Control over Signal-to-Noise Ratio Constrained Channels: Stabilization and Performance

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***Abstract*— Previous papers have considered the problem of using linear time invariant control to stabilize an unstable plant over a signal-to-noise ratio constrained communication channel, and have shown that this problem reduces to one of minimizing the H_2 norm of the complementary sensitivity function. Different techniques were used to derive the state and output feedback results, and it is not straightforward to characterize the performance that is achieved when output feedback is used with a nonminimum phase plant. In the present paper, a unified treatment of the state and output feedback cases is obtained by posing the problem as one of LQG optimal control. Doing so allows us both to analyze the achieved performance, in terms of sensitivity reduction, as well as to incorporate performance into the problem statement. When performance is considered, the results have interpretations in terms of Wiener filtering.**

I. Introduction

Recent papers [1]–[3] have studied the problem of using linear time invariant (LTI) control to stabilize an unstable plant over an infinite bandwidth additive white Gaussian noise (AWGN) communication channel whose input must satisfy a power limitation. The ratio of this power constraint to the spectral density of the channel noise is termed the signal-to-noise ratio (SNR) of the channel. The power limited stabilization problem is thus feasible if and only if the H_2 norm of the transfer function from the channel noise to the channel input is bounded above by the maximum allowable SNR. It turns out that this transfer function is equal to the complementary sensitivity function of the feedback system. With state feedback, its minimal H_2 norm is determined solely by the unstable plant poles, which thus determine the minimal SNR compatible with closed loop stability. On the other hand, with output feedback the minimal SNR will be greater than that in the state feedback case if the plant has nonminimum phase zeros or a time delay [4].

References [1]–[3], as well as much of the related literature (e.g., [5], [6]), are concerned solely with the problem of *stabilization*. Other important issues, such as those of *performance* and *robustness*, are not addressed. It is thus desirable to assess the performance and robustness of the stabilizing controllers designed in [1]–[3], as well as to design controllers to achieve a given level of performance or robustness in addition to closed loop stabilization, subject to the channel SNR limitation. To develop such results by extending the approach of [1]–[3] is problematic, in part because the state and output feedback results were derived using different techniques.

The purpose of the present paper is to propose a unified framework within which to derive the results of [1]–[3]. To do so, we shall follow the approach of Stein and Athans [7], and show that the desired H_2 minimization may be performed using an appropriately formulated linear quadratic Gaussian (LQG) optimal control problem. An important advantage of this framework is that it allows us to consider the problem of using LTI control to achieve performance as well as stabilization over an SNR limited communication channel. By "performance" we mean the shape of the sensitivity function, which embodies many key properties of a feedback system, including sensitivity to parameter variations, stability robustness, and the response to disturbances.

There is a large literature on the relation between communications and control. See for example, [8]–[10].

Terminology: Denote by $\mathbb{C}^-, \overline{\mathbb{C}}^-, \mathbb{C}^+$ and $\overline{\mathbb{C}}^+$ respectively the open-left, closed-left, open-right and closed-right halves of the complex plane $\mathbb{C}$. A square matrix $A \in \mathbb{R}^{n\times n}$ is called Hurwitz if all its eigenvalues are in $\mathbb{C}^-$. The expectation operator is denoted by $\mathscr{E}$. A rational transfer function of a continuous-time system is termed minimum phase if all its zeros lie in $\overline{\mathbb{C}}^-$, and is nonminimum phase if it has zeros in $\mathbb{C}^+$. Given $G(s)$, the transfer function of a continuous-time system, we say that $G(s) \in H_2$ if $G(s)$ is strictly proper and stable; i.e., all its poles lie in $\mathbb{C}^-$. The H_2 norm of $G(s)$, denoted by $\|G\|_{H_2}$, satisfies $\|G\|_{H_2}^2 = (1/2\pi)\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega$.

II. Previous Results

Consider the problem of stabilizing an unstable continuous-time plant by using feedback over a noisy continuous-time communication channel. Let the plant have state equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x \in \mathbb{R}^n, u \in \mathbb{R} \\ y(t) &= C_y x(t), \quad y \in \mathbb{R}, \end{aligned}$$

where (A, B, C_y) is assumed to be a minimal realization of the transfer function P_{yu}. We assume an infinite bandwidth AWGN channel with input-output relation

$$u_r(t) = u_s(t) + n(t), \tag{1}$$

where $u_s(t)$ is the channel input, $u_r(t)$ is the channel output, and $n(t)$ is zero-mean white Gaussian noise with power spectral density Φ. If the feedback system is stable then, after

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transients die away, the channel input will be stationary, and will be assumed to satisfy the power constraint $\mathscr{E}\{u_s^2(t)\} < \mathscr{P}$. By a mild abuse of terminology, we shall refer to $\mathscr{P}/\Phi$ as the signal-to-noise ratio of the channel [3].

