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Channel signal-to-noise ratio constrained feedback control: performance and robustness

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Abstract: The limitations in performance and robustness imposed by explicitly considering a communication channel in a control loop have received increased attention in recent years. Previous results in the literature have stated these limitations in terms of a minimal transmission data rate necessary for stabilisation. In this paper a signal-to-noise ratio (SNR) approach is used to study two specific cases: (i) performance in terms of model matching and (ii) robustness against a multiplicative uncertainty in the plant model. The analysis performed leads to closed-form expressions that allow the quantification of the extra SNR required in both cases.

1 Introduction

Limitations to stabilisability, performance and robustness in the area of control over networks have been topics of increased interest in recent years (see [1, 2] and the references therein). The most general results on stabilisability use information theoretic arguments to quantify the lowest channel transmission data rate necessary and sufficient for closed-loop stability [2–5].

For linear plant models, in [3, Theorem 2.1] and [4, Proposition 3.1] it is proved that if the plant is to be stabilised, then the transmission data rate has to satisfy a lower bound given by the $\log_2$ sum of the open loop unstable eigenvalues of the plant model.

Performance has been studied in terms of the variance of the state of a plant with stochastic input disturbance in [2, Theorem 1] and in [4, Corollary 3.2]. It is shown in [2, 4] that when the transmission data rate approaches the lower bound for stabilisability, the plant state variance tends to infinity, with no regard of the disturbance process variance. In [6, Theorem 7.3] an extension of the well-known Bode Integral [7, 8] is presented for the case of a plant with a stochastic input disturbance. From [6] it is possible to argue the obtainable (or not-obtainable thereof) performance as frequency attenuation of a sensitivity-like function for the closed-loop. As the channel transmission data rate approaches the lower bound for stabilisability, the Bode Integral for the sensitivity-like extension will be lower bounded by zero, which implies that disturbance attenuation at any frequency is impossible. Such loss of disturbance attenuation is consistent with the unboundedness of the state variance shown in [2, 4]. In summary, we have that the transmission data rate constraint required for some level of performance will be more severe than if just stabilisability is required, in agreement with an observation in [9, Remark 1].

Robustness has been recently studied by means of information theoretic arguments in [10, Theorem 3.4] and in [11, Theorem 3.3] and in the context of quantised systems in [12, Theorem 2]. In [10, Theorem 3.4] an upper bound on the plant state *m*th moment is presented as a sufficient condition for the existence of a stabilising feedback for a discrete-time stochastic scalar plant subject to uncertainties and a communication channel with a stochastic transmission data rate. In [11], on the other hand, a necessary condition is introduced for the stabilisability (and observability) of a linear discrete-time stochastic plant (subject to frequency-bounded uncertainties) as a lower bound on the channel capacity [11, Theorem 3.3], which is then explicitly computed for a scalar plant [11, Remark 3.8B]. Finally in [12, Theorem 2] the construction of an encoder and controller (decoder) is presented such that the obtained design guarantees the stability of a discrete-time

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linear unstable plant with uncertainties over a transmission data rate constrained communication channel.

A drawback of some of these general results is the lack of tightness in the obtained bounds ([4, Corollary III.2], [2, Theorem 1], [6, Theorem 7.3]) and the difficulty of implementing usually nonlinear solutions for the encoder and decoder involved in the communication channel [3, 11, 12]. Moreover, most of the contributions in the area of control over networks are for discrete-time systems. However, the plant is usually a continuous-time process, with continuous-time disturbances and model uncertainties. Also, even if analogue plant non-minimum phase (NMP) zeros can be removed by sampling ([13, §4]), the underlying limitation imposed by the NMP zeros will still remain [14, Remark 1]. Finally, few results in the literature include time delay ([15]) due to its infinite dimensional challenging characteristic in continuous-time.

In the present paper we follow the line of research proposed in [15, 16] and neglect any message encoding and decoding in the communication link, which is then reduced to the channel model itself. The analysis introduced in [15, 16] considers an additive white Gaussian noise (AWGN) channel model, casting the stabilisability problem of a linear time invariant (LTI) unstable plant as one of lower bounding the channel signal-to-noise ratio (SNR). In [17] we extended the formulation of [15, 16] to the case of an additive coloured Gaussian noise (ACGN) channel with a bandwidth limitation. Such bandwidth limitation may be imposed, for example, to avoid interference between different channels, while the coloured noise assumption is more realistic for a general communication channel. In the present paper, motivated by the poor performance of the infimal SNR solution for stabilisability, we consider quantifying the channel SNR for performance as disturbance rejection, and robustness as model uncertainty.

