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Quantized Control for Stochastic System

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Abstract: This paper considers the coarsest quantization control problem. Different from the previous works [4] where the systems are restricted to be deterministic, we focus on the feedback quantization control for general stochastic systems with multiplicative noises. It is showed that the coarsest quantizer that stabilize the multiplicative-noise stochastic system in mean-square sense is logarithmic, and the quantization density of which is larger than the results obtained in [4] for deterministic systems for the deterioration of multiplicative noises. Also, we explore that the solvability of the quantizer density is related to a special stochastic linear control problem.

Key Words: Mean-square Stability, Stochastic, Multiplicative Noise, Quantization, Feedback Control

1 Introduction

Linear feedback control plays an important role in modern control theory, and has been investigated by many researchers. With the developments of networks, the problem of time-delay, packets-dropouts and quantization arise in the digital transmission channel for the limited date bits can deteriorate the performance so that can not be neglected. Control using quantized information can be traced back to 1960's [16] [18] [17] [1] [19] and so on. The development of modern network control system has brought a resurgent interests in quantized feedback control, recent works on quantized feedback control include [4] [11] [12]. The research on quantized feedback control can be categorized depending on whether the quantizer is static or dynamic. Static quantizer is memoryless nonlinear function, while the dynamic quantizer use memory, and thus can be more complex and powerful. Most of the research about static quantizer use the uniform quantizer and logarithmic quantizer, while the uniform quantizer can keep the information well when the signal falls into the dynamic range of quantizer, the number of quantization levels required for a given quantization step-size increases linearly as the dynamics range increases. [5] considers the uniform quantizer and propose a new control design methodology, which relies on the possibility of changing the sensitivity of the quantizer while the system evolves to stabilize a linear time-invariant control systems with saturating quantized measurements. [7] investigates the quantized H_∞ control problem for discrete-time systems with random packet losses, [6] studies the asymptotic characteristics of uniform scalar quantizers that are optimal with respect to mean-squared error, while [2] has shown that the classical linear quantization approach is invalid in the case of the resolution is coarse and the open-loop dynamics is unstable.

In [3], it has been proved that there exists a critical positive date rate that below which there does not exist any quantizer and feedback control that can stabilize the unstable system, and the coarsest quantizer is proved to be logarithmic. [4] reveals that for single input linear time-invariant system, the coarsest quantizer is logarithmic and its quantization density is related to the unstable roots of the system matrix. In [8], it is showed that the quantized error of the logarithmic quantizer can be modeled as multiplicative noise related to the quantization density and the problem of quantized control can be transformed into the classical robust control problem. As for the stochastic systems, [13] considers the stochastic system with addictive noises under the assumption that the noises is bounded. A logarithmic quantizer with finite levels to guarantee the system achieve practical stability has been designed. The least quantization rate for system with the feedback subject to the Bernoulli packets dropouts is proved to has the relation with the packets dropout rate and the unstable roots of the system matrix in [11]. [12] considers the minimum date rate for mean-square stability of linear systems over a lossy channel which is modeled as a time-homogenous binary Markov process. The minimum date rate for scalar systems is given in terms of the magnitude of the unstable mode and the transition probabilities of the Markov chain. Necessary and sufficient conditions are provided for scalar systems.

It should be noted that most of the aforementioned works are only concerned with the deterministic system or stochastic systems with bounded additive noises or some special multiplicative noises. In this paper, we consider the coarsest quantization density for the mean-square stabilization for a general stochastic system with multiplicative noises. Similar to the case of deterministic system, the coarsest quantizer is proved to be logarithmic, and the quantization density is larger than in the case of deterministic system because of the deterioration of the multiplicative noises. This paper is organized in the following way, section II formulates the quantized feedback problem. Section III presents the solution of the problem and generalize it to exponentially mean-square stability. Finally, section IV draws some conclusions of this paper.

