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Infinite Horizon LQG Control with Fixed-Rate Quantization for Scalar Systems*

Li Chai
School of Information Science and Engineering
Wuhan University of Science and Technology
Qingshan, Wuhan, 430081, China
chaili@wust.edu.cn

Minyue Fu
School of Control Science and Engineering
Shandong University
Jinan, Shandong Province, China

***Abstract*— We study the infinite-horizon LQG control systems with the constraint that the measurement signal is quantized by a fixed-rate quantizer before going into the controller. It has been shown recently that only weak separation principle holds for the LQG control system with communication channels. In this paper, we study the problem of quantized LQG for a scalar system. An adaptive fixed-rate quantizer is designed to achieve the mean-square stability and the good long term average performance. The long term average cost is divided into two parts. The first part depends on the classical LQG cost, and the second part depends on the distortion of the quantizer. For a quantizer with a fixed bit rate of R (per sample), we show that the quantization distortion order is $R2^{-2R}$ for a large R.**

***Index Terms*— Quantized feedback control, linear quadratic Gaussian control, fixed-rate quantization, quantized estimation.**

I. INTRODUCTION

Recently there has been extensive research on quantized feedback control systems [1], [2], [3] see the survey paper [4]. While lots of results deal with stability and stabilization problems with logarithm quantization in a determinate setup [1], some researchers studied the problem of linear quadratic Gaussian control (LQG) with quantization data [2], [5], [6], [7], [8]. More recently, the weak separation principle is clearly established for finite-horizon LQG control, where a linear predictive code (LPC) with memoryless fixed-rate quantizer is given and separation principle is shown to hold under a high resolution quantization assumption and a mild rank condition[6].

In this paper, we extend the result in [6] to infinite-horizon LQG. For the infinite horizon, the key difficulty is that feedback LPC with memoryless fixed-rate quantizer can not even guarantee stochastic stability, let alone the performance. This is caused by the saturation effect of the finite-support of the quantizer, as shown in [3]. In this paper, we propose a simple LPC scheme with adaptive fixed-rate quantizer for quantized LQG control. We show that the mean-square stability of the quantized feedback system is achieved, and the average distortion is in the order of $N^{-2}\ln N$, where $N = 2^R$, and R is the quantization bit rate.

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II. PROBLEM STATEMENT AND SEPARATION PRINCIPLE

Consider a discrete-time system described by state-space equation

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t \\ y_t &= Cx_t + v_t, \end{aligned} \tag{1}$$

where x_t is the state, u is the control input, and y is the measured output. Assume that w_t and v_t are Gaussian with zero mean and covariances $W > 0$ and $V > 0$, respectively. The cost function is defined as

$$J = \lim_{T\to\infty} \sup \frac{1}{T}\mathcal{E}\left(\sum_{t=0}^{T-1} \begin{bmatrix} x_t \\ u_t \end{bmatrix}^T \begin{bmatrix} Q & H \\ H^T & S \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix}\right), \tag{2}$$

where $S > 0$, $Q \geq 0$, and $Q - HS^{-1}H^{'} \geq 0$.

The problem is to design an observer based controller, and an R-bit uniform quantizer to minimize the cost J.

Let P be the solution of the following Ricatti equation

$$P = Q + A^{'}PA - (B^{'}PA + H)'(S + B^{'}PB)^{-1}(B^{'}PA + H). \tag{3}$$

Define

$$K = -(S + B^{'}PB)^{-1}(B^{'}PA + H). \tag{4}$$

Let the optimal observer based controller is given by $u_t = K\hat{x}_t^q$, where K is the feedback gain matrix, and $\hat{x}_t^q$ is the quantized value of the estimated state from the following Kalman filter

$$\begin{aligned} \hat{x}_t &= \hat{x}_{t|t-1} + \Gamma(y_t - C\hat{x}_{t|t-1}) \\ \hat{x}_{t+1|t} &= A\hat{x}_t + Bu_t. \end{aligned} \tag{5}$$

where $\Gamma = EC^T(CEC^T + V)^{-1}$ and E is the solution of the following Ricatti equation

$$E = AEA^T - AEC^T(CEC^T + V)^{-1}CEA^T + W. \tag{6}$$

The weak separation principle stated below suggests that optimal quantized LQG control can be achieved by first

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constructing the optimal estimate $\hat{x}_t$, which is independent of the cost function, then quantizing it and the optimal control is given by $K\hat{x}_t^q$ [6].

