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A Class of Weak Kharitonov Regions for Robust Stability of Linear Uncertain Systems

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Abstract—**In this note, the Kharitonov's theorems are generalized to the problem of so-called weak Kharitonov regions for robust stability of linear uncertain systems. Given a polytope of (characteristic) polynomials P and a stability region D in the complex plane, P is called D-stable if the zeros of every polynomial in P is contained in D. It is of interest to know whether the D-stability of the vertexes of P implies the D-stability of P. A simple approach is developed which unifies and generalizes many known results on this problem.**

I. Introduction

Consider a family of characteristic polynomials P associated with a linear dynamic system containing parameter perturbations

$$P \doteq \left\{ p(s,q) = \sum_{i=0}^{n} a_i(q)s^i : q \in Q \right\}, \qquad a_n(q) \neq 0, \forall q \in Q \tag{1}$$

where

$$q \doteq [q_1, q_2, \cdots, q_m]^T \tag{2}$$

is the vector of *perturbation parameters* with each q_i varying in the *bounding rectangle*

$$Q_i \doteq \{t_i + jw_i : \underline{t}_i \le t_i \le \bar{t}_i, \underline{w}_i \le w_i \le \bar{w}_i\} \subset C, \tag{3}$$

$$Q \doteq Q_1 \times Q_2 \times \cdots \times Q_m \tag{4}$$

is the bounding set of q, and $a_i(q)$ is the ith coefficient of $p(s,q)$. It is assumed that $a_i(q)$ are affine functions of q and that each Q_i contains zero. Under these assumptions, we can rewrite $p(s,q)$ in (1) as

$$p(s,q) = p_0(s) + \sum_{i=1}^{m} q_i p_i(s) \tag{5}$$

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where $p_0(s)$ is the *nominal polynomial* which is obtained from $p(s,q)$ by setting $q = 0$, and $p_i(s)$ are the *perturbation polynomials*, obtained from $p(s,q) - p_0(s)$ by setting $q_i = 1$ and $q_k = 0$, $\forall k \neq i$. Accordingly, the family of polynomials P in (1) becomes the so-called *polytope of polynomials* and can be rewritten as

$$P = \left\{ p_0(s) + \sum_{i=1}^{m} q_i p_i(s) : q_i \in Q_i, \qquad i = 1, 2, \cdots, m \right\}. \tag{6}$$

Many systems with parameter variations can be captured by the aforementioned description. A simple example is the so-called *interval polynomial* for which

$$p(s,q) = p_0(s) + \sum_{i=0}^{n} q_i s^i, \tag{7}$$

i.e., each coefficient of the polynomial lies in a given interval which reflects the inaccuracy of the coefficient due to modeling or estimation error. Another trival example is

$$p(s,q) = p(s,[k,\tau]) = s^2 + (2 + \tau + k)s + \tau + k$$

which is the characteristic polynomial of the unity feedback system with open-loop transfer function equal to $k(s+1)/(s+2)(s+\tau)$, where k and τ are uncertain gain and time constant, respectively. For further engineering motivation of this type of polynomials, the reader is referred to, among numerous papers and books, [1]-[3] and the references therein.

For convenience, we denote

$$p \doteq (p_0(s), p_1(s), \cdots, p_m(s)). \tag{8}$$

The set of vertex polynomials of P is given by

$$V_P \doteq \{p(s,q) : q_i \in \{q_{i1}, q_{i2}, q_{i3}, q_{i4}\}, \qquad i = 1, 2, \cdots, m\} \tag{9}$$

where q_{i1}, q_{i2}, q_{i3}, and q_{i4} are the vertexes of Q_i. Note that if the perturbation parameter q_i is purely real, the Q_i becomes an interval and the number of its vertexes is dropped to two.

Given the family of (characteristic) polynomials as in (1) and a stability region D in C (the complex plane), it is of interest to determine whether the zeros of every polynomial in P are contained in D. The stability regions are usually subsets of C_- (the open left-half plane) for continuous-time systems, or subsets of the open unit disk for discrete-time systems.

We now give the definitions of D-stability, anti-D-stability, and weak Kharitonov regions. In the following, D^c and ∂D denote the complement and the boundary of D, respectively.

