Stabilization of Non-Minimum Phase Plants over Signal-to-Noise Ratio Constrained Channels*

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Abstract

We recently considered feedback stabilization over a Signal to Noise Ratio (SNR) constrained channel. The results were examined for the state feedback and minimum phase cases, with links to bit-rate limited control. In this paper, we extend this analysis to Non-Minimum Phase (NMP) plants, and show that for Linear Time Invariant (LTI) control, NMP zeros further constrain the ability to stabilize over an SNR limited channel. This differs from the situation of bit-rate limited stabilization where NMP zeros do not play a role. We show that by considering linear time varying feedback the effect of NMP zeros in SNR limited stabilization may be eliminated.

1 Introduction

A number of recent references have discussed issues related to feedback control over communication links (e.g.[11, 3, 6, 8, 9, 10]). These and other references discuss a number of problems related to control over communications channels including quantization effects, bit-rate limitations, bandwidth constraints, variable time delays and missing data.

In our recent paper [2], we considered an alternative view point based on a signal to noise ratio limitation in the feedback channel. We were able to demonstrate for the case of an additive white Gaussian noise channel that there are demands on the channel SNR required for stabilization of an unstable plant. For the case of either state feedback, or output feedback from a minimum phase plant, the bound on the required SNR depends only on the plant poles. Furthermore, we noted an interesting link between our SNR based result and a related bit-rate limited result along the lines of Nair and Evans in [9].

These previous results on bit-rate limited stabilization apply in the output feedback case, without any restriction on the plant being minimum phase. This motivates us to consider extensions of the SNR limitations to the nonminimum phase case.

2 Problem Formulation

Following [2], we consider control over a communication link as illustrated in Figure 1.

We consider the plant to be Linear Time Invariant (LTI) with minimal state space model:[1]

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

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[1]Note that a mathematically precise treatment of the continuous-time stochastic system would require use of *Ito calculus*, etc. on the stochastic differential equation $dx = Ax\,dt + B\,du$. Under appropriate stationarity assumptions, this formulation reduces to the analysis here [1, §4].

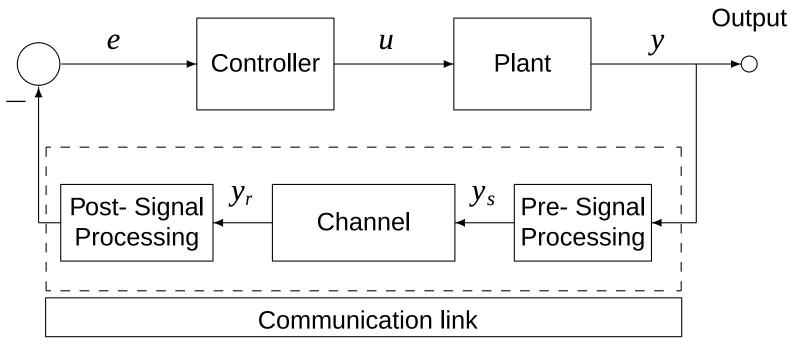

Figure 1: Control system with feedback over a communication link

and transfer function

$$P(s) = C\,(sI - A)^{-1}\,B \tag{2}$$

Our channel model is a simplified AWGN model with power constraint as shown in Figure 2.

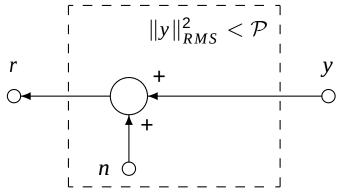

Figure 2: Continuous-time AWGN channel with an input power constraint

Note that the noise, $n(t)$, is considered to be white, with (constant) spectral density, Φ; and we define

$$\|y_s(t)\|^2_{RMS} \triangleq E\left[y_s'(t)y_s(t)\right] \tag{3}$$

We assume a given 'power constraint', $\mathcal{P}$, such that we require

$$\|y_s(t)\|^2_{RMS} < \mathcal{P} \tag{4}$$

Let $y_s(t)$ have spectral density, $\Phi_{y_s}(\omega)$, then the power constraint (4) can be expressed as:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left(\Phi_{y_s}(\omega)\right) d\omega < \mathcal{P} \tag{5}$$

For simplicity of exposition, we restrict attention to the case of a SISO plant.

3 Continuous-Time LTI Output Feedback

Suppose that we restrict attention to the case where the controller, pre and post compensators are continuous LTI systems with transfer functions $K_C(s)$, $K_{pre}(s)$ and $K_{post}(s)$ (respectively), all of which are free to the designer. Because of the SISO LTI nature of these compensators, we can reduce Figures 1 and 2 in this case to Figure 3, where $K(s) \triangleq K_C(s)K_{pre}(s)K_{post}(s)$.

We assume that $K(s)$ is a proper rational transfer function and that the feedback loop of Figure 3 is internally stable.