Lemma 1: Consider the system (1), the cost function (2), the quantized feedback controller $K\hat{x}_t^q$, with K given by (4) and $\hat{x}_t$ given by (5), and the R-bit fixed-rate quantization. Then, the quantized LQG controller is optimal if $\hat{x}_t^q$ is obtained by quantizer that minimizes the following distortion function

$$D = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} \mathcal{E}[(\hat{x}_t - \hat{x}_t^q)^{'}\Omega(\hat{x}_t - \hat{x}_t^q)] \qquad (7)$$

where $\Omega = K^{'}(S + B^{'}PB)K$. The corresponding cost function is given by

$$J = J_{LQG} + \min D = tr(PW) + tr(\Omega E) + \min D.$$

An LPC type quantization scheme is proposed in [6] as below

$$\begin{aligned} \hat{x}_t^q &= (A+BK)\hat{x}_{t-1}^q + \varepsilon_t^q \\ \varepsilon_t &= \Gamma(y_t - c\hat{x}_{t|t-1}) + A(\hat{x}_{t-1} - \hat{x}_{t-1}^q) \end{aligned} \qquad (8)$$

where $\hat{x}_{-1}^q = 0$, $\varepsilon_0 = \Gamma v_0$, and ε_t^q is the quantized value of ε_t. Under high resolution quantization and a mild rank condition, it is shown that the complete separation principle holds for finite-horizon LQG control system, which means ε_t^q can be quantized by memoryless quantizer, and the controller and estimator are the same as in Lemma 1. However, the quantization scheme (8) with memoryless quantizer can not guarantee stability if A is unstable, let alone the LQG performance. Hence we have to choose another type of quantizer, such that the quantized feedback system is stable and maintains a good LQG performance.

III. Main result

In this paper, we only deal with a scalar system. That is, the state x_t, the control input u_t and the output y_t are all scalar signals. Define $\eta_t = Ax_t - A\hat{x}_t$. Combining the system (1), the controller $u_t = K\hat{x}_t^q$, the state estimator (5) and the quantizer (8) together, we obtain the following equations

$$\begin{aligned} \eta_{t+1} &= (A - A\Gamma C)\eta_t + \Gamma C w_t + \Gamma v_{t+1} + w_t && (9) \\ \hat{x}_{t+1} &= (A+BK)\hat{x}_t - BK(\varepsilon_t - \varepsilon_t^q) \\ &\quad + \Gamma C\eta_t + \Gamma C w_t + \Gamma v_{t+1} && (10) \\ \varepsilon_{t+1} &= \Gamma C\eta_t + \Gamma C w_t + \Gamma v_{t+1} + A(\varepsilon_t - \varepsilon_t^q). && (11) \end{aligned}$$

Remark 1: 1) Since $\eta_0 = (A - A\Gamma C)x_0 - A\Gamma v_0$, then η_t is Gaussian for any t if the initial state x_0, w_t and v_t are all Gaussian.
2) $A - A\Gamma C$ is stable, and

$$\sigma_{\eta_{t+1}}^2 = (A - A\Gamma C)^2\sigma_{\eta_t}^2 + (\Gamma C + 1)^2\sigma_w^2 + \Gamma^2\sigma_v^2.$$

So

$$\lim_{t\to\infty} \sigma_{\eta_t}^2 = \frac{(\Gamma C+1)^2\sigma_w^2 + \Gamma^2\sigma_v^2}{1 - (A - A\Gamma C)^2}.$$

3) Denote $z_{t+1} = \Gamma C\eta_t + \Gamma C w_t + \Gamma v_{t+1}$, then z_{t+1} is Gaussian with zero mean and variance σ_{t+1}^2, where

$$\sigma_{t+1}^2 = \Gamma^2 C^2(\sigma_{\eta_t}^2 + \sigma_w^2) + \Gamma^2\sigma_v^2. \qquad (12)$$

Denote σ_z^2 as

$$\begin{aligned} \sigma_z^2 \; : &= \lim_{t\to\infty} \sigma_t^2 = \Gamma^2 C^2 \frac{(\Gamma C+1)^2\sigma_w^2 + \Gamma^2\sigma_v^2}{1 - (A - A\Gamma C)^2} \\ &\quad + \Gamma^2 C^2\sigma_w^2 + \Gamma^2\sigma_v^2 \end{aligned} \qquad (13)$$