Definition 1.1 [1], [4]: Given an open set $D \subset C$, a polynomial $p(s)$ is called D-stable (respectively, anti-D-stable) if every zero of $p(s)$ is contained in D (respectively, D^C, including ∂D). A family of polynomials P is called D-stable (respectively anti-D-stable) if every polynomial in P is D-stable (respectively anti-D-stable).

Definition 1.2: Let p be given in (8). A set $D \subset C$ is called a weak Kharitonov region with respect to p if the following condition holds: For an arbitrary bounding set Q of the form (4) and (3), P in (6) is D-stable if and only if V_P in (9) is D-stable.

The notion of weak Kharitonov region comes from the seminal work by Kharitonov [5], [6] where he considered the special case for which $D = C_-$ and P is an interval polynomial as in (7). He showed that P is C_--stable if and only if V_P is C_--stable, and furthermore, if and only if eight special vertex polynomials in V_P

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are C_--stable or four special ones when the coefficients of the interval polynomials are purely real. Since the later result requires checking much less number of polynomials than the former one and this number is independent of the polynomial degree, it is often referred to as the strong version while the former result the weak version.

The objective of this paper is as follows: given a family of polynomials P as in (6) and a stability region $D \subset C$, determine whether D is a weak Kharitonov region. The most pertinent results to this note are those by Petersen [7], [4], Soh and Berger [8], Soh [9], Hollot and Bartlett [10], Kraus *et al.* [11], and Bialas and Garloff [12]. In [7], the regions in C_- which can be mapped onto C_- by the so-called *strongly admissible* rational functions [13] are considered and a number of interesting regions are found to be weak Kharitonov regions. In [4], it is shown that C_- is a weak Kharitonov region if $p_i(s)$ are all anti-D-stable. In [8], [9], some sectors in the left-half plane are proven to be weak Kharitonov regions provided that the polynomial coefficients are real. In [10], [11], some special conditions on $p_i(s)$ are found for the open unit disk to be a weak Kharitonov region. In [12], Polynomials with even or odd perturbations, i.e., each $p_i(s)$ is either an even or odd polynomial, are shown to be Hurwitz if the vertex polynomials are Hurwitz.

In this note, a new approach to the problem of weak Kharitonov regions is developed using the concept of *decreasing phase property* for stability region D defined as follows.

Definition 1.3: Given a stability region $D \subset C$ and the polynomial vector $\boldsymbol{p}$ in (8), $\boldsymbol{p}$ is said to hold the *decreasing phase property* if, for an arbitrary nth order D-stable polynomial $f(s)$ and $1 \leq i \leq m$, arg $p_i(s)/f(s)$ is monotonously descreasing except at $p_i(s) = 0$ when s traverses on ∂D in the counterclockwise direction (or, for short, monotonously descreasing on ∂D).

We now end our introduction with a key theorem which links the problem of weak Kharitonov regions to the decreasing phase property discussed previously.

Theorem 1.1 (see the Appendix for Proof): Let an open set $D \subset C$ and $\boldsymbol{p} = (p_0(s), p_1(s), \cdots, p_m(s))$ be given. Then D is a weak Kharitonov region with respect to $\boldsymbol{p}$ if $\boldsymbol{p}$ holds the descreasing phase property.

II. Weak Kharitonov Regions

In this section, Theorem 1.1 is used to derive a number of useful weak Kharitonov regions for both continuous-time and discrete-time systems. These results unify and generalize many known results in [4]–[12].

Theorem 2.1: Any (rotated) open-half plane

$$D = \{x + jy: a + bx + cy < 0\}, \qquad a, b, c \in \boldsymbol{R} \quad (10)$$

is a weak Kharitonov region with respect to $\boldsymbol{p}$ in (8) for any $p_0(s)$ if $p_i(s)$, $i = 1, 2, \cdots, m$ are anti-D-stable.

Proof: Suppose $p_i(s)$, $i = 1, 2, \cdots, m$ are anti-D-stable. Let $f(s)$ be an arbitrary nth order D-stable polynomial. It is straightforward to see that arg $p_i(s)/f(s)$ is monotonously decreasing on ∂D because arg $p_i(s)$ (respectively, arg $f(s)$) are monotonously nonincreasing (respectively, increasing). Therefore, it follows from Theorem 1.1 that D is a weak Kharitonov region with respect to $\boldsymbol{p}$. □

Remark: The aforementioned theorem is an extension to the main result in [4] where $D = C_-$ is considered.