Because the closed loop system is asymptotically stable, the output sent on the communication channel $y_s(t)$ resulting from the input noise $n(t)$ is a stationary stochastic process with Gaussian distribution. By using the power spectral density of $y_s(t)$ as in the previous section,

$$\begin{aligned}\|y_s\|^2_{RMS} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left[T(j\omega)T'(-j\omega)\right]\Phi\, d\omega \\ &= \|T\|^2_{H_2}\Phi,\end{aligned} \tag{6}$$

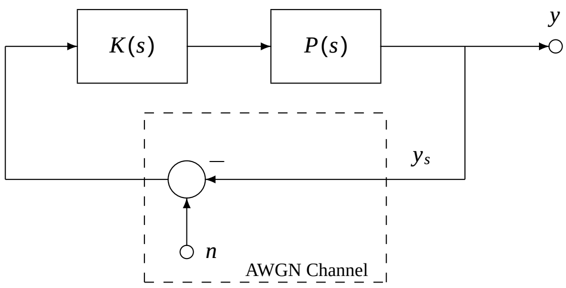

Figure 3: Simplified continuous-time feedback loop over an AWGN channel

where $T(s)$ is the closed loop transfer function between $n(t)$ and $y_s(t)$ in Figure 3

$$T(s) = \frac{P(s)K(s)}{1 + P(s)K(s)}, \tag{7}$$

and $\|T\|_{H_2}$ is the H_2 norm of $T(s)$. Note that $\|T\|_{H_2}$ is finite because $T(s)$ is stable and strictly proper. Thus to find the lowest achievable value of $\|y\|_{RMS}$ we have to find the lowest achievable value of $\|T\|_{H_2}$ over the class of all stabilizing controllers.

If the plant $P(s)$ is unstable, $\|T\|_{H_2}$ has a positive lower bound that cannot be further reduced by any choice of the controller, as we show in the following proposition, which generalizes a similar result in [2].

Proposition 3.1. *Consider the feedback loop of Figure 3. Assume that the plant $P(s)$ is proper; has n_p poles $p_k, k = 1, 2, \ldots, n_p$ in $\mathbb{C}^+$; has n_z zeros $z_\ell, \ell = 1, 2, \ldots, n_z$ in $\mathbb{C}^+$; and that $K(s)$ is such that the closed-loop is internally asymptotically stable. Then,*

$$\|T\|_{H_2}^2 \geq \left(\sum_{k=1}^{n_p} 2\,\mathrm{Re}\{p_k\} \right) + \eta. \tag{8}$$

where η is a function of the zeros and poles in $\mathbb{C}^+$ only, given by

$$\eta = \sum_{\ell=1}^{n_z} \sum_{k=1}^{n_z} \left(\frac{\gamma_\ell \bar{\gamma}_k}{(z_\ell + \bar{z}_k)} \right) \tag{9}$$

where

$$\gamma_\ell \triangleq \left(1 - B_p^{-1}(z_\ell)\right)(z_\ell + \bar{z}_\ell) \prod_{k \neq \ell} \frac{(z_\ell + \bar{z}_k)}{(z_\ell - z_k)} \tag{10}$$

and $B_p(s)$ is the Blaschke product of the ORHP plant poles:

$$B_p(s) = \prod_{k=1}^{n_p} \frac{s - p_k}{s + \bar{p}_k}. \tag{11}$$

Proof. See Appendix A. □

Note that this result shows the additional cost term η in the achievable H_2 norm of T when the plant is both unstable and NMP. Inequality (8), together with (6), in turn gives the signal to noise constraint for stabilization

$$\frac{\mathcal{P}}{\Phi} > \left(\sum_{k=1}^{n_p} 2\,\mathrm{Re}\{p_k\} \right) + \eta \tag{12}$$

For plants that are both unstable and NMP, the term η in (12) is always nonnegative and delineates additional restrictions compared to the minimum phase case, where $\eta = 0$ (see for example [2], (14)). Note that in the case of a single (real) NMP zero, i.e., when $n_z = 1$, we have:

$$\eta = 2z\left(1 - B_p^{-1}(z)\right)^2 \tag{13}$$

The following example illustrates the potentially very significant effects of NMP zeros.

Example 3.1. *Consider the case where the plant transfer function is*

$$P(s) = \frac{-(s-2)}{(s+1)(s-1)} \tag{14}$$

Clearly, we have one CRHP zero, $z = 2$ *and one CRHP pole,* $p = 1$*. Proposition 3.1 gives that*

$$\begin{aligned} \inf_{K(s)\ Stabilising} \|T\|_{H_2}^2 &= 2 + \frac{\gamma^2}{2z} \\ &= 18 \end{aligned} \tag{15}$$

since

$$\gamma = (1 - B_p^{-1}(z))2z = -8 \tag{16}$$

Note that a more explicit derivation of various terms in the Youla parameterization defined later in (38),(39) yields:

$$\begin{aligned} B_p &= \left(\frac{s-1}{s+1}\right) \\ N &= -\frac{(s-2)}{(s+1)^2} \\ X &= 4 \\ Y &= \left(\frac{s+7}{s+1}\right) \end{aligned}$$

and optimal H_2 *controller variables*

$$\begin{aligned} Q^* &= \frac{2(s+1)}{s+2} \\ K^* &= \frac{6(s+1)}{s+10} \end{aligned}$$

which gives rise to an optimal closed loop with

$$T^* = \frac{-6(s-2)}{(s+1)(s+2)} = \frac{18}{(s+1)} - \frac{24}{(s+2)}$$

and it can indeed be verified that $\|T^*\|_{H_2} = \sqrt{18}$.