The quantized LQG control problem becomes to design a quantizer to ε_t given by (11) such that the distortion is minimized. This is a classical problem called quantization for autoregressive sources, about which there is extensive literature [9], [10], [11], [12]. A systematic analysis of the optimal fixed-rate uniform scalar quantization is given for a class of memoryless distributions in [13]. Explicit asymptotic formulas are presented for the distortion and optimal quantizer length approximation, about Gamma distribution, of which Gaussian is a special case. However the results can not be used directly to the quantized LQG control problem since they are based on a key assumption that $|A| \leq 1$. The system we considered is generally unstable, i.e. $|A| > 1$. In this paper, we propose an adaptive fixed-rate quantization scheme based on results in [13].

From now on, we consider the following scalar system

$$\varepsilon_{t+1} = z_{t+1} + A(\varepsilon_t - \varepsilon_t^q) := z_{t+1} + As_t, \qquad (14)$$

where $|A| > 1$, z_{t+1} is Gaussian with variance σ_{t+1}, and s_t denotes the quantization error with the probability distribution function $p_{s_t}(s)$. Let L_t denotes the length of the uniform fixed-rate quantizer at time t, that is, the support of the quantizer is $(-L_t, L_t]$. Denote the PDF of ε_{t+1} as $h_{t+1}(x)$.

Theorem 1: For the scalar system (14), at each time $t+1$, assume the quantization rate R large enough such that $N \gg a$, where $N = 2^R$. Let the uniform fixed-rate quantization scheme is as follows

$$L_{t+1} = \begin{cases} L_{t+1,1} \approx (4\sigma_{t+1}^2 \ln N + A^2 L_t^2)^{\frac{1}{2}} \\ \qquad \text{if } |\varepsilon_t - \varepsilon_t^q| > \frac{\Delta_t}{2} \\ L_{t+1,2} \approx (4\sigma_{t+1}^2 \ln N + N^{-2} L_t^2)^{\frac{1}{2}} \\ \qquad \text{if } |\varepsilon_t - \varepsilon_t^q| \leq \frac{\Delta_t}{2} \end{cases} \qquad (15)$$

where $L_0 \approx 2\sigma_0\sqrt{\ln N}$. Then the distortion satisfies

$$D_{t+1} \approx \frac{4\sigma_{t+1}^2 \ln N}{3N^2} + \frac{A^2}{N^2} D_t. \qquad (16)$$

Combining Lemma (1), Theorem 1 and the equation (13), we obtain the following main result on quantized LQG control.

Theorem 2: Consider the system (1), the cost function (2), the quantized feedback controller $K\hat{x}_t^q$, with K given by (4) and $\hat{x}_t$ given by (5), and the R-bit fixed-rate quantization defined by (15). When $2^R \gg A$, the whole cost function is given by

$$J = J_{LQG} + D_o, \tag{17}$$

where $D_o \approx \frac{4\sigma_z^2 \ln N}{3N^2}$, and σ_z^2 is defined by (13).

IV. Proof of Theorem 1

In order to prove Theorem 1, we need a series of lemmas. For notational simplicity, we denote the PDF of a Gaussian random variable as

$$\rho(\mu, \sigma^2, x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{18}$$

Lemma 2: Let x and y be two independent random variables, and let $z = x + y$. Assume the PDF $p_x(x)$ of x is continuously differentiable. Then $p_z(z)$ is continuously differentiable and

$$p_z{}'(z) = \int_{-\infty}^{\infty} p_x^{'}(z-y) p_y(y) dy.$$

The proof is given in Appendix.

Lemma 3: For the scalar system (14), the PDF $h_t(x)$ of ε_t satisfies the following properties.

Property 1. $h_t(x)$ is symmetric, with finite variance and infinite support.

Property 2. $x h_t(x) \to 0$ as $x \to 0$ and as $x \to \infty$.

Property 3. $h_t(x)$ is continuously differentiable.

Property 4. As $x \to \infty$, $h_t'(x)$ is of the same order as some negative monotonically nondecreasing function.

Property 5. $\limsup_{x\to 0} |h_t'(x)| < \infty$.

The proof is omitted.

For a random variable X, let τ be defined as

$$\tau = \lim_{y\to\infty} T(y)$$

where $T(y) = \frac{E(X|X>y)}{y}$. It is shown in [13] that if τ exists and all moments of the density $p(x)$ are finite, then $\tau = 1$. We summarized some results in [13] by Lemma 4-6, which will be used.