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2 Problem Formation

Consider the discrete-time multiplicative-noise stochastic system:

$$x(t+1) = Ax(t) + Bu(t) + [A_0x(t) + B_0u(t)]\, w(t), \quad (1)$$

where $x(t) \in R^n$ is system state with the initial state as $x(0) = x_0$, $u(t) \in R$ is control input, and $w(t) \in R$ is white noise with zero mean and variance as σ^2. Throughout this paper , we assume $w(t)$ satisfying

$$E(w(t)w^T(s)) = \sigma^2\delta_{ts}, \quad (2)$$

and is uncorrelated with initial state $x(0)$, δ_{ts} is the kronecker function. For the convenience of discussions, we denote

$$Ev(x(t)) \triangleq Ex^T(t)Px(t),\ P = P^T > 0, \quad (3)$$

$$\nabla Ev(x(t)) \triangleq Ev(x(t+1)) - v(x(t)). \quad (4)$$

The time t shall be dropped out in the case of without causing the confusion.

Definition 1 *The discrete-time stochastic system (1) is quadratically mean-square stabilized, i.e., there exists a state feedback control $u(t) = Kx(t)$ (where K is a constant matrix) and symmetric positive matrix $R = R^T > 0$, such that for any initial state $x_0 \in R^n$, the closed-loop system:*

$$x(t+1) = (A + BK)x(t) + (A_0 + B_0K)x(t)w(t), \quad (5)$$

with the initial state as $x(0) = x_0$ satisfying

$$\nabla Ev(x(t)) = -x^T(t)Rx(t) < 0. \quad (6)$$

Definition 2 *The discrete-time stochastic system (1) is exponentially quadratic mean-square stabilized with convergence rate α, i.e., there exists a control $u(t) = Kx(t)$ and symmetric positive matrix $R > 0$, where K is a constant matrix, satisfying that for any initial state $x_0 \in R^n$, the closed-loop system:*

$$x(t+1) = (A + BK)x(t) + (A_0 + B_0K)x(t)w(t), \quad (7)$$

with the initial state as $x(0) = x_0$ satisfying

$$\nabla Ev(x(t)) = -\alpha x^T(t)Rx(t) < 0. \quad (8)$$

Remark 1 *The following discrete-time stochastic system:*

$$x(t+1) = Ax(t) + A_0x(t)w(t),\ x(0) = x_0, \quad (9)$$

is called mean square stable, if for any initial state $x_0 \in R^n$,

$$\lim_{t\to\infty} E\|x(t)\|^2 = 0,$$

When the system is mean-square stable, we also call that (A, A_0) is stable.

Remark 2 *It can be proved that if the discrete-time stochastic system (1) is quadratically mean-square stabilized, then it is mean-square stable.*

Consider a function $f : X \to U$, where $X \subset R^n$ and $U \subset R$, f introduce a partition of X satisfying $f(x) = -f(x)$, and has only finite or countable values in U, we call f is a quantizer. In this paper we consider only quantizers that are symmetric with respect to the origin and with an infinite countable number of levels. The quantization density is defined as the density of the quantizer f introduced in the state space X. The problem to be solved in this paper is stated as following:

Problem 1: For the system (1) and the Lyapunov function $Ev(x) = Ex^TPx$, where $P = P^T > 0$, find a set $U = \{u_i \in R, i \in Z\}$ and a function $f : X \to U$, such that

- $f(-x) = -f(x)$,
- For any $x \in X, x \neq 0$,

$$\nabla Ev(x) = Ev(Ax + Bf(x)) - v(x) < 0. \quad (10)$$

In the above f is called the quantizer.