Note, however, that this value is substantially larger than that obtained with state feedback (i.e., when NMP zeros have no effect), in which case $\|T^*\|_{H_2} = \sqrt{2}$. □

We therefore see, both by Proposition 3.1, and by Example 3.1, that there is an extra term, η, present in the NMP case. Therefore, the restriction to using LTI feedback control in the NMP case appears to give additional constraints on the required channel SNR for stabilization, not present in the minimum phase case. Note also that this is not the case in the results of [8], where only the plant poles influence the bit rate required for stabilization. This difference between these two cases may be understood as a consequence of restricting attention to LTI control and compensation. It may also be seen as an indirect effect of H_2 optimization, which yields not only stability, but some measure of performance as well. In the next section, we examine the use of time varying linear control for stabilization.

4 Linear Time Varying Feedback Stabilization

We now turn to consider a specific type of Linear Time Varying (LTV) feedback to try to avoid the additional restrictions imposed by plant NMP zeros. In the subsequent development for simplicity, we make the following assumption[2].

[2] Note that this assumption is required in the following development since for simplicity we restrict attention to the case of a zero order hold input

Assumption 1 (Nonzero steady state plant gain). *We assume that* $P(0) \neq 0$, *that is* $\begin{bmatrix} A & B \\ C & O \end{bmatrix}$ *is full rank.* □

We now propose a form of feedback based on sampling, filtering, resetting and other combinations of LTI and LTV operations. Suppose that the controller is an identity operator, and that the post processing in Figure 1 is as illustrated in Figure 4.

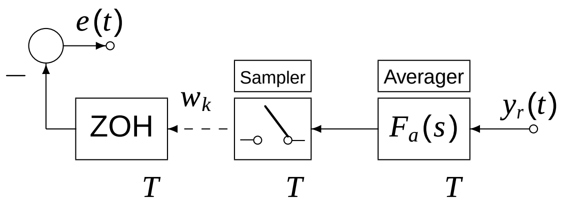

Figure 4: Proposed post signal processing for LTV feedback stabilization

In Figure 4 the various blocks are defined as follows:

Averager: the $Averager$ is an LTI filter with transfer function, $\frac{1-e^{-sT}}{sT}$, which can be described in the time domain as:

$$y_{rf}(t) \triangleq \frac{1}{T} \int_{t-T}^{t} y_r(\tau) d\tau \tag{17}$$

Sampler: the $Sampler$ follows standard definitions wherein:

$$w_k \triangleq y_{rf}(kT) \ \ ; \ \ k = 0, 1, 2, \ldots \tag{18}$$

Zero Order Hold: the Zero Order Hold (ZOH) follows the standard definition:

$$e(t) \triangleq -w_k \ \ ; \ \ t \in [kT, (k+1)T) \tag{19}$$

The pre signal processing process is more complicated and is depicted in Figure 5.

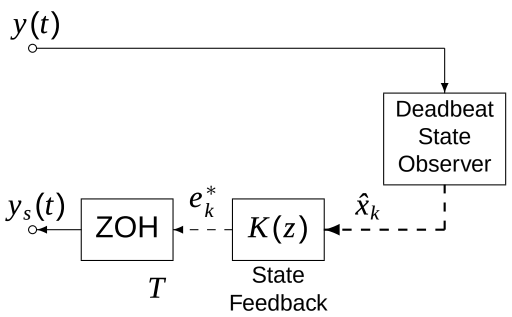

Figure 5: Proposed pre signal processing for LTV stabilization

In Figure 5, the ZOH block is equivalent, with the appropriate variable changes, to that defined before in (19). The state feedback block is a possibly dynamic linear operation:

$$w_k^* = K(z)\hat{x}_k, \tag{20}$$

where, with some abuse in notation, $K(z)$ represents the LTI discrete-time operator with discrete transfer function $K(z)$.

Deadbeat State Observer: the deadbeat state observer takes continuous output measurements, $y(t)$, and produces sampled data estimates of the extended state, $X = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$ according to:

$$
\begin{aligned}
&(a) \quad z(kT^+) = 0 \quad ; \quad k = 0, 1, \ldots && (21)\\
&(b) \quad \dot{z}(t) = -\mathcal{A}'z(t) + \mathcal{C}'y(t) \ \text{ for } \ t \in (kT, (k+1)T) && (22)\\
&(c) \quad \hat{X}_k = \begin{pmatrix} \hat{x}_k \\ \hat{u}_{k-1} \end{pmatrix} = W_T^{-1} z(kT^-) && (23)
\end{aligned}
$$

where W_T is the finite time Grammian

$$
W_T = \int_0^T e^{-\mathcal{A}'t}\mathcal{C}'\mathcal{C}e^{-\mathcal{A}t}dt \tag{24}
$$

and

$$
\mathcal{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \ ; \ \mathcal{C} = \begin{bmatrix} C & 0 \end{bmatrix}. \tag{25}
$$

The overall block diagram for this stabilization problem is therefore as shown in Figure 6.