Lemma 4: For a Gaussian distribution with zero mean and variance σ, we have (i) $\tau = 1$. (ii) the optimal quantization length is $L \approx 2\sigma\sqrt{\ln N}$. (iii) For the uniform fixed-rate quantizer with support $(-L, L]$, the distortion satisfies

$$\lim_{N\to\infty} \frac{D^{over}}{D^{gran}} = 0 \tag{19}$$

$$D \approx D^{gran} \approx \frac{\Delta_N^2}{12} = \frac{L^2}{3N^2}. \tag{20}$$

Lemma 5: For $\rho(0, \sigma^2, x)$, define $W_{\sigma^2}(y)$ as

$$W_{\sigma^2}(y) = \frac{1}{\sqrt{2\pi}\sigma} \int_y^{\infty} (x-y)^2 e^{-\frac{x^2}{2\sigma^2}} dx.$$

Then

$$\lim_{N\to\infty} \frac{D_L^{over}}{2W(L)} = 1 \tag{21}$$

$$W_{\sigma^2}(y) = \frac{2\sigma^5}{\sqrt{2\pi}} y^{-3} e^{-\frac{y^2}{2\sigma^2}} (1 + o_y(1)), \tag{22}$$

where $o_y(1)$ tends to zero as y tends to infinity.

Lemma 6: For any source density whatsoever

$$\lim_{N\to\infty} \frac{D^{gran}}{\Delta_N^2/12} = 1.$$

We are now ready to state the main technical lemmas for the proof of Theorem 1.

Lemma 7: For the scalar system (14), let the quantizer be defined as (15), the PDF $p_{s_t}(s)$ of $s_t = \varepsilon_t - \varepsilon_t^q$ is given by

$$p_{s_t}(s) = \begin{cases} h_t(-s + L_t + \frac{\Delta_t}{2}) & s < -\frac{\Delta_t}{2} \\ p_{s_k}(s) & |s| \le \frac{\Delta_t}{2} \\ h_t(-s - L_t - \frac{\Delta_t}{2}) & s > \frac{\Delta_t}{2}, \end{cases} \tag{23}$$

where $h_t(s)$ is the PDF of ε_t. Furthermore, when $|s| > \frac{\Delta_t}{2}$, $p_{s_t}(s)$ satisfies

$$p_{s_t}(s) \le \frac{1}{N^2} \rho(0, \frac{L_t^2}{4\ln N}, s) \tag{24}$$

for both $L_t = L_{t,1}$ and $L_t = L_{t,2}$, where $\rho(0, \frac{L_t^2}{4\ln N}, s)$ is defined as (18).

Lemma 8: For the scalar system (14), let the quantizer be defined as (15), the saturation probability satisfies

$$\Pr(|\varepsilon_t - \varepsilon_t^q| > \frac{\Delta_t}{2}) \le N^{-2} \tag{25}$$

for any $t \ge 0$.

Lemma 9: For the scalar system (14), let the quantizer be defined as (15), we have

$$D_{t+1}^{gran} \approx \frac{4\sigma_{t+1}^2 \ln N}{3N^2} + \frac{A^2}{N^2} D_t^{gran}, \ \forall t \ge 0. \tag{26}$$

and

$$\lim_{N\to\infty} \frac{D_{t+1}^{over}}{D_{t+1}^{gran}} = 0, \ \forall t \ge 0 \tag{27}$$

Theorem 1 follows from Lemma 9 directly by using the fact that $D_{t+1} = D_{t+1}^{over} + D_{t+1}^{gran}$.

V. Numerical Example

Consider the system (14) with A=8. We use the quantization scheme (15) with $R = 3$, $R = 4$, $R = 5$, $R = 6$, and $R = 8$. For each cases, $D_t = \frac{1}{T}(\varepsilon_t - \varepsilon_t^q)^2$ is plotted from $t = 1$ to $t = 5000$ in Fig. 1. When $R = 2$, the system is unstable. When $R = 8$, $N = 256$ and $\frac{\ln(N)}{N^2} = 8.4 \times 10^{-5}$. When $R = 6$, $N = 64$ and $\frac{\ln(N)}{N^2} = 0.001$. Those data are consistent with Fig. 1. Table 1 shows the total times that ε_t^q achieves the bound (i.e. it saturates) for different R. We see that the larger R is, the smaller the number of saturation times.

Table 1: Saturation times of different R.