Suppose first that the channel input is a stabilizing state feedback, $u_s(t) = -Kx(t)$, and that the control input is given by $u(t) = u_r(t)$. Define the resulting *state feedback sensitivity and complementary sensitivity functions*

$$S_{sf} = (1+K\phi B)^{-1}, \quad T_{sf} = K\phi B(1+K\phi B)^{-1}, \tag{2}$$

where $\phi(s) \triangleq (sI-A)^{-1}$. Then the power in the channel input is given by $\mathscr{E}\{u_s^2(t)\} = \|T_{sf}\|_{H_2}^2 \Phi$.

Denote the set of all stabilizing state feedback gains by $\mathscr{K}_{sf} = \{K : A-BK \text{is Hurwitz}\}$. It is shown in [1], [3] that

$$\inf_{K\in\mathscr{K}_{sf}} \|T_{sf}\|_{H_2}^2 = 2\sum_{i=1}^{N_p} \operatorname{Re}\{p_i\}, \tag{3}$$

where $\{p_i; i = 1,\ldots,N_P\}$ denote the $\mathbb{C}^+$ eigenvalues of A.

Suppose next that dynamic output feedback $U(s) = -C(s)Y(s)$ is used to stabilize the plant, and define the resulting *sensitivity and complementary sensitivity functions*

$$S = (1+CP_{yu})^{-1}, \quad T = CP_{yu}(1+CP_{yu})^{-1}. \tag{4}$$

The power in the channel input is given by $\mathscr{E}\{u_s^2(t)\} = \|T\|_{H_2}^2\Phi$, and thus the problem of computing the minimal SNR required for stabilization with output feedback reduces to that of minimizing $\|T\|_{H_2}$. If P_{yu} is minimum phase, then the minimal value of $\|T\|_{H_2}$ is also given by (3). Otherwise, it is shown in [2], [3] that the minimal value of $\|T\|_{H_2}$ is *strictly greater* than that achievable with state feedback.

A typical design specification requires that sensitivity be small at low frequencies, converge to one at high frequencies, and not be excessively large at intermediate frequencies [11]. It is thus of interest to determine the sensitivity properties of the feedback systems that achieve stabilization with a minimal SNR. Doing so is straightforward in the state feedback case.

Lemma 1 *Assume that A has no eigenvalues on the $j\omega$-axis. Then there exists a state feedback gain K^* that achieves the infimum in* (3)*, and the resulting sensitivity function S_{sf}^* defined by* (2) *must satisfy* $|S_{sf}^*(j\omega)| = 1$, $\forall\omega$.

Proof: The first claim follows from the proof of Theorem 2.1 in [3]. The rest follows by noting that (i) S_{sf}^* must have zeros at the unstable open loop poles, (ii) S_{sf}^* has poles only at the mirror images of the unstable open loop poles [12, Theorem 3.11], and (iii) $K\phi B$ is strictly proper. ■

It is hardly surprising that the optimal solution to (3) does not exhibit sensitivity reduction at any frequency, because performance was not explicitly considered in the cost function.

The solution to the state feedback problem in [1], [3] was obtained by solving a deterministic minimum energy regulation problem, and it is the well-known structure of the optimal regulator that allows us to characterize the resulting sensitivity function. The solution to the output feedback problem in [2], [3], on the other hand, was obtained by using a Youla parametrization. Except in the minimum phase case, it is not easy to characterize performance properties of the optimal solution.

In the next section, we propose a unified framework within which to derive both the state and output feedback results from [1]–[3]. As we shall see in the sequel, this approach allows us to characterize performance in the output feedback case, as well as to incorporate performance penalties into the cost function.

III. SNR Minimization via LQG Optimal Control

It was shown by Stein and Athans [7] that H_2 minimization problems involving the sensitivity and complementary sensitivity functions of a feedback system may be solved using an appropriately formulated Linear Quadratic Gaussian (LQG) optimal control problem.

A. The LQG Problem

We now state the LQG problem using notation that enables us to present the results of [7]. Consider the system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Ed(t)\\ y(t) &= C_y x(t) + \mu n(t)\\ z(t) &= Hx(t)\end{aligned} \tag{5}$$

where d and n are each unit intensity Gaussian white noise processes, u is a control signal, z a performance output, and y a measured output. We assume that u, d, n, y, and z are all scalar, and denote the open loop transfer functions from u and d to y and z by P_{yu}, P_{zu}, P_{yd}, and P_{zd}.