3 Problem Solution

In this paper we shall adopt some notations defined in the earlier work of [4] as the following:

- $Q(EV)$ denotes all quantizer that solve the Problem 1;
- for $g \in Q(EV)$ and $0 < \varepsilon \leq 1$, denote $\sharp gQ(EV)$ as the number of quantization in the interval $[\epsilon, 1/\epsilon]$, i.e.,

$$\eta_g = \lim sup_{\varepsilon\to 0} \frac{\sharp gQ(EV)}{-ln\varepsilon}, \quad (11)$$

  η_g is the quantization density of g.
- Moreover, a quantizer f is said to be coarsest for $Ev(x)$ if it has the smallest density of quantization, i.e.,

$$f = \arg \inf_{g\in Q(EV)} \eta_g. \quad (12)$$

Notice our aim is to derive the coarsest quantization density when using the quantized feedback for system (1), so we shall first derive all the quantized feedback that can make the system mean-square stable. The following lemma gives us all the admissible control.

Lemma 1 *Let $Ev(x) = Ex^TPx$, where $P = P^T > 0$, is a Lyapunov function for system (1). For any $x \in X, x \neq 0$, all the u that make the system (1) mean-square stable are characterized by*

$$U(x) = \{u \in R \mid u_1(x) < u < u_2(x)\}, \quad (13)$$

where

$$u_{1,2}(x) = K_{GD}x(t) \pm \sqrt{\frac{x^T(t)Qx(t)}{B^TPB + \sigma^2B_0^TPB_0}}, \quad (14)$$

and

$$K_{GD} = -\frac{B^TPA + \sigma^2B_0^TPA_0}{B^TPB + \sigma^2B_0^TPB_0}, \quad (15)$$

while

$$Q = P - A^TPA - \sigma^2A_0^TPA_0 + \Delta. \quad (16)$$

where $\Delta = \frac{(A^TPB+\sigma^2A_0^TPB_0)^TQ^{-1}(A^TPB+\sigma^2A_0^TPB_0)}{B^TPB+\sigma^2B_0^TPB_0}$.

Proof: Set the equation

$$\nabla Ev(x) = 0, \tag{17}$$

the admissible U is characterized by the open interval between the roots of the following second order equation in u,

$$(B^TPB+\sigma^2B_0^TPB_0)u^2(t)+2x^T(t)(A^TPB+\sigma^2A_0^TPB_0) \times u(t)+x^T(t)(A^TPA+\sigma^2A_0^TPA_0-P)x(t)=0, \tag{18}$$

since $B^TPB + \sigma^2B_0^TPB_0 > 0$, solve the above equation we get the boundary of control set is $u_1(x)$ and $u_2(x)$ as in (14) and the admissible control set as (13). Notice $Ev(x) = Ex^TPx$ is the Lyapunov function, so $Q > 0$. The proof is completed. ■

It is necessary for the proof to check that the admissible control set (13) with the following properties,

- P1: $U(\alpha x) = \alpha U(x),\ \ \alpha > 0$;
- P2: $u_1(x) = -u_2(x),\ $ if $K_{GD}x = 0$.

It will be seen that the above properties play an important role in the analysis of the coarsest quantizer. Firstly, from P2 we know that when $x \perp K_{GD}^T$, $u = 0$ can be used to ensure that the Lyapunov function decreases along trajectories. Thus, the coarsest quantization density can be deduced in the direction of K_{GD}. In fact, consider the subspace spanned by

$$Y_{GD} = \{x \in X; x = y\frac{K_{GD}^T}{K_{GD}K_{GD}^T}, y \in R\}. \tag{19}$$

For the quantizer $g : X \to U$, $g \in \mathcal{Q}(V)$, we make the restriction of g on Y_{GD} as

$$h:\ Y_{GD} \longrightarrow U_{Y_{GD}}$$
$$h(x) = \{g(x)|x \in Y_{GD}\},$$

and the extension g^{GD} of h given by

$$g^{GD} : X \longrightarrow U_{Y_{GD}}$$
$$g^{GD}(x) = h(K_{GD}x).$$

In view of P2, it follow that $g^{GD} \in Q(EV)$. Further note $U_{Y_{GD}} \subseteq U$, so $\sharp g^{GD}[\varepsilon] \le \sharp g[\varepsilon]$, for any $0 < \varepsilon < 1$. Thus we only need to find the coarsest quantizer in the direction of $K_{GD}^{'}$, that is

$$\inf_{g \in E(QV)} \eta_g = \inf_{g^{GD} \in E(QV)} \eta_{g^{GD}}. \tag{20}$$

Now we draw the main results of this paper.