Corollary 2.1 [7]: Any open-half plane

$$D = \{x + jy: x < -a + by\}, \qquad a \geq 0, b \in \boldsymbol{R} \quad (11)$$

is a weak Kharitonov region with respect to $\boldsymbol{p} = (p_0(s), 1, s, \cdots, s^n)$ for any $p_0(s)$ of nth-order. In particular, $\boldsymbol{C}_-$ is a weak Kharitonov region with respect to the aforementioned $\boldsymbol{p}$ [5].

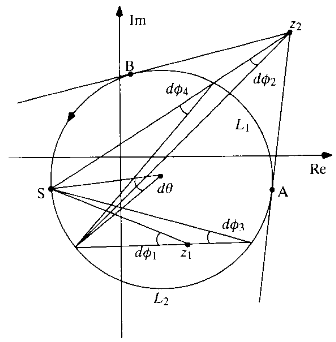

Fig. 1.

Theorem 2.2 [7], [8]: Any open region

$$D = \{x + jy: a_1 - b_1 |x| < y < -a_1 + b_1 |x|, x < -a_3\}, \qquad a_i \geq 0, b_i > 0 \quad (12)$$

is a weak Kharitonov region with respect to $\boldsymbol{p} = (p_0(s), 1, s, \cdots, s^n)$ for any $p_0(s)$ of nth-order.

Proof: The proof is essentially identical to that of Theorem 2.1. □

Theorem 2.3: Any open circular region

$$D = \{s: s = c + \rho \exp(j\theta): 0 \leq \rho < r, 0 \leq \theta \leq 2\pi\}, \qquad c \in \boldsymbol{C}, r > 0 \quad (13)$$

is a weak Kharitonov region with respect to $\boldsymbol{p}$ in (8) for any $p_0(s)$ if $p_i(s)$, $i = 1, 2, \cdots, m$ are anti-D-stable.

Proof: Let $f(s)$ be any nth-order D-stable polynomial. From Theorem 1.1, it is sufficient to show that arg $p^i(s)/f(s)$, $1 \leq i \leq m$, is monotonously decreasing when s traverses on ∂D. Let z_1 and z_2 be any zeros of $f(s)$ and $p_i(s)$, respectively, see Fig. 1. We claim that arg $(s - z_2)/(s - z_1)$ is monotonously descreasing. To see this, we divide ∂D into L_1 and L_2 according to the tangent points A and B in Fig. 1. When s traverses on L_1, arg $(s - z_2)/(s - z_1)$ is obviously decreasing because arg $(s - z_1)$ is increasing and arg $(s - z_2)$ is decreasing. Now suppose s traverses on L_2 and θ is increased by $d\theta$. Note that both arg $(s - z_1)$ and arg $(s - z_2)$ are increased. Therefore, we need to prove that the increment $d\phi_1$ of arg $(s - z_1)$ is greater than the increment $d\phi_2$ of arg $(s - z_2)$. This is not difficult to see from Fig. 1 because $d\phi_1 > d\phi_3$, $d\phi_2 \leq d\phi_4$, and $d\phi_3 = d\phi_4 = d\theta/2$. Consequently, arg $(s - z_2)/(s - z_1)$ is monotonously decreasing on L_2. Hence, our claim holds. We then conclude that arg $p_i(s)/f(s)$ is monotonously descreasing on ∂D because the number of zeros of $p_i(s)$ is no more than that of $f(s)$. □

Corollary 2.2 [7]: Any open circular regions

$$D = \{x + jy: (x + a)^2 + y^2 < r^2\}, \qquad 0 \leq r \leq a \quad (14)$$

and

$$D = \{x + jy: (x - a)^2 + y^2 < r^2\}, \qquad 0 \leq r \leq a \leq 1/2 \quad (15)$$

are weak Kharitonov regions with respect to $\boldsymbol{p} = (p_0(s), 1, s, \cdots, s^n)$ for any $p_0(s)$ of nth-order.