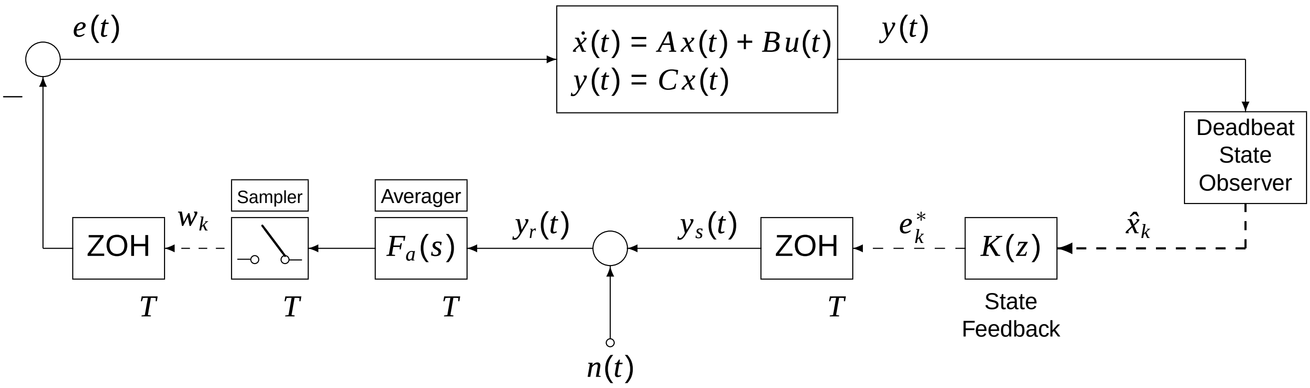

Figure 6: LTV Output feedback equivalent loop

We now demonstrate that under suitable assumptions, Figure 6 simplifies to a discrete time, LTI, state feedback problem. We start by examining the properties of the channel and adjacent ZOHs, filter and sampler.

Proposition 4.1. *Consider the system shown in Figure 6, with averager, sampler and ZOH as defined in* (17)–(19). *Then*

$$
w_k = w_{k-1}^* + n_k \tag{26}
$$

where w_{k-1}^* is obtained from (20) *and n_k is a white noise process with zero mean and variance $E(n_k^2) = \Phi/T$.*

Proof. Straightforward from noting that

$$
\begin{aligned}
w_k &= y_{rf}(kT)\\
&= \frac{1}{T}\int_{(k-1)T}^{kT} y_r(t)dt\\
&= \frac{1}{T}\int_{(k-1)T}^{kT} y_s(t)dt + \frac{1}{T}\int_{(k-1)T}^{kT} n(t)dt\\
&= w_{k-1}^* + n_k.
\end{aligned}
$$

The statistical properties of n_k follow from those of $n(t)$ and the averaging filter. □

In addition, for the ideal stabilization[3] question posed, we can also establish some properties of the deadbeat observer as follows:

[3]This ideal stabilization question implies that we have a perfect, noise and error free LTI model relation $u(t)$ to $y(t)$. This is of course impractical and serves only as a starting point for the proposed analysis.

Proposition 4.2. *Consider the system of Figure 6 with deadbeat observer designed as in (21)-(23). Define* $u_k = -e_k$*, then under Assumption 1, assuming* (A, C) *is observable then* W_T^{-1} *is well defined and*

$$\begin{aligned} \hat{x}_k &= x_k = x(kT) \\ \hat{u}_{k-1} &= u_{k-1} \end{aligned}$$

Proof. See Appendix B. □

Note therefore that under the conditions of Propositions 3.1 and 4.2, the system can be expressed as:

$$x_{k+1} = A_T x_k + B_T u_k \tag{27}$$

$$u_k = -K(z) x_{k-1} - n_k \tag{28}$$

where A_T, B_T are the appropriate ZOH discretization of A, B ; $A_T = e^{AT}$; $B_T = \int_0^T e^{A\tau} B d\tau$.

Therefore, in the case of ideal stabilization, we are able to reduce the problem to a discrete time delayed state feedback problem. Note that $E[y_s^2(t)] = E[(Kx_k)^2]$ and so we have reduced the continuous output feedback stabilization problem to that of solving:

$$\inf_{K\text{ : Stabilizing}} E\left[(Kx_k)^2\right] = \inf_{K\text{ : Stabilizing}} \|T_k(z)\|_{H_2}^2 \,\Phi/T \tag{29}$$

where $T_k(z) = -K(z)(zI - A_T + B_T K(z)) z^{-1})^{-1} B_T$ is the transfer function from n_k to $K(z)x_k$.

Proposition 4.3. *Consider the delayed state feedback system (27), (28). Then*

$$\inf_{K(z)\ \textit{Stabilizing}} \|T_k(z)\|_{H_2}^2 = \left(\prod_{i=1}^{n_p} |\phi_i|^2\right) - 1 + \Omega \tag{30}$$

where

$$\Omega = \left(\prod_{i=1}^{n_p} |\phi_i|^2\right) \left|\sum_{i=1}^{n_p} \frac{|\phi_i|^2 - 1}{\phi_i}\right|^2 \tag{31}$$

and ϕ_i *are unstable eigenvalues of* A_T.

Proof. See Appendix C. □

We can interpret Proposition 4.3 in terms of SNR constraints directly as follows.

Corollary 4.4. *The system of Figure 1, with time-varying feedback as in figure 4, can be stabilized without exceeding the power constraint (4), if and only if:*

$$\frac{\mathcal{P}}{\Phi} > \frac{\left(\prod_{i=1}^{n_p} |\phi_i|^2\right) - 1 + \Omega}{T} \tag{32}$$

□

Note that Ω represents the additional cost, compared to [2] (Section 3), due to time delay in the feedback. We also note that delay (e.g., due to encoding and decoding) is not explicitly considered in [8] or [9] and therefore there is no term equivalent to Ω.