The LQG cost has the form

$$J_{LQG} = \mathscr{E}\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T z^2(t) + \rho^2 u^2(t) dt\right\} \tag{6}$$

We seek to minimize (6) using dynamic output feedback, $U(s) = -C(s)Y(s)$. Substituting this control law into (6) and applying Parseval's theorem reduces the LQG cost to

$$J_{LQG} = \frac{1}{\pi}\int_0^\infty \operatorname{trace}\left(M(j\omega)M^T(-j\omega)\right)d\omega, \tag{7}$$

where

$$M \triangleq \begin{bmatrix} P_{zd} - P_{zu}SCP_{yd} & -\mu P_{zu}SC \\ \rho SCP_{yd} & \mu\rho SC\end{bmatrix}, \tag{8}$$

and S is given by (4).

We saw in Section II that the problem of minimizing the power in the channel input using output feedback reduces to that of minimizing $\|T\|_{H_2}$. We now present two ways to solve the latter problem by considering special cases of the LQG cost (7).

B. Loop Transfer Recovery at the Plant Input

Consider first the problem of minimizing (6) using constant feedback of noise free state measurements, $u(t) = -Kx(t)$. In this case the resulting cost has the form

$$J_{sf} = \frac{1}{\pi}\int_0^\infty \operatorname{trace}\left(M_{sf}(j\omega)M_{sf}^T(-j\omega)\right)d\omega, \tag{9}$$

where

$$M_{sf} = \begin{bmatrix} P_{zd} - P_{zu} S_{sf} K\phi E \\ \rho S_{sf} K\phi E \end{bmatrix}, \tag{10}$$

and S_{sf} is given by (2). We can find K to minimize (9) by making the following observations. Suppose that $E = B$ in system (5), so that the disturbance enters the system at the actuator. Then it is easy to show that (9) reduces to

$$J_{sf} = \frac{1}{\pi}\int_0^\infty \left(|P_{zu}(j\omega)|^2 |S_{sf}(j\omega)|^2 + \rho^2 |T_{sf}(j\omega)|^2\right) d\omega, \tag{11}$$

and thus the minimization problem involves a tradeoff between the H_2 norms of the state feedback sensitivity and complementary sensitivity functions (2). Setting $H = 0$ and $\rho = 1$ thus reduces the problem of minimizing (11) to that of minimizing $\|T_{sf}\|_{H_2}$.

We now use the approach of [7] to pose the output feedback problem in the same framework. Again set $E = B$, and let $\mu \to 0$, so that the measurement noise vanishes. Then M defined in (8) satisfies

$$M \to \begin{bmatrix} P_{zd} - P_{zu} C(I + P_{yu} C)^{-1} P_{yd} & 0 \\ -\rho C(I + P_{yu} C)^{-1} P_{yd} & 0 \end{bmatrix},$$

and the LQG cost (6) reduces to an H_2 tradeoff between the sensitivity and complementary sensitivity functions (4),

$$J_{LQG} = \frac{1}{\pi}\int_0^\infty \left(|P_{zu}(j\omega)|^2 |S(j\omega)|^2 + \rho^2 |T(j\omega)|^2\right) d\omega. \tag{12}$$

Once again we see that, by setting $H = 0$ and $\rho = 1$, we are able to solve the problem of minimizing $\|T\|_{H_2}$ by using LQG techniques. If A has no eigenvalues on the $j\omega$-axis, then we compute the state feedback gain $K = B^T P$, where P is the stabilizing solution to the Riccati equation

$$0 = A^T P + PA - PBB^T P. \tag{13}$$

The optimal compensator has the form $C(s) = K(sI - A + BK + LC_y)^{-1} L$, where L is the solution to the optimal estimation problem with a process disturbance entering at the plant input ($E = B$) in the limit as measurement noise that becomes small, $\mu \to 0$.

The procedure just described is referred to as "loop transfer recovery (LTR) at the plant input" [7]. If $P_{yu}(s)$ is minimum phase, the input sensitivity and complementary sensitivity functions will approach those with state feedback,

$$S(s) \to S_{sf}(s), \qquad T(s) \to T_{sf}(s), \tag{14}$$

where convergence is pointwise in frequency. It follows that if the plant is minimum phase, then the minimal SNR required for stabilization with output feedback is identical to that required with state feedback.