Theorem 2.4: Any open parabolic region

$$D = \{x + jy: (ax)^2 - (by)^2 > 1, x < 0\}, \qquad a, b > 0 \quad (16)$$

is a weak Kharitonov region with respect to $p(s) = (p_0(s), 1, s, \cdots, s^n)$ for any $p_0(s)$ of nth-order.

Proof: The proof is essentially identical to that of Theorem 2.1. □

Theorem 2.5 [9]: Any region

$$D = \{x + jy: x < 0, a|x| < |y| < b|x|\}, \qquad b \geq a > 0 \tag{17}$$

is a weak Kharitonov region with respect to $p(s) = (p_0(s), 1, s, \cdots, s^n)$ for any $p_0(s)$ of nth-order provided that the parameters q_i, $i = 1, 2, \cdots, n$ and the coefficients of $p_0(s)$ are real.

Proof: Let $f(s)$ be any nth-order D-stable polynomial with zeros given by $z_1, z_1^*, z_2, z_2^*, \cdots$, where z_k^* denotes the complex conjugate of z_k, $\text{Im}(z_k) > 0$. By Theorem 1.1, it is sufficient to show that $\arg s^i/f(s)$ is monotonously descreasing when s traverses on ∂D for any $0 \leq i \leq n$. Note that ∂D is given by $|y| = b|x|$ or $|y| = a|x|$. We first observe that $\arg s^i$ is fixed on ∂D. Therefore, we only need to show that $\arg(s - z_k)(s - z_k^*)$ is monotonously increasing on each ∂D. This holds trivially on $|y| = b|x|$. On $y = a|x|$, this holds because $\arg(z - z_k)$ increases faster than $\arg(z - z_k^*)$ decreases. A similar argument applies to $y = -a|x|$. Therefore, $\arg s^i/f(s)$ is monotonously descreasing on ∂D. □

Theorem 2.6: Every open convex region $D \subset C$ is a weak Kharitonov region with respect to $(p_0(s), 1)$ for any $p_0(s)$.

Proof: Let $f(s)$ be any D-stable polynomial. It is obvious that $\arg 1/f(s)$ is monotonously descreasing on ∂D. Therefore, it follows from Theorem 1.1 that D is a weak Kharitonov region with respect to $(p_0(s), 1)$. □

Theorem 2.7 [12]: Let $\boldsymbol{p}$ be given in (8) satisfying the following condition: for each $i = 1, 2, \cdots, m$, either $\text{Re}\, p_i(j\omega) \equiv 0$ or $\text{Im}\, p_i(j\omega) \equiv 0$. Then C_- is a weak Kharitonov region with respect to $\boldsymbol{p}$.

Proof: Let $f(s)$ be an arbitrary nth-order C_--stable polynomial. From Theorem 1.1, it is sufficient to show that $\arg p_i(j\omega)/f(j\omega)$ is monotonously descreasing when ω increases except at $p_i(j\omega) = 0$, $i = 1, 2, \cdots, m$. This is obvious because $\arg f(j\omega)$ is monotonously increasing and $p_i(j\omega)$ is either purely real or purely imaginary without phase change. □

Theorem 2.8 [11]: The open unit disk is a weak Kharitonov region with respect to

$$\boldsymbol{p} = \big(p_0(s), 1 + s^n, 1 - s^n, s + s^{n-1}, s - s^{n-1}, \cdots, s^{[n/2]} + s^{n-[n/2]}, s^{[n/2]} - s^{n-[n/2]}\big)$$

for any $p_0(s)$ of nth-order, where $[\cdot]$ denote the integer part.

Proof: Let D be the open unit disk and $f(s)$ be any nth-order D-stable polynomial. Note that $\partial D = \{\exp(j\theta): 0 \leq \theta \leq 2\pi\}$. By applying Theorem 1.1, it is sufficient to show that $\arg(s^i \pm s^{[n-i]})/f(s)$ is monotonously descreasing for any $i \leq [n/2]$ when s traverses on ∂D. Note that