	$R=3$	$R=4$	$R=5$	$R=6$	$R=8$
ε^q saturate	2142	1298	514	259	97
ε^q not saturate	2857	3702	4486	4741	4903

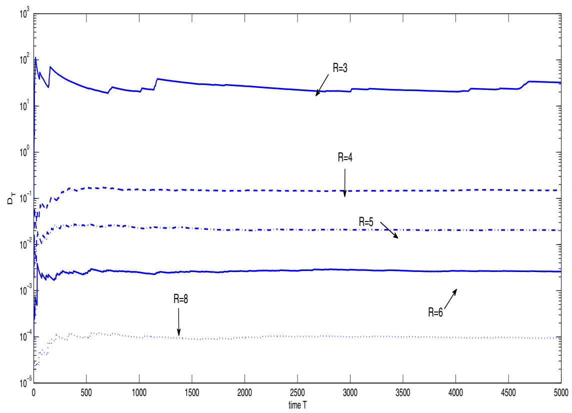

Fig. 1 Distortion for different R.

VI. Conclusion

In this paper, we have studied the infinite-horizon quantized LQG control problem. Under high resolution quantization framework, an adaptive fixed-rate quantization scheme is proposed to achieve the stochastic stability and the LQG performance. We have shown that the average quantization distortion has the order of $R2^{-2R}$ under high resolution quantization, which is the same with that of LPC scheme with memoryless quantizer.

Appendix

A1. Proof of Lemma 2

Proof: Since x and y are independent,

$$f(x,y) = p_x(x)p_y(y).$$

Note that

$$\begin{aligned} F_Z(z) &= \Pr\{x+y\le z\} = \int_{-\infty}^{\infty} F_X(z-y)p_y(y)dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} p_x(x)p_y(y)dxdy. \end{aligned}$$

Then

$$\begin{aligned} h(z) &:= \frac{dF_Z(z)}{dz} = \frac{d}{dz}\int_{-\infty}^{\infty}\int_{-\infty}^{z-y} p_x(x)p_y(y)dxdy \\ &= \int_{-\infty}^{\infty}\frac{d}{dz}\left[\int_{-\infty}^{z-y} p_x(x)dx\right]p_y(y)dy \\ &= \int_{-\infty}^{\infty} f(z-y)p_y(y)dy. \end{aligned}$$

For arbitrary small $\delta>0$, we have

$$\begin{aligned} &\lim_{\delta\to 0}\frac{p_z(z+\delta)-p_z(z)}{\delta} \\ =\ &\lim_{\delta\to 0}\frac{1}{\delta}\int_{-\infty}^{\infty}\left[p_x(z+\delta-y)-p_x(z-y)\right]p_y(y)dy \\ =\ &\lim_{\delta\to 0}\frac{1}{\delta}\int_{-\infty}^{\infty}[p_x^{'}(z-y)\delta+o(\delta)]p_y(y)dy \\ =\ &\int_{-\infty}^{\infty} p_x^{'}(z-y)p_y(y)dy. \end{aligned}$$

The second equality follows from the continuously differentiable property of $p_x(x)$. Similarly, the following holds

$$\lim_{\delta\to 0}\frac{p_z(z-\delta)-p_z(z)}{-\delta} = \int_{-\infty}^{\infty} p_x^{'}(z-y)p_y(y)dy.$$

Therefore, $p_z(z)$ is differentiable, and

$$p_z^{'}(z) = \int_{-\infty}^{\infty} p_x^{'}(z-y)p_y(y)dy.$$

■

A2. Proof of Lemma 7

Proof:

We proceed by induction. At $t=0$, recall that $L_0 = 2\sigma_0\sqrt{\ln N}$. Hence

$$p_{s_0}(s) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma_0}e^{-\frac{(s-L_0-\frac{\Delta_0}{2})^2}{2\sigma_0^2}} & s<-\frac{\Delta_0}{2}, \\ p_{s_t}(s) & |s|\le\frac{\Delta_0}{2}, \\ \frac{1}{\sqrt{2\pi}\sigma_0}e^{-\frac{(s+L_0+\frac{\Delta_0}{2})^2}{2\sigma_0^2}} & s>\frac{\Delta_0}{2}. \end{cases}.$$

Note that

$$e^{-\frac{(L_0+\frac{\Delta_0}{2})^2}{2\sigma_0^2}} \le e^{-\frac{L_0^2}{2\sigma_0^2}} = \frac{1}{N^2}.$$