We show that if one requires performance as shaping of the loop sensitivity function, or faces robustness against multiplicative uncertainty in the plant model then, necessarily, the required SNR will be greater than that required for stabilisability. Specifically, we characterise, in a closed-form expression, the sensitivity function that arises from the infimal SNR solution for stabilisability. The extra SNR requirement is then quantified as the squared H_2 norm of the difference between the sensitivity function due to the performance (or robustness) requirement and the sensitivity function imposed by the infimal SNR solution.

The paper is organised as follows: in Section 2 we review the continuous-time output feedback stabilisability problem (and its solution) over an ACGN channel with bandwidth limitation. Section 3 presents solutions that impose an extra SNR requirement, discussed first in the framework of a desired performance and then as a consequence of multiplicative uncertainty in the plant model. We also provide numerical examples to illustrate these extra SNR requirements. Concluding remarks on the obtained results are presented in Section 4.

A preliminary version of the present results has been communicated in [18].

Terminology: Let $\mathbb{C}^-, \bar{\mathbb{C}}^-, \mathbb{C}^+$ and $\bar{\mathbb{C}}^+$ denote, respectively, the open-left, closed-left, open-right and closed-right halves of the complex plane $\mathbb{C}$. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^+$ the set of positive real numbers, $\mathbb{R}_o^+$ the set of non-negative real numbers and $\mathbb{R}^-$ the set of real negative numbers. A continuous-time signal is denoted by $x(t)$, $t \in \mathbb{R}_o^+$, and its Laplace transform by $X(s)$, $s \in \mathbb{C}$. Where the meaning is clear from the context, we will omit the argument of $x(t)$ or $X(s)$. The expectation operator is denoted by ε. A rational transfer function of a continuous-time system is minimum phase if all its zeros lie in $\bar{\mathbb{C}}^-$, and is non-minimum phase if it has zeros in $\mathbb{C}^+$. Given $P(s)$, the transfer function of a continuous-time system, we say that $P(s) \in H_2$ if $P(s)$ is strictly proper and stable; i.e, all its poles lie in $\mathbb{C}^-$. We say that $P(s)$ is in RH_∞ if $P(s)$ is a proper and real rational stable transfer function. The squared H_2 norm of $P(s)$, denoted by $\|P\|_{H_2}^2$, is $\|P\|_{H_2}^2 = (1/2\pi)\int_{-\infty}^{\infty} |P(j\omega)|^2 \mathrm{d}\omega$. The class of all stabilising controllers $C_o(s)$ of an unstable plant $G_o(s)$ is denoted by $\mathcal{K}$. For a complex number a, $\bar{a}$ represents its complex conjugate. The power of a stationary stochastic signal $u(t)$ is defined by $\|u\|_{\mathrm{Pow}}^2 \triangleq \varepsilon\{u^2(t)\}$.

2 Brief review of the SNR constrained stabilisation solution

Consider the feedback loop in Fig. 1 where the problem is to stabilise a continuous-time plant with time delay $\tau_o \in \mathbb{R}_o^+$

$$G_o(s) = G_1(s)\mathrm{e}^{-s\tau_o}$$

where $G_1(s)$ is a rational transfer function with relative degree $n_g \geq 0$, which contains m different unstable poles ($p_i \in \mathbb{C}^+, i = 1, \ldots, m$), and q different NMP zeros ($z_j \in \mathbb{C}^+, j = 1, \ldots, q$). The assumption of distinct zeros and poles in $\mathbb{C}^+$ simplifies the derivation of the results, but it is not essential to them.

We assume the channel model to be the ACGN channel with bandwidth limitation, as in Fig. 1. The signals

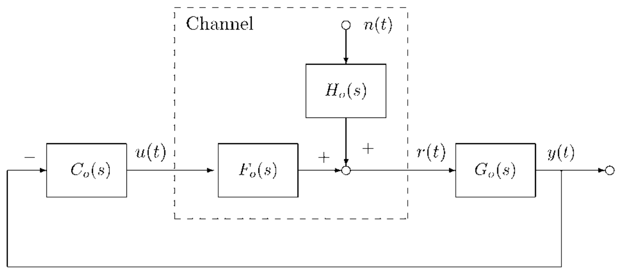

Figure 1 *Stabilisation via output feedback over an ACGN channel with bandwidth limitation*