$$\begin{aligned}\exp(j\theta) &\pm \exp(j(n-i)\theta) \\ &= \exp\left(j\frac{n}{2}\theta\right)\left(\exp\left(j\left(i - \frac{n}{2}\right)\theta\right) \pm \exp\left(j\left(\frac{n}{2} - i\right)\theta\right)\right) \\ &= \begin{cases} 2\cos\left(\left(i - \frac{n}{2}\right)\theta\right)\exp\left(j\frac{n}{2}\theta\right) \text{ or} \\ 2j\sin\left(\left(i - \frac{n}{2}\right)\theta\right)\exp\left(j\frac{n}{2}\theta\right)\end{cases}\end{aligned}$$

and its phase is either $n\theta/2$ or $(-\pi + n\theta)/2$. Let z_k, $k = 1, 2, \cdots, n$ be the zeros of $f(s)$, $s = \exp(j\theta)$ and suppose θ is increased by $d\theta$, as shown in Fig. 2. Then, $\arg s^i \pm s^{[n-i]}$ is increased by $n\,d\theta/2$. On the other hand, $\arg f(s)$ is increased by more than $n\,d\theta/2$ because $d\theta_k > d\phi_k = d\theta/2$ (see Fig. 2). Consequently, $\arg(s^i \pm s^{[n-i]})/f(s)$ is monotonously descreasing on ∂D. □

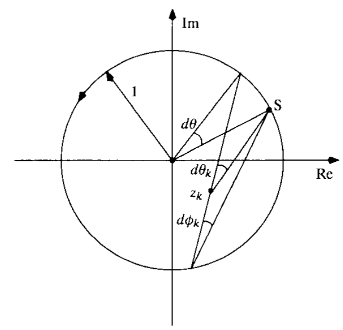

Fig. 2.

Theorem 2.9 [10]: The unit disk is a weak Kharitonov region with respect to $p = (p_0(s), 1, s, \cdots, s^{[n/2]})$ for any $p(s)$ of nth-order.

Proof: The proof of this Theorem is exactly the same as that of Theorem 2.8 except that $s^i \pm s^{[n-i]}$ is replaced by s^i and that $\arg s^i$ is increased by only $i\,d\theta$ rather than $n\,d\theta$ when θ is increased by $d\theta$. □

To summarize, the weak Kharitonov regions are tabulated in Tables I and II for continuous-time and discrete-time systems, respectively. It should be noted, however, more weak Kharitonov regions can be constructed by 1) applying Theorem 1.1 on other special uncertain polynomial (e.g., low-order polynomials); 2) using the fact that the intersection of weak Kharitonov regions is a weak Kharitonov region [7]; 3) relaxing the requirement of the descreasing phase property in Theorem 1.1.

Appendix
Proof of Theorem 1.1

The following lemma is essential in the proof of Theorem 1.1.

Lemma 1: Given an open stability region $D \subset$ C and nth-order D-stable polynomials $f_0(s)$ and $f_0(s) + f_1(s)$ with positive leading coefficients. Suppose $\arg f_1(s)/f_0(s)$ is monotonously descreasing on ∂D. Then, the polynomial

$$f(s, \alpha) = f_0(s) + \alpha f_1(s)$$

is D-stable for all $0 < \alpha < 1$.

Proof: Let $\Gamma \subset C$ denote the trajectory of $f_1(s)/f_0(s)$ as s traverses on ∂D, i.e.,

$$\Gamma = \{f_1(s)/f_0(s): s \in \partial D\}.$$

Since $f_0(s)$ is D-stable and $\deg f_1(s) \leq \deg f_0(s)$, Γ is a bounded and closed curve. Therefore, $\arg f_1(s)/f_0(s)$ being monotonously descreasing implies that Γ encloses the origin. On the other hand, the point $-1 + j0$ is not encircled by Γ because $f_0(s) + f_1(s)$ is D-stable (principle of argument). Consequently, using the facts that $\arg f_1(s)/f_0(s)$ is monotonously descreasing again and that Γ encloses the origin, the interval $(-\infty, -1]$ is not enclosed by Γ. In particular, the point $-1/\alpha + j0$ is not enclosed by Γ. Therefore, $f_0(s) + \alpha f_1(s)$ is D-stable (principle of argument). □

Proof of Theorem 1.1: Suppose V_P is D-stable. Define, for $i = 1, 2, \cdots, m$

$$g_{2i-1}(s) = (\bar{t}_i - \underline{t}_i)p_i(s)$$
$$g_{2i}(s) = j(\bar{\omega}_i - \underline{\omega}_i)p_i(s)$$