Theorem 1 *The coarsest quantizer is logarithmic and the quantization density is determined by:*

$$\rho = \frac{u_2}{u_1} = \frac{\sqrt{\Delta}-1}{\sqrt{\Delta}+1}. \tag{21}$$

Proof: Let

$$z(t) = Q^{1/2}x,$$

then the boundary points of the control set can be changed into

$$u_{1,2}(x) = K_{GD}Q^{-1/2}z(t) \pm \sqrt{\frac{z^T(t)z(t)}{B^TPB+\sigma^2B_0^TPB_0}}.$$

Denote $K_{GD}Q^{-1/2} = \bar{K}_{GD}$ and decompose z into the following form:

$$z(t) = Q^{-\frac{1}{2}}(A^TPB+\sigma^2A_0^TPB_0)\alpha + w\beta,$$

where $w \perp Q^{-\frac{1}{2}}(A^TPB+\sigma^2A_0^TPB_0)$, hence

$$u_{1,2}(\alpha,\beta) = K_{GD}Q^{-1}(A^TPB+\sigma^2A_0^TPB_0)\alpha \pm \sqrt{\Delta + \frac{\beta^2}{B^TPB+\sigma^2B_0^TPB_0}}. \tag{22}$$

Denote the set $U(\alpha,\beta) = \{u_1(\alpha,\beta) < u < u_2(\alpha,\beta)\}$, then $U(\alpha,\beta)$ is the minimal set when $\beta = 0$. Let

$$\rho = inf_{U(\alpha,0)\bigcap U(1,0)\neq\emptyset}\alpha. \tag{23}$$

Since the boundary points corresponding to $U(1,0)$ are

$$u_1 = K_{GD}Q^{-1}(A^TPB+\sigma^2A_0^TPB_0) + \sqrt{\Delta}, \tag{24}$$

$$u_2 = K_{GD}Q^{-1}(A^TPB+\sigma^2A_0^TPB_0) - \sqrt{\Delta}, \tag{25}$$

it turns out to be that ρ admits

$$\rho = inf_{\alpha u_1 > u_2}\alpha, \tag{26}$$

or (21). When all the values of $K_{GD}x \in (\rho\alpha_0,\ \alpha_0)$, $u_0 = \beta_u$ is the common value of u to be used for the system to be mean-square stabilized. Let