If the plant is nonminimum phase, then it is known that the "recovery" in (14) cannot occur. Instead, S and T converge to transfer functions determined by the locations of the NMP plant zeros. It is shown in [13] that

$$S(s) \to S_{sf}(s)(1 + \Delta(s)), \tag{15}$$

where

$$\Delta(s) \triangleq K\phi(s)\left(B - B_m B_z(s)\right), \tag{16}$$

$B_z(s)$ is a Blaschke product of NMP plant zeros, and B_m is chosen so that $C_y \phi(s) B = C_y \phi(s) B_m B_z(s)$. A procedure for computing B_m may be found in [13], where it is also shown that $\Delta(s)$ has poles only at the mirror images of the NMP plant zeros. Furthermore,

$$T(s) \to S_{sf}(s) K\phi(s) B_m B_z(s).$$

In case there is only one nonminimum phase plant zero, at $s = z$, the results simplify to

$$\begin{aligned} S(s) &\to S_{sf}(s)\left(1 + \frac{2zL_{sf}(z)}{s+z}\right) \\ T(s) &\to T_{sf}(s) - S_{sf}(s)\frac{2zL_{sf}(z)}{s+z}, \end{aligned} \tag{17}$$

where $L_{sf}(s) \triangleq K\phi(s)B$.

Example 2 Consider the unstable nonminimum phase plant from [2, Example 3.1], $P(s) = (2-s)/(s^2-1)$, with state variable realization

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_y = \begin{bmatrix} -3/2 & 1/2 \end{bmatrix}.$$

Solving (13) reveals that $K = \begin{bmatrix} 0 & 2 \end{bmatrix}$, and thus that $T_{sf}(s) = 2/(s+1)$. It then follows from (17) that $T(s) \to (-6s + 12)/((s+1)(s+2))$, which agrees with [2, Example 3.1]. ■

The preceding discussion has shown that minimizing $\|T\|_{H_2}$ by using an equivalent LQG problem allows us to derive explicit expressions for the optimal sensitivity and complementary sensitivity functions, and thus we can characterize performance and robustness properties of the resulting feedback system even if the plant is nonminimum phase. It remains to derive an explicit expression for the optimal cost. With the assumptions that $\rho = 1$ and $E = B$, the optimal cost for a specific value of μ is given by [12]

$$J^*_{LQG}(\mu) = B^T PB + K\Sigma(\mu) K^T, \tag{18}$$

where P solves the minimum energy control Riccati equation (13) and $\Sigma(\mu)$ denotes the solution to the filtering Riccati equation

$$0 = A\Sigma(\mu) + \Sigma(\mu)A^T + BB^T - (1/\mu^2)\Sigma(\mu) C_y^T C_y \Sigma(\mu). \tag{19}$$

The first term on the right hand side of (18) is simply the state feedback cost given by (3). The second term may be evaluated by considering the limiting value of $\Sigma(\mu)$ as $\mu \to 0$. For minimum phase plants, this limit is equal to zero. For nonminimum phase plants, the limit is nonzero, and may be computed explicitly in terms of the NMP zero locations by applying formulas dual to those for the cheap control optimal regulator cost. An expression for the latter appears in the dissertation [14], was derived independently by Shaked [15], and reported in [16].

Example 3 Let us return to Example 2, and consider the limiting value of the optimal estimation cost $\Sigma(\mu)$ with $E = B$ and $\mu \to 0$. This cost may be obtained by solving the dual

Riccati equation (19). Dualizing the results of [14] shows that $\lim_{\mu\to 0}\Sigma(\mu)=\Sigma_0$, where

$$\Sigma_0 = \frac{1}{2z} x_m^T B B^T x_m \xi \xi^T,$$

$z=2$, $x_m = (-zI-A)^{-1}C_y^T$, and ξ is the first row of the matrix $\begin{bmatrix} x_m & C_y^T \end{bmatrix}^{-1}$. This yields $\lim_{\mu\to 0} J^*_{LQG}(\mu) = 18$, which agrees with [2, Example 3.1]. ■

C. Loop Transfer Recovery at the Plant Input

The results of [7] may be used to obtain an alternate approach to minimizing the complementary sensitivity function. Consider the problem of designing an observer,

$$\dot{\hat{x}} = (A - LC_y)\hat{x} + Ly,$$

for the state of system (5), denote the state estimate by $\hat{x}$, and the estimation errors in x and z by $\tilde{x} = x - \hat{x}$ and $\tilde{z} = H\tilde{x}$. Many properties of an observer are dual to those of state feedback, including *sensitivity and complementary sensitivity functions* dual to those in (2):

$$S_{obs} \triangleq (1 + C_y \phi L)^{-1}, \quad T_{obs} \triangleq C_y \phi L (1 + C_y \phi L)^{-1}. \tag{20}$$