TABLE I
WEAK KHARITONOV REGIONS FOR CONTINUOUS-TIME SYSTEMS

p	D	condition
$(p_0(s), p_1(s), \cdots, p_m(s))$	(10), (13)	$p_i(s)$ is anti-D-stable
	$\mathbb{C}_-$	Re $p_i(j\omega) \equiv 0$ or Im $p_i(j\omega) \equiv 0$, $1 \le i \le m$
$(p_0(s), 1, s, \cdots, s^n)$	(11), (12), (14), (16)	none
	(17)	real parameters and coefficients
$(p_0(s), 1)$	any open convex set	none

TABLE II
WEAK KHARITONOV REGIONS FOR DISCRETE-TIME SYSTEMS

p	D	condition
$(p_0(s), p_1(s), \cdots, p_m(s))$	the open unit disk or any open circular region inside of it	$p_i(s)$ is anti-D-stable
$(p_0(s), 1, s, \cdots, s^n)$	(15)	none
$(p_0(s), 1, s, \cdots, s^{\lfloor n/2 \rfloor})$	the open unit disk	none
$(p_0(s), 1 \pm s^n, \cdots, s^{\lfloor n/2 \rfloor} \pm s^{n-\lfloor n/2 \rfloor})$	the open unit disk	none

$$\alpha_{2i-1} = \begin{cases} \dfrac{t_i - \underline{t}_i}{\bar{t}_i - \underline{t}_i} & \bar{t}_i \neq \underline{t}_i \\ 0 & \bar{t}_i = \underline{t}_i \end{cases}$$

$$\alpha_{2i} = \begin{cases} \dfrac{\omega_i - \underline{\omega}_i}{\bar{\omega}_i - \underline{\omega}_i} & \bar{\omega}_i \neq \underline{\omega}_i \\ 0 & \bar{\omega}_i = \underline{\omega}_i \end{cases}$$

$$f_0(s) = p_0(s) + \sum_{i=1}^{m} (\underline{t}_i + j\underline{\omega}_i) p_i(s)$$

and, for any $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{2m})^T$ and $1 \le l \le 2m$

$$f_l(s, \alpha) = f_0(s) + \sum_{k=1}^{l} \alpha_k g_k(s).$$

Note that

$$f_l(s, \alpha) = f_{l-1}(s, \alpha) + \alpha_l g_l(s)$$

and that any polynomial in P can be expressed by $f_{2m}(s, \alpha)$ for some α with all $0 \le \alpha_k \le 1$, $k = 1, 2, \cdots, 2m$. From the decreasing phase property of $\boldsymbol{p}$, we know that, for any nth-order D-stable polynomial $f(s)$, arg $g_k(s)/f(s)$ is monotonously descreasing on ∂D, $k = 1, 2, \cdots, m$.

Given an arbitrary polynomial $f_{2m}(s, \alpha) \in P$, we need to prove that $f_{2m}(s, \alpha)$ is D-stable by *reductio ad absurdum*. That is, we assume $f_{2m}(s, \alpha)$ is not D-stable and show that there exists some vertex polynomial of P which is also not D-stable. Indeed, according to Lemma 1, $f_{2m}(s, \alpha)$ being not D-stable implies that either $f_{2m-1}(s, \alpha)$ or $f_{2m-1}(s, \alpha) + g_{2m}(s)$ is not D-stable. Without loss of generality, we may assume that $f_{2m-1}(s, \alpha)$ is not D-stable. Using Lemma 1 again, we further obtain that either $f_{2m-2}(s, \alpha)$ of $f_{2m-2}(s, \alpha) + \alpha_{2m-1} g_{2m-1}(s)$ is not D-stable. Continuing with this process repeatedly, we will eventually have either $f_0(s)$ or another vertex polynomial of P to be not D-stable. This conclusion contradicts the assumption that V_P is D-stable. Therefore, $f_{2m}(s, \alpha)$ must be D-stable. Since $f_{2m}(s, \alpha)$ is an arbitrary polynomial in P, D must be a weak Kharitonov region with respect to $\boldsymbol{p}$. □

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