$$\alpha_0 = \beta_\alpha = \alpha\Delta \tag{27}$$

for some α, then

$$u_0 = \alpha\Delta - \sqrt{\alpha^2\Delta}. \tag{28}$$

Substituting β_α in the last equality for α we obtain

$$u_0 = \beta_\alpha(1 - \frac{1}{\sqrt{\Delta}}), \tag{29}$$

and since

$$\sqrt{\Delta} = \frac{1+\rho}{1-\rho}, \tag{30}$$

we have that $\beta_\alpha = \frac{1+\rho}{2\rho}\beta_u$ or $\alpha_0 = \frac{1+\rho}{2\rho}u_0$. Now, from the previous derivation we know that for all the $x \in \bar{\Omega}_0^+ = \{x \in X : \rho\alpha_0 < K_{GD}x < \alpha_0\}$, $u_0 = \beta_u$ guarantees the non-increasing of Ev. Moreover, we know that for any Ω_0 which can be associated to u_0 by any quantizer in $Q(EV)$, $\Omega_0 \bigcap Y_{GD} \subset \bar{\Omega}_0^+ \bigcap Y_{GD}$. Thus, from lemma 1, of which the scaling Property P1 is a consequence, we have that $u_1 = \rho u_0$ guarantees the non-increasing of Ev for all the $x \in \rho\bar{\Omega}^+ = \bar{\Omega}^+$. Furthermore, $u = u_1$ is the smallest value that can be an immediate predecessor of u_0. Fig 1 gives a visual proof of this statement and helps seeing that if $u_1 < \rho u_0$ then there is a gap in the covering of K_{GD}, that cannot be covered by any set Ω associated with any value of control u which is either $u > u_0$ or $u < u_1$ The same arguments can be repeated for u_2 and by introduction for

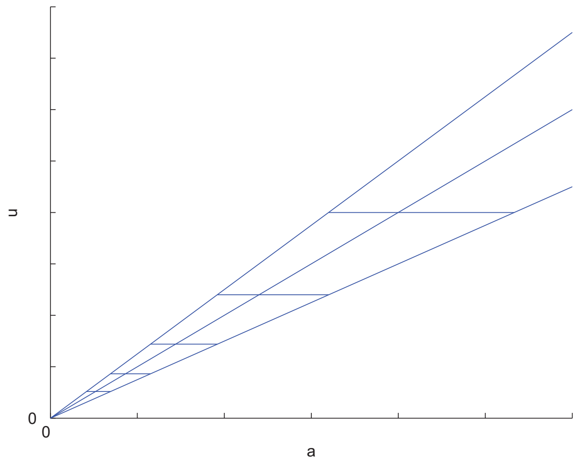

Fig. 1: The optimal partition of u

u_2 and by induction for any u_i. $\bar{\Omega}_i^+$ in the theorem statement are derived from the corresponding $\bar{\Omega}_i^+$ by replacing the the non strict inequality $\leq$ with a strict one, so that f is a well-defined function, and Ω_{zero} is the natural closure of the partition. Thus, we have that the sequence for both positive and negative control values is given by

$$\pm u_{i+1} = \pm u_i \rho,\ i \in Z\ with\ u_0 = \beta_u, \tag{31}$$

and the resulting quantization in the state-space is given by

$$\pm \alpha_{i+1} = \pm \alpha_i \rho,\ i \in Z\ with\ \alpha_0 = \beta_u. \tag{32}$$

From the structure of f it follows that :

$$\eta_f = \frac{2}{ln \frac{2}{\rho}} < \propto, \tag{33}$$

and from the construction we have that for any $g \in EQ(v)$

$$\sharp f[\delta] \leq \sharp g[\delta]\ for\ all\ 0 < \delta \leq 1. \tag{34}$$

Finally, since for any $0 < \varepsilon < 1 - \rho$ and $\rho_\varepsilon = \rho + \varepsilon$, f constructed as f but with ρ instead of ρ_ε belongs to $EQ(v)$, we have that f is the coarsest for $Ev(x(t))$. The proof is completed.■

Remark 3 *When the covariance σ^2 of the noise $w(t)$ is 0, which means that the noise $w(t) = 0$, the quantization density is the same as the deterministic case in [4]. The quantization density for stochastic system with multiplicative noise is larger than that for the deterministic system, this is easy to see that we need more bits to trade off its deterioration aroused by the noises.*

Remark 4 *Denote*

$$\Delta = \gamma^2, \tag{35}$$

and from (30) we can derive that

$$\rho = \frac{\gamma - 1}{\gamma + 1}, \tag{36}$$

which is monotonically increasing over $\gamma \geq 0$, thus minimizing ρ is equivalent to minimizing γ.

What follows is to characterize the coarsest quantizer by searching over all quadratic stochastic control Lyapunov functions, and derive the optimal $P > 0$ such that $Ev(x) = Ex^T Px$ is the lyapunov function and ρ^* is the minimum. We will show that the solvability of the optimal quantizer density is related to solving a special stochastic linear quadratic regulator problem.