The response of $\tilde{z}$ to the process disturbance and measurement noise satisfies

$$\tilde{Z} = \left(P_{zd} - H\phi L S_{obs} P_{yd}\right) D + \mu H \phi L S_{obs} N,$$

and thus the mean square estimation error is given by

$$\mathscr{E}\{\tilde{z}^2(t)\} = \frac{1}{\pi}\int_0^\infty M_{obs}(j\omega) M_{obs}^T(-j\omega)\, d\omega, \tag{21}$$

where

$$M_{obs} = \begin{bmatrix} \left(P_{zd} - H\phi L S_{obs} P_{yd}\right) & \mu H \phi L S_{obs} \end{bmatrix}.$$

Suppose that the performance output is identical to the measured output, $H = C_y$. Then (21) reduces to

$$\mathscr{E}\{\tilde{z}^2(t)\} = \frac{1}{\pi}\int_0^\infty \left(|S_{obs}(j\omega)|^2 |P_{yd}(j\omega)|^2 + \mu^2 |T_{obs}(j\omega)|^2\right) d\omega, \tag{22}$$

and thus the problem of finding an observer gain to minimize the mean square estimation error involves a tradeoff between the H_2 norms of the transfer functions (20).

We now use the results of [7] to connect the cost function (6) to the estimation error (22). Suppose again that $H = C_y$ and let the control cost satisfy $\rho \to 0$. Then the matrix (8) satisfies

$$M \to \begin{bmatrix} (I + P_{yu}C)^{-1} P_{yd} & -\mu (I + P_{yu}C)^{-1} P_{yu} C \\ 0 & 0 \end{bmatrix},$$

and the LQG cost (7) for observer based output feedback becomes a frequency weighted tradeoff between S and T:

$$J_{LQG} = \frac{1}{\pi}\int_0^\infty \left(|S(j\omega)|^2 |P_{yd}(j\omega)|^2 + \mu^2 |T(j\omega)|^2\right) d\omega. \tag{23}$$

If we suppose that $E = 0$ and $\mu = 1$, then the LQG cost (23) reduces to the H_2 norm of T, the transfer function that describes the response of the plant output to the measurement noise. It follows that we may minimize $\|T\|_{H_2}$ with output feedback by first solving the estimation problem with zero process noise, and then applying the LTR procedure by tuning the state feedback gain.

The procedure described above is referred to as "loop transfer recovery at the plant output", and is dual to the procedure for LTR at the plant input. If P_{yu} is minimum phase, then it is known that as the control cost becomes vanishingly small the output sensitivity and complementary sensitivity functions satisfy $S(s) \to S_{obs}(s)$ and $T(s) \to T_{obs}(s)$. It follows that the optimal LQG cost with output feedback is identical to the optimal state estimation cost. If P_{yu} is nonminimum phase, then results dual to those described in Section III-B show that the recovery no longer takes place, and provide a procedure for computing the limiting values of S and T in terms of the observer gain and the NMP plant zeros.

D. Comparison

We have just outlined two procedures for using LQG optimal control to minimize the channel SNR required for stabilization, corresponding to the two block diagrams in Figure 1. The procedure described in Section III-B, and depicted in Figure 1(a), uses minimal energy state feedback applied to state estimates from an observer designed by assuming a fictitious process disturbance entering at the plant input, $E = B$, and tuned by letting the measurement noise approach zero. Note that the configuration in Figure 1(a) models a communication link between the controller and the plant actuators, and that the channel noise enters the feedback system in the same way as does the fictitious process disturbance. It is this situation that is considered in [1]–[3].

The alternate procedure, described in Section III-C and depicted in Figure 1(b), assumes that the channel is used to communicate the plant output to the controller, as in [6]. The channel noise thus enters the system in the same way as does measurement noise in the LQG problem. The assumptions that no process disturbance is present ($E = 0$) and that $H = C_y$ imply that the estimator is designed solely to minimize the effect of the channel noise upon the channel input. The state feedback is then tuned by allowing the control weighting to approach zero.

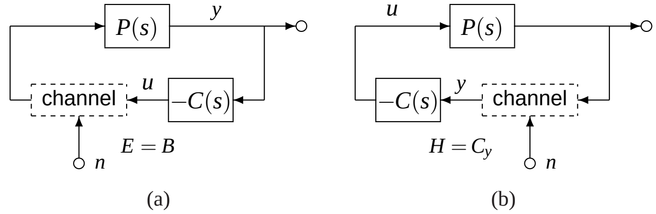

Fig. 1. Recovery at the Input, $E = B$, vs. Recovery at the Output, $H = C_y$.