Also, due to the sampled nature of the discrete time plant, $\phi_i = e^{p_i T}$, and we can now evaluate the behaviour of (32), with fast sampling.

Proposition 4.5.

$$\lim_{T\to 0} \left\{ \frac{\left(\prod_{i=1}^{n_p} |\phi_i|^2\right) - 1 + \Omega}{T} \right\} = 2 \sum_{i=1}^{n_p} \operatorname{Re}\{p_i\} \tag{33}$$

Proof. First note that

$$\begin{aligned}\lim_{T\to 0}\left\{\frac{\prod_{i=1}^{n_p}|\phi_i|^2-1}{T}\right\} &= \lim_{T\to 0}\left\{\frac{\exp\left(2\sum_{i=1}^{n_p}\text{Re}\{p_i\}\,T\right)-1}{T}\right\}\\ &= 2\sum_{i=1}^{n_p}\text{Re}\{p_i\}\end{aligned} \tag{34}$$

Also, the remaining term has a limit which can be expressed as:

$$\begin{aligned}\lim_{T\to 0}\left\{\frac{\Omega}{T}\right\} &= \lim_{T\to 0}\left\{\left(\prod_{i=1}^{n}|\phi_i|^2\right)\frac{\left|\sum_{i=1}^{n_p}\frac{|\phi_i|^2-1}{\phi_i}\right|^2}{T}\right\}\\ &= \lim_{T\to 0}\left(\prod_{i=1}^{n}|\phi_i|^2\right)\lim_{T\to 0}\left\{\frac{\left|\sum_{i=1}^{n_p}\frac{|\phi_i|^2-1}{\phi_i}\right|^2}{T}\right\}\\ &= 1\times\left|\lim_{T\to 0}\sum_{i=1}^{n_p}\left(\frac{|\phi_i|^2-1}{\phi_i\sqrt{T}}\right)\right|^2\\ &= \left|\sum_{i=1}^{n_p}\lim_{T\to 0}\left(\frac{e^{2\,\text{Re}\{p_i\}T}-1}{e^{p_iT}\sqrt{T}}\right)\right|^2\\ &= 0\end{aligned} \tag{35}$$

and the desired result follows. □

We then have the main result of this section.

Theorem 4.6. *Suppose that the SNR constraint satisfies*

$$\frac{\mathcal{P}}{\Phi} > \sum_{k=1}^{n_p} 2\,\text{Re}\{p_k\}. \tag{36}$$

Then there exists a sufficiently small T such that the system is stabilizable, under the SNR constraint, by the time varying feedback scheme of figure 4. □

Proof. Follows immediately from Corollary 4.4 and Proposition 4.5. □

We therefore see that by using the appropriate time varying operators, the effect of NMP zeros on ideal SNR limited feedback stabilization may be removed.

Example 4.1. *We return to the plant* (14) *introduced in Example 3.1. Suppose that now we use the the LTV output feedback loop of Figure 6 for stabilisation.*

For this plant we compute the SNR constraint bound

$$\delta(T) \triangleq \frac{\prod_{i=1}^{n_p}|\phi_i|^2-1+\Omega}{T} \tag{37}$$

on the RHS of (32). *Because the plant discretisation has a single unstable pole at $z = e^T$, the bound* (37) *reduces to*

$$\begin{aligned}\delta(T) &= \frac{e^{2T}-1+\Omega}{T}, \quad \textit{where } \Omega = (e^{2T}-1)^2, \textit{ from } (31)\\ &= \frac{e^{2T}-1+(e^{2T}-1)^2}{T}\\ &= \frac{e^{2T}(e^{2T}-1)}{T}.\end{aligned}$$

Thus, we see that

$$\lim_{T \to 0} \delta(T) = 2,$$

i.e., for faster sampling rates, the SNR constraint for the LTV output feedback equivalent loop of Figure 6 asymptotically relaxes to that for the LTI state feedback *loop, in which the lowest SNR constraint bound is* 2 *(Equation* (12) *with* $\eta = 0$*).*

Figure 7 plots the bound $\delta(T)$ *for sampling rates* T *ranging in* $(0, 0.5s]$ *(in solid line). For reference, we also plot (in dashed line) the SNR constraint bound obtained in the LTI discrete-time state feedback case [2].*