Theorem 2 *The optimal P corresponding to ρ^* is given by the semi-positive-definite solution of the following Riccati equation:*

$$\begin{aligned} &P - A^T PA - \sigma^2 A_0^T PA_0 + (A^T PB + \sigma^2 A_0^T PB_0) \\ &\times (B^T PB + \sigma^2 B_0^T PB_0 + 1)^{-1} (A^T PB + \sigma^2 A_0^T PB_0)^T \\ &= 0, \end{aligned} \tag{37}$$

which is also the kernel matrix to the special stochastic LQR problem

$$J(x_0, u) = min \sum_{t=0}^{\infty} E(u^2(t)), \tag{38}$$

corresponding to the minimum energy control that mean-square stabilizes the system (1).

Proof: In view of Remark 4 we know that minimizing ρ is equivalent to minimizing γ. Now consider the following equivalent problem:

$$\gamma^* = inf \gamma \tag{39}$$

subject to
$(A^T PB+\sigma^2 A_0^T PB_0)^T Q^{-1} (A^T PB+\sigma^2 A_0^T PB_0) \leq \gamma^2$,
given that $Q^{-1/2}(A^T PB + \sigma^2 A_0^T PB_0) \in R^n$, then the following implications follow immediately,

$$\begin{aligned} &\frac{(A^T PB+\sigma^2 A_0^T PB_0)^T Q^{-1} (A^T PB+\sigma^2 A_0^T PB_0)}{B^T PB+\sigma^2 B_0^T PB_0} \leq \gamma^2 \\ &Tr\{ \frac{Q^{\frac{-1}{2}} (A^T PB+\sigma^2 A_0^T PB_0)}{(B^T PB + \sigma^2 B_0^T PB_0)} \\ &\quad \times (A^T PB+\sigma^2 A_0^T PB_0)^T Q^{-1/2} \} \leq \gamma^2 \\ &\Leftrightarrow \lambda_{max} \frac{(Q^{\frac{-1}{2}} (A^T PB+\sigma^2 A_0^T PB_0)}{(B^T PB+\sigma^2 B_0^T PB_0)} \\ &\quad \times (A^T PB + \sigma^2 A_0^T PB_0)^T Q^{-1/2}) \leq \gamma^2 \\ &\Leftrightarrow Q^{\frac{-1}{2}} (A^T PB+\sigma^2 A_0^T PB_0)(B^T PB+\sigma^2 B_0^T PB_0)^{-1} \\ &\quad \times (A^T PB + \sigma^2 A_0^T PB_0)^T Q^{\frac{-1}{2}} \leq \gamma^2 I \\ &\Leftrightarrow (A^T PB + \sigma^2 A_0^T PB_0)(B^T PB + \sigma^2 B_0^T PB_0)^{-1} \\ &\quad \times (A^T PB + \sigma^2 A_0^T PB_0)^T \leq \gamma^2 Q \\ &\Leftrightarrow (A^T PB + \sigma^2 A_0^T PB_0)(B^T PB + \sigma^2 B_0^T PB_0)^{-1} \\ &\quad \times (A^T PB+\sigma^2 A_0^T PB_0)^T \leq \gamma^2 (P - A^T PA - \sigma^2 A_0^T PA_0 \\ &\quad + \frac{(A^T PB + \sigma^2 A_0^T PB_0)(A^T PB + \sigma^2 A_0^T PB_0)^T}{B^T PB + \sigma^2 B_0^T PB_0}) \\ &\Leftrightarrow P - A^T PA - \sigma^2 A_0^T PA_0 + (A^T PB + \sigma^2 A_0^T PB_0) \\ &\times (\frac{\gamma^2 (B^T PB+\sigma^2 B_0^T PB_0)}{\gamma^2 - 1})^{-1} (A^T PB+\sigma^2 A_0^T PB_0)^T \\ &\geq 0. \end{aligned} \tag{40}$$

