation.

Proof: Let the controller $K(s)$ be of the following state-space realization:

$$\dot{\xi}(t) = A_c\xi(t) + B_cy(t) \quad (3.4a)$$

$$u(t) = C_c\xi(t). \quad (3.4b)$$

It should be noted that there is no loss of generality to assume the controller to be strictly proper. This follows from the fact that the design of a proper H_∞ controller can be converted to the design of a strictly proper one [15]. Now, letting $x_c = [x^T \ \xi^T]^T$, the closed-loop system of (2.1) with the controller (3.4) is given by the state-space equations

$$\dot{x}_c(t) = [\bar{A} + \bar{H}F(t)\bar{E}]x_c(t) + \bar{B}_1w(t) \quad (3.5a)$$

$$z(t) = \bar{C}_1x_c(t) \quad (3.5b)$$

where

$$\bar{A} = \begin{bmatrix} A & B_2C_c \\ B_cC_2 & A_c + B_cD_{22}C_c \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} H_1 \\ B_cH_2 \end{bmatrix},$$

$$\bar{E} = [E_1 \quad E_2C_c], \quad \bar{C} = [C_1 \quad D_{12}C_c], \quad \bar{B}_1 = \begin{bmatrix} B_1 \\ B_cD_{21} \end{bmatrix}.$$

Also, the closed-loop system of (3.1) with the controller (3.4) is of the form

$$\dot{x}_c(t) = \bar{A}x_c(t) + \left[\sqrt{\epsilon}\varrho\bar{H} \quad \gamma^{-1}\bar{B}_1\right]\tilde{w}(t) \quad (3.6a)$$

$$\tilde{z}(t) = \begin{bmatrix} \frac{1}{\sqrt{\epsilon}}\bar{E} \\ \bar{C}_1 \end{bmatrix} x_c(t) \quad (3.6b)$$

where $\bar{A}$, $\bar{B}_1$, $\bar{H}$, $\bar{E}$, and $\bar{C}_1$ are as in (3.5). The desired result now follows immediately from Corollary 3.1. ▽ ▽ ▽

Remark 3.1: Theorem 3.1 established the equivalence between the robust H_∞ control problem for the system (2.1) and the scaled H_∞ control problem for the system (3.1). Therefore, a complete solution to the robust H_∞ control problem can be obtained via existing H_∞ control techniques. Moreover, Theorem 3.1 also allows us to parameterize all strictly proper linear dynamic output feedback controllers that solve the robust H_∞ control problem. □

Next, we discuss the quadratic stabilization problem of system (2.1) via linear dynamic output feedback. The result to be given is similar to that in [8, theorem 3.4]. However, we include a more general structure of parameter uncertainties which allows for uncertainties in the output equation.

Similar to the robust H_∞ control, quadratic stabilization of the system (2.1) will be shown to be equivalent to the H_∞ control of the following system:

$$(\Sigma_5): \quad \dot{x}(t) = Ax(t) + H_1\tilde{w}(t) + B_2u(t) \quad (3.7a)$$

$$\tilde{z}(t) = E_1x(t) + E_2u(t) \quad (3.7b)$$

$$y(t) = C_2x(t) + H_2\tilde{w}(t) + D_{22}u(t) \quad (3.7c)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $\tilde{w}(t) \in \mathbb{R}^i$ is the disturbance input, $y(t) \in \mathbb{R}^r$ is the output, $\tilde{z}(t) \in \mathbb{R}^j$ is the controlled output, and A, H_1, B_2, E_1, E_2, C_2, H_2, and D_{22} are the same as in the system (2.1). The equivalence between quadratic stabilization and H_∞ control is given in the following theorem. (The proof is similar to that of Theorem 3.1 and thus is omitted.)

Theorem 3.2: The system (2.1) is quadratically stabilizable via a given linear dynamic output feedback controller $K(s)$ if and only if the closed-loop system corresponding to (3.7) and $K(s)$ is stable with disturbance attenuation ϱ^{-1}.

Remark 3.2: In view of Theorem 3.2, it results that the separation principle for H_∞ control also carries over to quadratic stabilization of (2.1) via linear dynamic output feedback. Moreover, Theorem 3.2 also allows us to parameterize all strictly proper linear dynamic output feedback controllers which quadratically stabilize (2.1). □

IV. Conclusions

This note has developed a linear dynamic output H_∞ control technique for systems subject to time-varying parameter uncertainties in both the state and the output equations. Based on the notion of quadratic stability with disturbance attenuation, the problem of robust H_∞ control and quadratic stabilization via linear dynamic output feedback have been solved.

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Stability in Feedback Systems with Tapered and Other Special Input Spaces

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***Abstract*—In this note the effect of weighted input and output spaces on the stability of feedback systems is examined. For a weighted space, the recent past of a signal is emphasized and the remote past is deemphasized. Tapered spaces are a subclass of weighted spaces. Results concerning weighted spaces used on linear systems are given. The circle conditions are recovered for feedback systems with exponentially tapered spaces.**

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I. Introduction

In this note, we analyze the effect weighted input and output spaces have on the stability of a feedback system. Tapered spaces (a subclass of weighted spaces) have the property that for a bounded subspace of time functions, given a tolerance ϵ, a finite time interval δ may be found such that a time function may be determined up to a time T, to within ε, just by observing it over $[T - \delta, T]$ rather than the entire past. For a continuous input–output system for which a tapered input space is appropriate, the effect on the system from the remote past part of the input is small. Such systems are in some sense "highly" causal. In the analysis of what may be called "natural systems," for example, the inputs may be measured but only over a finite time span, while the system will have been "running" for a long time previous to when the measurements are taken. Tapered input spaces are reasonable for systems which are better modeled with, or would have a better design with, the property that stimuli in the remote past do not have as much effect as recent stimuli.

Weighted spaces of some form have been discussed in connection with many topics. For example, Boyd and Chua [2] show that if a particular weighted input space is used (sufficiently a tapered input space), approximations by Volterra series of nonlinear operators can be made over an infinite time interval. They site Volterra and Wiener having considered weighted spaces for physical reasons. They state that weighted spaces are "an old assumption whose full power has not been used" [2]. Michael and Miller analyze the stability of a coupled core nuclear reactor with the variables representing core powers living in weighted spaces [4]. Weighted spaces have been used in connection with Kalman filtering [1]. Use of weighted input spaces is one method of preventing divergence in a Kalman filter.

The input and output spaces in this note are examples of "fitted families of seminorms," developed by Root; see [6]. This approach is more general (inclusive) than the procedure where time-projection operators are used on the spaces of time functions. Weighted (or tapered) spaces of time functions are more easily handled by the fitted families approach than by time projections.

In Section III, it is shown that the small gain theorem, which guarantees the stability of a feedback system, can be generalized to hold for weighted input and output spaces. The bounding space stability and BIBO stability of a linear system with weighted input and output spaces is analyzed in Section IV. A frequency domain result is presented. These results are applied to feedback systems. A Nyquist stability test and a circle test are recovered for systems with exponentially tapered input and output spaces. In Section V, an example is presented where we use the circle conditions, as they are generalized here, to compute how long it takes the output of a system with effort limited controller to die down after a pulse is applied to the input. Some of the results in this note appeared in the conference paper [5].

II. Preliminaries

An input–output system (Y, F, U) is, in the context of this note, a mapping F from an input space U to an output space $Y(y = F(u))$, where U is a translation invariant subset of a normed linear space A of time functions, and Y (also translation invariant) is either a Banach space B or an extended space B^e of time functions. A proper definition of the spaces A, B, and