IV. Performance Issues

The results described in [1]–[3] and Section III are concerned only with stabilization in the sense that the optimization problem is trivial if the plant is stable. There is considerable motivation to consider problems of performance in addition to stabilization. First, we have seen that the optimal sensitivity function obtained with state feedback, or output feedback and a minimum phase plant, is allpass and thus does not achieve sensitivity reduction or disturbance attenuation at any frequency. Second, consider the results of Nair and Evans [6], who study the problem of stabilization over a channel that is noise free but has a limited data rate. They show that the presence of a process disturbance renders attempts to communicate near the theoretical minimum rate problematic, in the sense that the state will exhibit large excursions even though it is guaranteed to remain finite [6, p. 418]. Evidently, requiring some measure of performance in addition to stabilization would necessitate communication at a higher bit rate. This argument is adapted in [17] to apply to the problem of communication over a discrete-time Gaussian noise channel. It is shown in [17, Corollary III.2] that if stabilization is achieved over a channel with capacity close to the theoretical minimum, then the response to disturbances will be very large.

The preceding considerations motivate us to consider performance by shaping the sensitivity function to influence the differential sensitivity, robustness, and disturbance response of the system. First, we will impose a performance penalty by adding a state weighting to the LQG cost criterion and extending the analysis of Section III-B. The dual version of this problem, to be discussed in Section IV-B, has an interpretation in terms of Wiener filtering and optimal estimation.

A. State Feedback with a Penalty on State Variables

Suppose that we wished to impose a performance penalty by letting H be nonzero. Then the results of [18] (see also [19]) show that the optimal cost for the *state feedback* problem with $\rho = 1$ is given by

$$J_{sf}^* = \frac{1}{\pi}\int_0^\infty \log(1+|P_{zu}(j\omega)|^2)d\omega + 2\sum_{i=1}^{N_p}\mathrm{Re}\{p_i\}. \quad (24)$$

It follows from the Kalman return difference equality [20] that the optimal sensitivity function S_{sf}^* must satisfy

$$1+P_{zu}^T(-s)P_{zu}(s) = \frac{1}{S_{sf}^*(-s)S_{sf}^*(s)}, \quad (25)$$

and thus (24) is equivalent to

$$J_{sf}^* = -\frac{2}{\pi}\int_0^\infty \log|S_{sf}^*(j\omega)|d\omega + 2\sum_{i=1}^{N_p}\mathrm{Re}\{p_i\}. \quad (26)$$

It follows from (26) that the optimal state feedback cost with a performance penalty imposed is greater than that for mere stabilization by an amount equal to the arithmetic inverse of the area under the log sensitivity integral curve. The additional cost has two components, one associated with the penalty imposed on z, and the other with that imposed on u. Since only the latter contributes to the required channel SNR, we now separate it from the total cost.

Equating (24) with (11), for $\rho = 1$, shows that

$$\frac{1}{\pi}\int_0^\infty \left(|P_{zu}(j\omega)|^2|S_{sf}^*(j\omega)|^2 + |T_{sf}^*(j\omega)|^2\right)d\omega = \frac{1}{\pi}\int_0^\infty \log(1+|P_{zu}(j\omega)|^2)d\omega + 2\sum_{i=1}^{N_p}\mathrm{Re}\{p_i\},$$

and thus

$$\|T_{sf}^*\|_{H_2}^2 = \frac{1}{\pi}\int_0^\infty \log(1+|P_{zu}(j\omega)|^2)d\omega + 2\sum_{i=1}^{N_p}\mathrm{Re}\{p_i\} - \frac{1}{\pi}\int_0^\infty |P_{zu}(j\omega)|^2|S_{sf}^*(j\omega)|^2 d\omega.$$

Applying (25) yields the main result

$$\|T_{sf}^*\|_{H_2}^2 = \frac{1}{\pi}\int_0^\infty \log(1+|P_{zu}(j\omega)|^2)d\omega - \frac{1}{\pi}\int_0^\infty \frac{|P_{zu}(j\omega)|^2}{1+|P_{zu}(j\omega)|^2}d\omega + 2\sum_{i=1}^{N_p}\mathrm{Re}\{p_i\}. \quad (27)$$

It is obvious from first principles that the right hand side of (27) must be at least as large as $2\sum_{i=1}^{N_p}\mathrm{Re}\{p_i\}$. Indeed, if the control signal used to stabilize the system and satisfy a performance objective were smaller than that required to stabilize alone, then this control signal would have been obtained as the solution to the minimum energy stabilization problem previously obtained. The dual estimation problem, to be considered in Section IV-B, yields an appealing alternate proof of this fact.