After rearrangement of (40) it follows that we have to find the smallest γ satisfying there is a quadratic lyapunov function $Ev = Ex^T Px$ with $P > 0$ such that

$$\begin{aligned}&P - A^T PA - \sigma^2 A_0^T PA_0 + (A^T PB + \sigma^2 A_0^T PB_0)\\&(B^T PB + \sigma^2 B_0^T PB_0 + \beta)^{-1}(A^T PB + \sigma^2 A_0^T PB_0)^T\\&\geq 0,\end{aligned} \tag{41}$$

where

$$\beta = \frac{B^T PB + \sigma^2 B_0^T PB_0}{\gamma^2 - 1}. \tag{42}$$

Notice that for a given fixed $\beta > 0$, (41) is a Riccati inequality. Since the inequality is not affected by positive scaling of P, we can assume that $\beta = 1$ without loss of generality. It is well-known that P satisfying (41), there holds $P \geq R$, where R is the solution of the following equation:

$$\begin{aligned}&R - A^T RA - \sigma^2 A_0^T RA_0 + (A^T RB + \sigma^2 A_0^T RB_0)\\&(B^T RB + \sigma^2 B_0^T RB_0 + 1)^{-1}(A^T RB + \sigma^2 A_0^T RB_0)^T\\&= 0.\end{aligned}$$

Furthermore, since

$$\frac{B^T PB + \sigma^2 B_0^T PB_0}{\gamma^2 - 1} = 1, \tag{43}$$

thus the problem is transformed into finding the optimal R. Riccati equation (43) shows that the optimal R is the corresponding answer to the following optimal control

$$min \sum_{i=0}^{\infty} Eu_i^2 = Ex^T(0)Rx(0), \tag{44}$$

■

Some reviews are made in order to deduce the optimal P, as illustrated in [9], under the assumptions that (A, A_0, B, B_0) is stabilizable and $(A, A_0/Q^{1/2})$ is observable, the stochastic LQR optimal control problem in infinite horizon time

$$J(x_0, u) = \sum_{t=0}^{\infty} E[x^T(t)Qx(t) + u^T(t)Su(t)], \tag{45}$$

where $Q = Q^T > 0, S = S^T > 0$, has a unique solution is equivalent to that the following algebraic Riccati equation

$$\begin{aligned}&Q = P - A^T PA - \sigma^2 A_0^T PA_0 + (A^T PB + \sigma^2 A_0^T PB_0)\\&(S + B^T PB + \sigma^2 B_0^T PB_0)(A^T PB + \sigma^2 A_0^T PB_0)^T,\end{aligned} \tag{46}$$

has a unique solution $P > 0$, where $S + B^T PB + \sigma^2 B_0^T PB_0 > 0$, and the optimal controller is uniquely determined by

$$\begin{aligned}&u^*(t) = Kx(t) = -(S + B^T PB + \sigma^2 B_0^T PB_0)^{-1}\\&\times(A^T PB + \sigma^2 A_0^T PB_0)^T x(t).\end{aligned}$$

Remark 5 *(38) is a special case of (45) with $Q = 0$, so the above assumptions of stabilizable and exactly observable do not hold, and thus the optimal performance index could not reach. Instead, we choose that $Q = \varepsilon I$, where ε can be chosen arbitrarily close to 0, then we can solve the equation with ε and get the associated quantization density arbitrarily close to the associated cost control problem (44).* ■