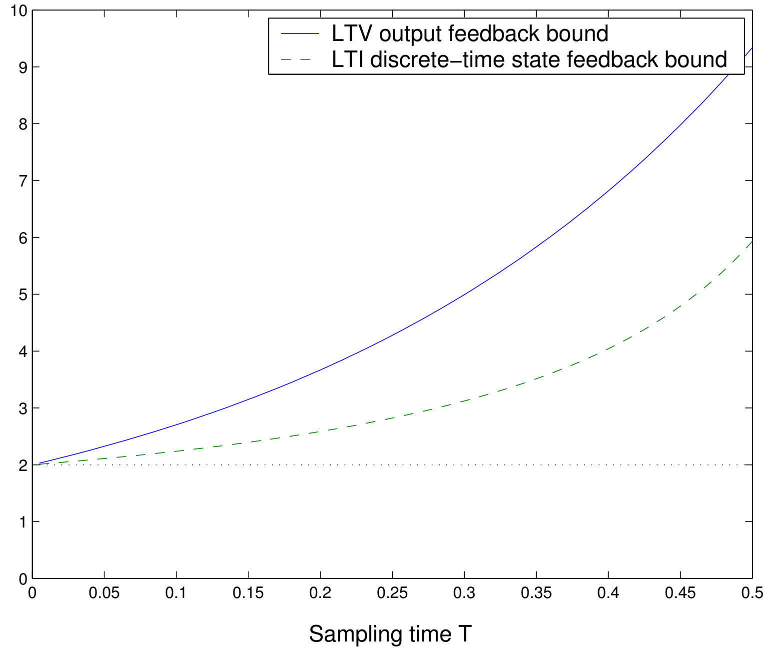

Figure 7: SNR constraint bounds

5 Conclusions

In this paper we have extended the results of [2] on SNR limited stabilization to the case of NMP plants. We have shown that for LTI control, NMP zeros may impose a significantly more difficult SNR requirement for stabilization. In the ideal case of no modelling errors, noise, or disturbances, this additional restriction may be essentially removed by the use of Linear Time Varying feedback strategies.

A Appendix - Proof of Proposition 3.1

We follow a similar style of proof to that in [2]. To compute the lowest value of $\|T\|_{H_2}$ over the class of all stabilizing controllers, we apply a technique used in Chen et al. [4]. This is based on considering subspaces of the set of rational transfer functions, $G(s)$:

$$L_2 = \left\{ T(s) : \int_{-\infty}^{\infty} |T(j\omega)|^2 d\omega < \infty \right\},$$

$$H_2 = L_2 \cap \left\{ T(s) : \text{analytic in } \bar{\mathbb{C}}^+ \right\}$$

and the orthogonal complement of H_2

$$H_2^\perp = L_2 \cap \left\{ T(s) : \text{analytic in } \bar{\mathbb{C}}^- \right\}$$

We start by deriving an expression for $T(s)$ based on a parameterisation of all stabilizing controllers. Represent $P(s)$ by a coprime factorization

$$P(s) = \frac{N(s)}{B_p(s)},$$

where $N(s) \in RH_\infty$ (the space of proper, stable, rational functions), and B_p is the Blaschke product defined in (11). By using the well-known Youla controller parameterisation [e.g. 5, §5.4], we can represent any stabilizing controller for the feedback loop in Figure 3 by[4]

$$K = \frac{X + B_p Q}{Y - NQ}, \tag{38}$$

where Q, X and Y are in RH_∞, with X and Y satisfying the Bezout identity

$$NX + B_p Y = 1. \tag{39}$$

By replacing (38) in (39), we find that T can be expressed as

$$T = \left(1 - B_p(Y - NQ)\right) = N(X + B_p Q). \tag{40}$$

Thus, from (40), the problem of finding the lowest value of $\|T\|_{H_2}$ over the class of stabilizing controllers reduces to that of finding

$$\inf_{Q \in RH_\infty} \|1 - B_p Y + B_p NQ\|_{H_2}. \tag{41}$$

Now, as in [2] and since since B_p is all pass:

$$\begin{aligned} \inf_{Q \in RH_\infty} \|1 - B_p Y + B_p NQ\|_{L_2} &= \inf_{Q \in RH_\infty} \left\|B_p^{-1} - Y + NQ\right\|_{L_2}^2, \\ &= \inf_{Q \in RH_\infty} \left\|\left(B_p^{-1} - 1\right) + (1 - Y + NQ)\right\|_{L_2}^2, \\ &= \left\|1 - B_p^{-1}\right\|_{L_2}^2 + \inf_{Q \in RH_\infty} \|1 - Y + NQ\|_{L_2}^2 \end{aligned} \tag{42}$$

where the last line follows since $(1 - B_p^{-1})$ is both strictly proper and anti-stable, and therefore is in $H_2^\perp$, and conversely, $(1 - Y + NQ)$ is strictly proper and stable and therefore in H_2.

It follows by the same arguments as in [2, Appendix A.2], that:

$$\left\|1 - B_p^{-1}\right\|_{L_2}^2 = 2 \sum_{k=1}^{n_p} \mathrm{Re}\{p_k\}. \tag{43}$$

It therefore remains to evaluate the last term on the RHS of (42), which we denote by η:

$$\eta \triangleq \inf_{Q \in RH_\infty} \|1 - Y + NQ\|_{L_2}^2. \tag{44}$$

Let $N(s) = B_z(s) N_m(s)$ where N_m has no CRHP poles or zeros, and B_z is the Blaschke product of the CRHP zeros of N:

$$B_z(s) = \prod_{\ell=1}^{n_z} \frac{(s - z_\ell)}{(s + \bar{z}_\ell)} \tag{45}$$

Because B_z is all-pass we can rewrite (44) as

$$\begin{aligned} \eta := \inf_{Q \in RH_\infty} \left\|B_z^{-1}(1 - Y) + N_m Q\right\|_{L_2}^2 \\ = \inf_{Q \in RH_\infty} \left\|\Gamma^\perp + \Gamma + N_m Q\right\|_{L_2}^2 \end{aligned} \tag{46}$$

[4] Dependency on s is suppressed to simplify notation when convenient.