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involved in the channel model are $u(t)$ the channel input, $r(t)$ the channel output and $n(t)$ a zero-mean white Gaussian noise process with power spectral density Φ. There are two possible locations for the ACGN channel (measurement path and actuation path); we consider here the actuation path location. Such a setting is common in practice and arises, for example, when actuators are far from the controller and have to communicate through a communication network. The channel transfer function $F_o(s)$, modelling the bandwidth limitation, is assumed to be stable, minimum phase and with relative degree $n_f \geq 0$. The minimum phase condition for the channel model $F_o(s)$ is without loss of generality since in the setting of Fig. 1 any NMP zero located in $F_o(s)$ can be relocated into $G_1(s)$. The channel transfer function $H_o(s)$ colouring the additive white Gaussian noise $n(t)$ is assumed to be stable, minimum phase and with relative degree $n_h \geq 0$.

The channel input is required to satisfy the power constraint

$$\mathcal{P} > \|u\|^2_{\text{Pow}} \tag{1}$$

for some predetermined input power level $\mathcal{P} > 0$. We assume that the closed-loop feedback system is stabilised, in the sense that for any distribution of initial conditions, the distribution of all closed-loop signals in Fig. 1 converges exponentially fast to a stationary distribution. Without loss of generality, we therefore consider the properties of the stationary distribution of the relevant signals. The power of the channel input signal satisfies then

$$\|u\|^2_{\text{Pow}} = \|T_{\text{un}}\|^2_{H_2}\Phi \tag{2}$$

where Φ is the power spectral density of the channel additive noise and $T_{\text{un}}(s)$ is the closed-loop transfer function

$$T_{\text{un}}(s) = -\frac{C_o(s)G_o(s)}{1 + C_o(s)G_o(s)F_o(s)}H_o(s) \tag{3}$$

relating the channel input with the channel additive noise. The channel input power constraint can be restated, from (1) and (2), as a constraint imposed on $\mathcal{P}/\Phi$ the channel SNR

$$\frac{\mathcal{P}}{\Phi} > \|T_{\text{un}}\|^2_{H_2} \tag{4}$$

With a slight abuse of notation the proposed SNR $\mathcal{P}/\Phi$ involves Φ, the power spectral density of the channel noise rather than its power. The choice of the channel additive noise power spectral density is justified since the channel additive Gaussian noise power $\|n\|^2_{\text{Pow}}$

$$\|n\|^2_{\text{Pow}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi \mathrm{d}\omega = \Phi\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega$$

is ill-defined in continuous-time.

We introduce now the terms $B_p(s)$ and $B_z(s)$ defined as

$$B_p(s) = \prod_{i=1}^{m}\frac{s - p_i}{s + \bar{p}_i}, \quad B_z(s) = \prod_{j=1}^{q}\frac{s - z_j}{s + \bar{z}_j} \tag{5}$$

containing, respectively, the $\mathbb{C}^+$ poles of $G_o(s)$ and the $\mathbb{C}^+$ zeros of $G_o(s)$. We also define the residue of $B_p^{-1}(s)$ at $s = p_i$ by

$$\text{Res}_{s=p_i}B_p^{-1}(s) := 2\text{Re}\{p_i\}\prod_{\substack{j=1\\ j\neq i}}^{m}\frac{p_i + \bar{p}_j}{p_i - p_j} \tag{6}$$

In [19] it is shown that for $\mathcal{K}$, the class of all stabilising controllers $C_o(s)$, the SNR $\mathcal{P}/\Phi$ required for stability satisfies the closed-form lower bound

$$\frac{\mathcal{P}}{\Phi} > \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{r_i\bar{r}_j}{p_i + \bar{p}_j}\mathrm{e}^{(p_i+\bar{p}_j)\tau_o} \tag{7}$$

representing the infimal in $\mathcal{K}$ of the squared H_2 norm $\|T_{\text{un}}\|^2_{H_2}$ in (4) and

$$r_i = \text{Res}_{s=p_i}B_p^{-1}(s)B_z^{-1}(p_i)F_o^{-1}(p_i)H_o(p_i) \tag{8}$$

The proof of (7) is given in [17] and is included in the Appendix 1 for completeness.

Formula (7) presents explicitly the main obstacles to feedback stability in terms of a limitation in the channel SNR, that is: unstable poles, NMP zeros and time delay. The effect of the ACGN channel with bandwidth limitation is to increase the infimal SNR required for stabilisability, through the gain value of the inverse of $F_o(s)$ at the plant unstable poles. The effect of the gain value of $H_o(s)$ on the infimal SNR required for stabilisability will depend on the frequency response of $H_o(s)$.