Therefore

$$p_{s_0}(s) \le \frac{1}{N^2}\rho(0,\sigma_0^2,s) = \frac{1}{N^2}\rho(0,\frac{L_0^2}{4\ln N},s).$$

Now we assume $p_{s_t}(s)$ satisfies (23) and (24). It follows from (14) that the PDF $h_{t+1}(x)$ of ε_{t+1} is

$$h_{t+1}(x) = \rho(0,\sigma_{t+1}^2,x) * p_{As_t}(s).$$

It is easy to see that the PDF $p_{s_{t+1}}(s)$ of $s_{t+1} = \varepsilon_{t+1} - \varepsilon_{t+1}^q$ satisfies (23), whenever $L_{t+1} = L_{t+1,1}$, or $L_{t+1} = L_{t+1,2}$.

Next we show that (24) also holds for both cases. At the case of $L_{t+1} = L_{t+1,1}$, it follows from (24) that

$$p_{As_t}(s) \leq \frac{1}{N^2}\rho(0, \frac{A^2L_t^2}{4\ln N}, s).$$

Hence, when $\left|\varepsilon_{t+1} - \varepsilon_{t+1}^q\right| > \frac{\Delta_{t+1,1}}{2}$, we have

$$\begin{aligned}
& p_{s_{t+1}}(s) \\
= \;& \rho(0, \sigma_{t+1}^2, s + L_{t+1,1} + \frac{\Delta_{t+1,1}}{2}) * p_{As_t}(\bar{s}) \\
\leq \;& \frac{1}{N^2}\rho(0, \sigma_{t+1}^2, s + L_{t+1,1} + \frac{\Delta_{t+1,1}}{2}) * \rho(0, \frac{A^2L_t^2}{4\ln N}, \bar{s}) \\
\leq \;& \frac{1}{N^2}\rho(0, \sigma_{t+1}^2 + \frac{A^2L_t^2}{4\ln N}, s + L_{t+1,1} + \frac{\Delta_{t+1,1}}{2}) \\
\leq \;& \frac{1}{N^4}\rho(0, \frac{L_{t+1,1}^2}{4\ln N}, s).
\end{aligned}$$

At the case of $L_{t+1} = L_{t+1,2}$, it follows from $|\varepsilon_t - \varepsilon_t^q| \leq \frac{\Delta_t}{2}$ that s_t is bounded. When $\left|\varepsilon_{t+1} - \varepsilon_{t+1}^q\right| > \frac{\Delta_{t+1,2}}{2}$, we have

$$\begin{aligned}
& p_{s_{t+1}}(s) \\
= \;& \rho(0, \sigma_{t+1}^2, s + L_{t+1,2} + \frac{\Delta_{t+1,2}}{2}) * p_{As_t}(\bar{s}) \\
= \;& \frac{1}{\sqrt{2\pi}\sigma_{t+1}} e^{-\frac{(s+L_{t+1,2}+\frac{\Delta_{t+1,2}}{2}-\alpha_t)^2}{2\sigma_{t+1}^2}} \int_{-\frac{|A|\Delta_t}{2}}^{\frac{|A|\Delta_t}{2}} p_{As_t}(s)ds \\
\leq \;& \frac{1}{N^2}\frac{1}{\sqrt{2\pi}\sigma_{t+1}} e^{-\frac{s^2}{2\sigma_{t+1}^2}} \\
\leq \;& \frac{1}{N^2}\rho(0, \frac{L_{t+1,2}^2}{4\ln N}, s),
\end{aligned}$$

where $\alpha_t \in [-\frac{|A|\Delta_t}{2}, \frac{|A|\Delta_t}{2}]$. The second equality follows from the integral mean value theorem. The proof is then completed by induction. ■

A3. Proof of Lemma 8

Proof:

Using (25), we have

$$\begin{aligned}
\Pr(|\varepsilon_t - \varepsilon_t^q| > \frac{\Delta_t}{2}) &= 2\int_{\frac{\Delta_t}{2}}^{\infty} p_{s_t}(s)ds \\
\leq \frac{1}{N^2}\int_{-\infty}^{\infty} \rho(0, \frac{L_t^2}{4\ln N}, s)ds &= \frac{1}{N^2}
\end{aligned}$$

holds for both $\Delta_t = \Delta_{t,1}$ and $\Delta_t = \Delta_{t,2}$. This completes the proof. ■

A4. Proof of Lemma 9

Proof:

Assume that $\Pr(L_{t+1} = L_{t+1,1}) = \alpha$. It follows from Lemma 6 that the granular distortion at time $t+1$ is

$$\begin{aligned}
D_{t+1}^{gran} \;&\approx\; \mathcal{E}(\frac{L_{t+1}^2}{3N^2}) = \Pr(L_{t+1} = L_{t+1,1})\frac{L_{t+1,1}^2}{3N^2} \\
&\quad + \Pr(L_{t+1} = L_{t+1,2})\frac{L_{t+1,2}^2}{3N^2} \\
&= \alpha\mathcal{E}(\frac{4\sigma_{t+1}^2\ln N + A^2L_t^2}{3N^2}) \\
&\quad + (1-\alpha)\mathcal{E}(\frac{4\sigma_{t+1}^2\ln N + N^{-2}L_t^2}{3N^2}) \\
&= \frac{4\ln N}{3N^2}\sigma_{t+1}^2 + (\frac{1}{N^2} + \alpha\frac{N^2A^2-1}{N^2})\mathcal{E}(\frac{L_t^2}{3N^2})
\end{aligned}$$

Using Lemma 8, we know that

$$\alpha = \Pr(\left|\varepsilon_{t+1} - \varepsilon_{t+1}^q\right| > \frac{\Delta_{t+1,1}}{2}) \leq N^{-2}.$$

Therefore

$$\begin{aligned}
D_{t+1}^{gran} \;&\leq\; \frac{4\ln N}{3N^2}\sigma_{t+1}^2 + (\frac{1}{N^2} + \frac{1}{N^2}\frac{N^2A^2-1}{N^2})\mathcal{E}(\frac{L_t^2}{3N^2}) \\
&= \frac{4\ln N}{3N^2}\sigma_{t+1}^2 + \frac{(A^2+1)}{N^2}(1+o(N))D_t^{gran} \\
&\approx \frac{4\ln N}{3N^2}\sigma_{t+1}^2 + \frac{(A^2+1)}{N^2}D_t^{gran}.
\end{aligned}$$

Under the high rate assumption, we have $\frac{(A^2+1)}{N^2} \ll 1$, hence D_t^{gran} is finite for all t, and

$$\lim_{t\to\infty} D_{t+1}^{gran} = \frac{4\ln N}{3N^2}\lim_{t\to\infty}\sigma_{t+1}^2 = \frac{4\ln N}{3N^2}\sigma_z^2.$$

This completes the proof of (26).

Using Lemma 5, the overload distortion can be computed as

$$\begin{aligned}
D_{t+1}^{over} \;&=\; \Pr(L_{t+1} = L_{t+1,1}) \\
&\quad \cdot\int_{L_{t+1,1}}^{\infty} (x - L_{t+1,1} + \frac{\Delta_{t+1,1}}{2})^2 h_{t+1}(x)dx \\
&\quad + \Pr(L_{t+1} = L_{t+1,2}) \\
&\quad \cdot\int_{L_{t+1,2}}^{\infty} (x - L_{t+1,2} + \frac{\Delta_{t+1,1}}{2})^2 h_{t+1}(x)dx \\
&\leq N^{-2}N^{-2}(\sigma_{t+1}^2 + A^2\sigma_{\varepsilon_t}^2) \\
&\quad + (1-N^{-2})N^{-2}(\sigma_{t+1}^2 + N^{-2}\sigma_{\varepsilon_t}^2) \\
&= N^{-2}\sigma_{t+1}^2 + (A^2+1)N^{-4}\sigma_{\varepsilon_t}^2(1+o(N)) \\
&\approx N^{-2}\sigma_{t+1}^2 + (A^2+1)N^{-4}(D_t^{gran} + D_t^{over}).
\end{aligned}$$

Therefore, we have

$$\frac{D_{t+1}^{over}}{D_{t+1}^{gran}} = \frac{\sigma_{t+1}^2 + (A^2+1)N^{-2}(D_t^{gran} + D_t^{over})}{4\sigma_{t+1}^2\ln N + (A^2+1)D_t^{gran}}. \quad (28)$$

It follows from Lemma 4 that $\lim_{N\to\infty}\frac{D_0^{over}}{D_0^{gran}} = 0$. By induction of (28), we have

$$\lim_{N\to\infty}\frac{D_{t+1}^{over}}{D_{t+1}^{gran}} = 0$$

for any t. This completes the proof.

■

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