where $B_z^{-1}(s)(1 - Y(s)) = \Gamma^{\perp}(s) + \Gamma(s)$ where Γ (and $\Gamma^{\perp}$) are in H_2 (and $H_2^{\perp}$) respectively. Since N_m is minimum phase, it follows that

$$\inf_{Q \in RH_\infty} \|\Gamma + N_m Q\|_{L_2}^2 = 0 \tag{47}$$

and therefore from (46):

$$\eta = \left\|\Gamma^{\perp}(s)\right\|_{H_2^{\perp}}^2 \tag{48}$$

The CRHP poles of $\Gamma^{\perp}(s)$ are precisely the CRHP plant zeros, z_ℓ, and therefore we can use residue calculus to show that

$$\Gamma^{\perp}(s) = \sum_{\ell=1}^{n_z} \left(\frac{\rho_\ell}{(s - z_\ell)} (1 - Y(z_\ell)) \right) \tag{49}$$

where ρ_ℓ is the residue of B_z^{-1} evaluated at $s = z_\ell$:

$$\rho_\ell = (z_\ell + \bar{z}_\ell) \prod_{k \neq \ell} \frac{(z_\ell + \bar{z}_k)}{(z_\ell - z_k)} \tag{50}$$

Therefore, noting from (39) and $N(z_\ell) = 0$ that $Y(z_\ell) = B_p^{-1}(z_\ell)$, and defining $\gamma_\ell = \rho_\ell \left(1 - B_p^{-1}(z_\ell)\right)$ it follows that:

$$\eta = \sum_{\ell=1}^{n_z} \sum_{k=1}^{n_z} \left(\frac{\gamma_\ell \bar{\gamma}_k}{(z_\ell + \bar{z}_k)} \right) \tag{51}$$

which completes the proof.

□

B Appendix - Proof of Proposition 4.2.

Firstly, we define an augmented continuous time state,

$$X(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{52}$$

Then note that for $t \in (kT, (k+1)T)$, since $u(t) = -e(t)$ is generated by a ZOH

$$\begin{aligned} \dot{X}(t) &= \begin{bmatrix} \dot{x}(t) \\ \dot{u}(t) \end{bmatrix} = \begin{bmatrix} Ax(t) + Bu(t) \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} X(t) \\ &= \mathcal{A} X(t) \\ y(t) &= Cx(t) = \mathcal{C} X(t) \end{aligned} \tag{53}$$

We now show that $(\mathcal{A}, \mathcal{C})$ is observable by contradiction and the PBH test (e.g., see [7, § 2.4.3]). Suppose that $(\mathcal{A}, \mathcal{C})$ is not an observable pair, then there exists nontrival v_1, v_2, λ such that

$$\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{54}$$

$$\text{and } \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \tag{55}$$

Equations (54) and (55) can be rewritten as

$$Av_1 + Bv_2 = \lambda v_1 \tag{56}$$

$$0 = \lambda v_2 \tag{57}$$

$$Cv_1 = 0. \tag{58}$$

Now (57) implies either $\lambda = 0$ or $v_2 = 0$. Suppose $v_2 = 0$. Then (56) and (58) reduce to: $Av_1 = \lambda v_1 \;\; ; \;\; Cv_1 = 0$ which implies that (A, C) is not observable. Therefore, suppose instead $v_2 \neq 0$ and $\lambda = 0$. Then (56) reduces to:

$$Av_1 + Bv_2 = 0 \tag{59}$$

which together with (58) violates Assumption 1. Thus by contradiction, $(\mathcal{A}, \mathcal{C})$, is observable.

Since $\mathcal{A}, \mathcal{C}$, is observable, the finite time observability Grammian, W_T, is full rank for any $T > 0$.

With z as defined in (21), (22), for $t \in ((k-1)T, kT)$:

$$z(t) = \int_{(k-1)T}^{t} e^{-\mathcal{A}'(t-\tau)}\mathcal{C}'y(\tau)d\tau \tag{60}$$

Also, over the same time interval,

$$\begin{aligned} y(\tau) &= \mathcal{C}e^{\mathcal{A}(\tau-(k-1)T)}X((k-1)T^+) \\ &= \mathcal{C}e^{\mathcal{A}(\tau-kT)}X(kT^-) \end{aligned} \tag{61}$$

and so combining (61) and (60)

$$\begin{aligned} z(kT^-) &= \int_{(k-1)T}^{kT} e^{-\mathcal{A}'(kT-\tau)}\mathcal{C}'\mathcal{C}e^{\mathcal{A}(\tau-kT)}X(kT^-)d\tau \\ &= W_T X(kT^-) \end{aligned} \tag{62}$$

□

C Appendix - Proof of Proposition 4.3

We follow similar lines to those in Appendix A.4 of [2]. Introduce the spaces rational of transfer functions

$$\begin{aligned} L_2(\mathbb{D}) &= \left\{ f(z) : \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{j\theta})|^2 d\theta < \infty \right\} \\ H_2(\mathbb{D}) &= L_2(\mathbb{D}) \cap \left\{ f(z) : \text{analytic in } \mathbb{D}^{\complement} \right\} \end{aligned}$$

and the orthogonal complement of $H_2(\mathbb{D})$

$$H_2^{\perp}(\mathbb{D}) = L_2(\mathbb{D}) \cap \{ f(z) : \text{strictly proper and analytic in } \mathbb{D} \},$$

where $\mathbb{D}$ is the unit disk in the complex plane $\mathbb{C}$.