Example 1: We continue the present exposition by studying the reduced case given by a minimum phase plant with only one real unstable pole p and no time delay, $\tau_o = 0$. The objective is to perceive what is the SNR demand as the filter $F_o(s)$ frequency response becomes flat. In order to do so we choose $F_o(s)$ to be a Butterworth filter of variable order n, while in the interest of clarity we consider $H_o(s) = 1$. The SNR required for stabilisability is then given by

$$\frac{\mathcal{P}}{\Phi} > 2pB_n(p/\omega_o)^2 \tag{9}$$

where $B_n(s)$ is the Butterworth polynomial in factorised form, [20, pp. 508–509], and ω_o is the -3 [dB] cut-off frequency defining the bandwidth of the communication channel.

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Now it is possible to observe, for example from Fig. 2, that increasing the order of filter $F_o(s)$ from 1 to 3 (i.e. a roll-off of -60 [dB] instead of -20 [dB]) will increase the SNR required for stabilisability. This can be analytically quantified by the factor

$$\frac{B_3(p/\omega_o)^2}{B_1(p/\omega_o)^2} = \left(\frac{p^2}{\omega_o^2} + \frac{p}{\omega_o} + 1\right)^2$$

Still, the above example only addresses the requirement for stabilisability. In order to investigate the closed-loop performance of the infimal SNR controller $\hat{C}_o(s)$ (which by intuition we expect to be poor), we present next the closed-form expression for the optimal output feedback sensitivity function $\hat{S}_o(s) = 1/(1 + G_o(s)\hat{C}_o(s))$.

Theorem 1 (infimal SNR sensitivity function): Consider an ACGN channel with bandwidth limitation, as in Fig. 1, and a stabilising proper controller $\hat{C}_o(s)$ which achieves the infimal SNR. The expression for the optimal closed-loop sensitivity function is then given by

$$\hat{S}_o(s) = 1 - e^{-s\tau_o} B_z(s) F_o(s) H_o^{-1}(s) \sum_{i=1}^{m} \left(\frac{r_i e^{p_i \tau_o}}{s + \bar{p}_i} \prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{s - p_j}{s + \bar{p}_j} \right) \tag{10}$$

Proof 1: Recall from the proof reported in the Appendix that the optimal Youla parameter $\hat{Q}_o(s)$ is given by

$$\hat{Q}_o(s) = -M_o^{-1}(s) N_o^{-1}(s) \Gamma(s) H_o^{-1}(s) \tag{11}$$

The infimal complementary sensitivity function is given by

$$\hat{T}_o(s) = \frac{F_o(s) G_o(s) \hat{C}_o(s)}{1 + F_o(s) G_o(s) \hat{C}_o(s)}$$

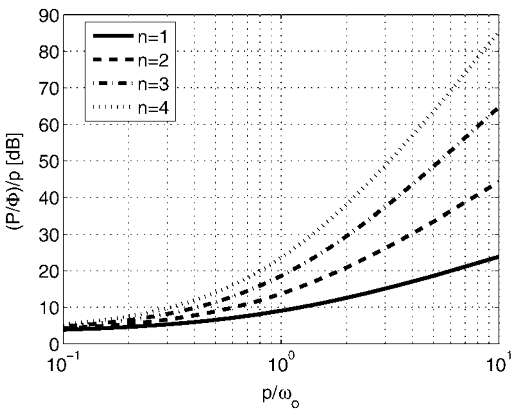

Figure 2 *SNR/p bound for stabilisability as a function of the factor p/ω_o*

Replacing $\hat{C}_o(s)$ as in (30) with the Youla parameter $\hat{Q}_o(s)$ and $F_o(s)G_o(s)$ defined as in (29) gives