Let us now consider the output feedback problem. It follows from (12) and (18) that, for $\rho = 1$, we have

$$J_{LQG}^* = B^TPB + K\Sigma(\mu)K^T = \frac{1}{\pi}\int_0^\infty \left(|P_{zu}(j\omega)|^2|S^*(j\omega)|^2 + |T^*(j\omega)|^2\right)d\omega, \quad (28)$$

where P is the solution to the Riccati equation for the state feedback problem, and $\Sigma(\mu)$ is the solution to the dual estimation Riccati equation (19). We have already observed that, in the minimum phase case, $\lim_{\mu\to 0}\Sigma(\mu) = 0$, $S^* \to S_{sf}^*$, and $T^* \to T_{sf}^*$. Hence, for a minimum phase plant, the optimal value of $\|T\|_{H_2}$ is the same as that with state feedback in (27).

For a nonminimum phase plant, the limiting estimation error is nonzero, $\Sigma_0 \triangleq \lim_{\mu\to 0}\Sigma(\mu) \neq 0$, and the results of [14] may be adapted to compute Σ_0 in terms of the nonminimum phase plant zeros. Substituting (15) into (28) and rearranging yields

$$\|T\|_{H_2}^2 = B^TPB + K\Sigma^*K^T - \frac{1}{\pi}\int_0^\infty \frac{|P_{zu}(j\omega)|^2|1+\Delta(j\omega)|^2}{1+|P_{zu}(j\omega)|^2}d\omega, \quad (29)$$

where B^TPB, the cost with state feedback, is given by (24). The state feedback gain K satisfies no special property unless we solve a specific state feedback control problem such as the minimum energy problem considered above.

B. A Dual Problem: Estimation with a Process Disturbance

Let us now consider the problem of state estimation with a process disturbance present. The counterparts to (24) and (26) for the optimal estimation problem are given by

$$\mathscr{E}\{\tilde{z}^2(t)\} = \frac{1}{\pi}\int_0^\infty \log\left(1+|P_{yd}(j\omega)|^2\right)d\omega + 2\sum_{i=1}^{N_p}\mathrm{Re}\{p_i\} \quad (30)$$

$$= -\frac{2}{\pi}\int_0^\infty \log|S^*_{obs}(j\omega)|d\omega + 2\sum_{i=1}^{N_p}\mathrm{Re}\{p_i\}. \quad (31)$$

Expression (30) for the optimal estimation error is due, in the case of a stable plant, to Yovits and Jackson [21]. It was extended to unstable open loop plants by Anderson and Mingori [18]. See also the discussion by Braslavsky, et al [19]. Expression (31) follows from the dual version of the return difference identity (25).

With output feedback, the optimal sensitivity and complementary sensitivity functions must satisfy (23) with $\mu = 1$,

$$J^*_{LQG} = \frac{1}{\pi}\int_0^\infty \left(|S^*(j\omega)|^2|P_{yd}(j\omega)|^2 + |T^*(j\omega)|^2\right)d\omega. \quad (32)$$

If the plant is minimum phase, then applying the recovery procedure implies that $S^* \to S^*_{obs}$ and $T^* \to T^*_{obs}$, and thus (31) and (32) must be equivalent. Setting these expressions equal to one another, and rearranging reveals that

$$\|T^*_{obs}\|^2_{H_2} = \frac{1}{\pi}\int_0^\infty \log\left(1+|P_{yd}(j\omega)|^2\right)d\omega - \frac{1}{\pi}\int_0^\infty \frac{|P_{yd}(j\omega)|^2}{1+|P_{yd}(j\omega)|^2}d\omega + 2\sum_{i=1}^{N_p}\mathrm{Re}\{p_i\}. \quad (33)$$

Recall that T_{obs} is the transfer function from channel noise to the channel input in Figure 1(b). Hence, (33) represents the transmission power required due to the response of the channel input to the channel noise. To provide an interpretation of (33), suppose that the plant were stable, so that the third term on the right hand side is equal to zero. Then the first term is the optimal estimation error associated with a causal Wiener filter [21]. The second term is the optimal estimation error for the infinite delay, noncausal Wiener filter that appears in the work of Bode and Shannon [22]. It is thus clear that the sum of these two terms must be nonnegative. Furthermore, a plot of $\log(1+x)$ vs. $x/(1+x)$ reveals that the difference between the two terms will be large whenever the gain of the transfer function from the process disturbance to the output, P_{yd} is large over a significant frequency range. The fact that more transmission power is required in this case is intuitively reasonable. Since the measurement noise is assumed to have unity variance, as the size of the process disturbance increases, the optimal estimator must rely more heavily on the measured system output, that has been transmitted over the noisy channel, to obtain the optimal state estimate.