Consider now the single input multiple output transfer function

$$P(z) = \frac{N(z)}{B_{d,p}(z)} = \frac{1}{z}(zI - A_T)^{-1}B_T$$

where N is a stable and proper rational transfer function matrix, and

$$B_{d,p}(z) = \prod_{i=1}^{m} \frac{z - \phi_i}{1 - z\bar{\phi}_i}$$

is here the discrete-time Blaschke product of all poles ϕ of P outside the unit disk $\mathbb{D}$. The class of all discrete-time stabilizing controllers for P in the feedback control of (27),(28) is parameterized by

$$K(z) = \frac{X(z) + B_{d,p}(z)Q(z)}{Y(z) - Q(z)N(z)},$$

where Q, X and Y are stable and proper rational functions which satisfy the Bezout identity

$$X(z)N(z) + B_{p,d}(z)Y(z) = 1 \tag{63}$$

Then we have that

$$\begin{aligned} T_k(z) &= X(z)N(z) + B_{d,p}(z)Q(z)N(z) \\ &= 1 - B_{d,p}(z)Y(z) + B_{d,p}(z)Q(z)N(z). \end{aligned} \tag{64}$$

Thus,

$$\begin{aligned}\|T_k\|^2_{H_2(\mathbb{D})} &= \inf_{Q(z)} \|1 - B_{d,p}(z)\,(Y(s) - Q(z)N(z))\|^2_{H_2(\mathbb{D})} \\ &= \inf_{Q(z)} \left\| \left(B_{d,p}(z)\right)^{-1} - \left(B_{d,p}(\infty)\right)^{-1} + \left(B_{d,p}(\infty)\right)^{-1} - Y(z) + Q(z)N(z) \right\|^2_{L_2(\mathbb{D})} \\ &= \left\| \left(B_{d,p}(z)\right)^{-1} - \left(B_{d,p}(\infty)\right)^{-1} \right\|^2_{L_2(\mathbb{D})} + \inf_{Q(z)} \left\| \left(B_{d,p}(\infty)\right)^{-1} - Y(z) + Q(z)N(z) \right\|^2_{L_2(\mathbb{D})}.\end{aligned} \tag{65}$$

As in [2] we can show that

$$\left\| \left(B_{d,p}(z)\right)^{-1} - \left(B_{d,p}(\infty)\right)^{-1} \right\|^2_{L_2(\mathbb{D})} = \left(\prod_{i=1}^{n_p} |\phi_i|^2 \right) - 1. \tag{66}$$

It remains therefore to evaluate second term on the RHS of (65). Assume that $N(z)$ has (minimum) relative degree r (note in fact that in our case, $r = 2$).

Then

$$\begin{aligned}\Omega &= \inf_{Q(z)} \left\| B^{-1}_{d,p}(\infty) - Y(z) + Q(z)N(z) \right\|^2_{L_2(\mathbb{D})} \\ &= \inf_{Q(z)} \left\| z^r \left(B^{-1}_{d,p}(\infty) - Y(z) \right) + Q(z)\,(z^r N(z)) \right\|^2_{L_2(\mathbb{D})}\end{aligned} \tag{67}$$

Since $Y(z)$ is analytic in $\mathbb{D}^c$, it admits a Taylor series expansion in z^{-1}. We also note from (63) and that $N(z)$ is strictly proper that $Y(\infty) = B^{-1}_{d,p}(\infty)$ and therefore

$$z^r \left(B^{-1}_{d,p}(\infty) - Y(z) \right) = \beta_1 z^{(r-1)} + \beta_2 z^{(r-2)} + \ldots + \beta_{r-1} z + Y^+(z) \tag{68}$$

where $Y^+(z)$ is stable and proper. Furthermore, from (63) and $N(z)$ relative degree r, $B^{-1}_{d,p} = -\beta_o - \beta_1 z - \beta_2 z^2 \ldots$; and so

$$\begin{aligned}\beta_o &= -\prod_{i=1}^{n_p} (\phi_i) \\ \beta_1 &= \left(-\prod_{i=1}^{np} (\phi_i) \right) \sum_{i=1}^{n_p} \left(\frac{|\phi_i|^2 - 1}{\phi_i} \right)\end{aligned} \tag{69}$$

Using (68) in (67), and noting that $z^\ell \in H_2^\perp(\mathbb{D})$ for $\ell = 1, 2, \ldots$ we obtain:

$$\begin{aligned}\Omega &= \left\| \beta_1 z^{(r-1)} + \beta_2 z^{(r-2)} + \ldots + \beta_{r-1} z \right\|^2_{l_2(\mathbb{D})} \\ &\quad + \inf_{Q(z)} \| Y^r(z) + Q(z)\,(z^r N(z)) \| \\ &= |\beta_1|^2 + |\beta_2|^2 + \ldots + |\beta_{r-2}|^2\end{aligned} \tag{70}$$

since $z^r N(z)$ is minimum phase and relative degree zero, and $z^\ell \perp z^k$ for $k \neq \ell$. The result then follows from the fact that the relative degree is 2, and using (70) and (66). □

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