Theorem 3 *The quantization density enabling the system (1) exponentially stable is characterized in the following way:*

$$\rho = \frac{\sqrt{\frac{(A^T PB+\sigma^2 A_0^T PB_0)^T Q^{-1}(A^T PB+\sigma^2 A_0^T PB_0)}{B^T PB+\sigma^2 B_0^T PB_0}} - 1}{\sqrt{\frac{(A^T PB+\sigma^2 A_0^T PB_0)^T Q^{-1}(A^T PB+\sigma^2 A_0^T PB_0)}{B^T PB+\sigma^2 B_0^T PB_0}} + 1} \tag{47}$$

where

$$\begin{aligned}&Q = \alpha P - A^T PA - \sigma^2 A_0^T PA_0\\&+\frac{(A^T PB + \sigma^2 A_0^T PB_0)(A^T PB + \sigma^2 A_0^T PB_0)^T}{B^T PB + \sigma^2 B_0^T PB_0}.\end{aligned} \tag{48}$$

Proof: For the exponentially mean-square stability, we know that if

$$EV(x(t+1)) < \alpha^2 EV(x(t)), \tag{49}$$

we can get that

$$E||x(t)||_2^2 < \alpha^t E||x(0)||_2^2. \tag{50}$$

The margin points of the set $u(t)$ that can make the system exponentially mean-square stable are

$$u_{1,2}(x) = K_{GD}x \pm \sqrt{\frac{x^T(t)Qx(t)}{B^T PB + \sigma^2 B_0^T PB_0}}, \tag{51}$$

where K_{GD} is given as in (15), and

$$\begin{aligned}&Q = \alpha P - A^T PA - \sigma^2 A_0^T PA_0\\&+\frac{(A^T PB + \sigma^2 A_0^T PB_0)(A^T PB + \sigma^2 A_0^T PB_0)^T}{B^T PB + \sigma^2 B_0^T PB_0}.\end{aligned} \tag{52}$$

Following the way of the proof of theorem 1, we can get (47). ■

Corollary 1 *The optimal P corresponding to the minimum ρ is given by the semi-positive-definite solution of the following Riccati equation:*

$$\begin{aligned}&\alpha P - A^T PA - \sigma^2 A_0^T PA_0 + (A^T PB + \sigma^2 A_0^T PB_0)\\&\times(B^T PB + \sigma^2 B_0^T PB_0 + 1)^{-1}(A^T PB + \sigma^2 A_0^T PB_0)^T\\&= 0,\end{aligned} \tag{53}$$

which is also the solution to the special linear quadratic regulator problem

$$\sum_{i=0}^{\infty} Eu_i^2 = Ex^T(0)Px(0), \tag{54}$$

corresponding to the minimum energy control that exponentially mean-square stabilizes the system:

$$x(t+1) = (\frac{A}{\sqrt{\alpha}} + \frac{A_0}{\sqrt{\alpha}}w(t))x(t) + (B + B_0 w(t))u(t) \tag{55}$$

with initial state $x(0) = x_0$.

It is interesting to consider a special case that $A_0 = A$, $B_0 = B$, then the system (1) becomes

$$x(t+1) = Ax(t)(1+w(t)) + Bu(t)(1+w(t)), \quad (56)$$

with initial state $x(0) = x_0$. In this case, we have the following interesting observation: if we define $P_1 = (1+\sigma^2)P$, then (35) becomes

$$\gamma^2 = \frac{B^T P_1 A Q^{-1} A^T P_1 B}{B^T P_1 B}, \quad (57)$$

and (41) becomes

$$\frac{P_1}{1+\sigma^2} - (A^T P_1 A + A^T P_1 B)(B^T P_1 B + 1)^{-1} \times (A^T P_1 B + A^T P_1 A)^T = 0. \quad (58)$$

Compared this with (53), we come to the conclusion that the solution above is equivalent to the exponential stabilization with quantized feedback without multiplicative noise and the exponential convergence rate $\alpha = \frac{1}{1+\sigma^2}$.

4 Conclusions

In this paper the coarsest quantizer for the feedback control of stochastic system with multiplicative noise have been discussed. It has been shown that the coarsest quantizer is logarithmic and the density of the quantizer can be characterized by the optimal root of a special linear quadratic regulator problem. The presented result is also extended to the exponentially mean-square stability.

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