V. Conclusions

In this paper we have used LQG optimal control theory to place earlier results on stabilization over signal-to-noise ratio constrained channels into a common framework. Doing so allows us to incorporate performance directly into the problem statement, and to obtain interesting interpretations in terms of classical concepts from Wiener filtering.

References

[1] J. H. Braslavsky, R. H. Middleton, and J. Freudenberg, "Feedback stabilization over signal-to-noise ratio constrained channels," in *Proceedings of the 2004 American Control Conference*, 2004.

[2] R. H. Middleton, J. H. Braslavsky, and J. Freudenberg, "Stabilization of non-minimum phase plants over signal-to-noise ratio constrained channels," in *Proceedings of the Asian Control Conference*, 2004.

[3] J. Braslavsky, R. Middleton, and J. Freudenberg, "Feedback stabilisation over signal-to-noise ratio constrained channels," Centre for Complex Dynamic Systems and Control, The University of Newcastle, Australia, Tech. Rep. EE05003, February 2005, submitted to the IEEE Transactions on Automatic Control. [Online]. Available: \url{ftp://warhol.newcastle.edu.au/pub/Reports/EE05003.pdf.gz}

[4] ——, "Effects of time delay on feedback stabilisation over signal-to-noise ratio constrained channels," in *Proceedings of the 16th IFAC World Congress*, Prague, Czech Republic, July 2005.

[5] G. Nair and R. Evans, "Exponential stabilisability of finite-dimensional linear systems with limited data rates," *Automatica*, vol. 39, no. 4, pp. 585–593, April 2003.

[6] G. N. Nair and R. J. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM Journal of Control and Optimization*, vol. 43, no. 2, pp. 413–436, July 2004.

[7] G. Stein and M. Athans, "The LQG/LTR procedure for multivariable feedback control design," *IEEE Transactions on Automatic Control*, vol. 32, no. 2, pp. 105–114, February 1987.

[8] N. Elia, "When Bode meets Shannon: Control-oriented feedback communication schemes," vol. 49, no. 9, pp. 1477–1488, September 2004.

[9] S. Tatikonda, A. Sahai, and S. M. Mitter, "Stochastic linear control over a communication channel," vol. 49, no. 9, pp. 1549–1561, September 2004.

[10] N. C. Martins and M. A. Dahleh, "Fundamental limitations of performance in the presence of finite capacity feedback," in *Proceedings of the 2005 American Control Conference*, June.

[11] M. M. Seron, J. H. Braslavsky, and G. C. Goodwin, *Fundamental Limitations in Filtering and Control*. Springer, 1997.

[12] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. Wiley-Interscience, 1972.

[13] Z. Zhang and J. S. Freudenberg, "Loop transfer recovery for nonminimum phase plants," *IEEE Transactions on Automatic Control*, vol. 35, no. 5, pp. 547–553, May 1990.

[14] Z. Zhang, "Loop transfer recovery for nonminimum phase plants and ill-conditioned plants," Ph.D. dissertation, University of Michigan, Ann Arbor, MI, 1990.

[15] U. Shaked, "Singular and cheap optimal control: the minimum and nonminimum phase cases," National Research Institute for Mathematical Sciences, Pretoria, Republic of South Africa, Technical Report TWISK 181, 1980.

[16] M. Grimble and M. Johnson, *Optimal Control and Stochastic Estimation: Theory and Applications*. John Wiley and Sons, 1988.

[17] J. S. Freudenberg, R. H. Middleton, and V. Solo, "The minimal signal-to-noise ratio required to stabilize over a noisy channel," February 2005, submitted to the 44th IEEE Conference on Decision and Control and European Control Conference ECC 2005.

[18] B. D. O. Anderson and D. L. Mingori, "Use of frequency dependence in linear quadratic control problems to frequency-shape robustness," *Journal of Guidance and Control*, vol. 8, no. 3, pp. 397–401, May-June 1985.

[19] J. H. Braslavsky, M. M. Seron, D. Q. Mayne, and P. K. Kokotovic, "Limiting performance of optimal inear filters," *Automatica*, vol. 35, pp. 189–199, 1999.

[20] B. D. O. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*. Englewood Cliffs, NJ: Prentice Hall, 1990.

[21] H. L. Van Trees, *Detection, Estimation, and Filtering Theory, Pt. I*. New York: John Wiley and Sons, 1968.

[22] H. W. Bode and C. A. Shannon, "A simplified derivation of linear least square smoothing and prediction theory," *Proceedings of the IRE*, vol. 38, pp. 417–426, 1950.