$$\hat{T}_o(s) = 1 - M(s)Y(s) + e^{-s\tau_o} N(s) M(s) \hat{Q}_o(s) \tag{12}$$

Replacing $\hat{Q}_o(s)$ as in (11) into (12) gives

$$\begin{aligned}
\hat{T}_o(s) &= 1 - M(s)Y(s) + e^{-s\tau_o} N(s) M(s) M_o^{-1}(s) N_o^{-1}(s) \\
&\quad \times \Bigg(\underbrace{-B_p^{-1}(s) N_o(s) X(s) H_o(s) + \sum_{i=1}^{m} \frac{r_i e^{p_i \tau_o}}{s - p_i}}_{-\Gamma(s)} \Bigg) H_o^{-1}(s) \\
&= 1 - M(s)Y(s) - e^{-s\tau_o} N(s) X(s) \\
&\quad + e^{-s\tau_o} B_z(s) B_p(s) \left(\sum_{i=1}^{m} \frac{r_i e^{p_i \tau_o}}{s - p_i} \right) F_o(s) H_o^{-1}(s) \\
&= e^{-s\tau_o} B_z(s) \sum_{i=1}^{m} \left(\frac{r_i e^{p_i \tau_o}}{s + \bar{p}_i} \prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{s - p_j}{s + \bar{p}_j} \right) F_o(s) H_o^{-1}(s)
\end{aligned}$$

where in the last line we used the Bezout identity $e^{-s\tau_o} N(s) X(s) + M(s) Y(s) = 1$. Finally, recalling that $\hat{S}_o(s) = 1 - \hat{T}_o(s)$ gives (10) □

In order to avoid unnecessary complications we will maintain, in what follows, the assumption introduced in Theorem 1 of a proper closed-loop controller, this is equivalent to assume the condition $n_g \geq n_f - n_h + 1$. The above assumption can be and the infimal SNR requirement will then be only arbitrarily approached, but not achieved. The optimal closed-loop sensitivity function expression becomes also more involved (see [21, Theorem 4.1] for more details).

Note that the sensitivity function $\hat{S}_o(s)$ from (10) corresponding to the infimal SNR required for stability will have poor disturbance rejection performance, as it essentially corresponds to a minimum energy control solution (see for example [13]). This can be seen easily for the case of a memoryless AWGN channel with a minimum phase plant with no time delay. Indeed, in this case, from (6) and (8) follows

$$-\sum_{i=1}^{m} \frac{r_i}{s - p_i} = 1 - B_p^{-1}(s)$$

By multiplying booth sides by $B_p(s)$ and rearranging we obtain

$$1 - B_p(s) \sum_{i=1}^{m} \frac{r_i}{s - p_i} = B_p(s) = \prod_{i=1}^{m} \frac{s - p_i}{s + \bar{p}_i} \tag{13}$$

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which equals $\hat{S}_o(s)$ accordingly to Theorem 1. The frequency response of such a sensitivity function is all-pass with magnitude one, and thus it does not achieve any disturbance rejection. This observation is consistent with the conclusion that follows from [6, Theorem 7.3] and the observation made in [9. Remark 1].

Quantifying the extra SNR needed when we require more than just stability is the focus of the next section.

3 Beyond stabilisability: SNR trade-offs

In the present section we address the problem of quantifying the channel SNR when the closed-loop sensitivity function is not the optimal sensitivity function $\hat{S}_o(s)$ described in Theorem 1

Such a case can arise when some required control design objectives have been combined into a target sensitivity function, or also for example when we have to deal with uncertainties in the plant model. Both situations will require a higher channel SNR in comparison with the infimal SNR solution for stabilisability reviewed in the previous section.

As a practical motivation consider the following sketched case: if we estimate that the channel input power satisfies (7) with equality and we have a multiplicative uncertainty in the plant model (or the knowledge of its bound), then the resulting channel input power $\|u\|^2_{\mathrm{Pow}}$ will be greater than the infimal power constraint $\hat{\mathcal{P}}$. This could affect the transmitter hardware (designed to satisfy $\hat{\mathcal{P}}$, but not $\|u\|^2_{\mathrm{Pow}}$), and in turn it could result in distortion or interference with other users nearby. Also, for example, if the channel transmitter is a remote wireless modem working on battery power, its operational time will be unavoidably reduced as the battery is drained at the increased rate imposed by $\|u\|^2_{\mathrm{Pow}}$ instead of the lower power level $\hat{\mathcal{P}}$.Thus, it is important to analyse the case of extra SNR requirement beyond stabilisability.

We perform our analysis by means of a sensitivity function $S_{\mathrm{ext}}(s)$ that represents the performance (or robustness) requirement. Observe that even though $S_{\mathrm{ext}}(s)$ will be different from $\hat{S}_o(s)$, it satisfies the interpolation conditions for internal stability

$$S_{\mathrm{ext}}(p_i) = 0\forall i = 1, \ldots, m, \quad S_{\mathrm{ext}}(z_j) = 1\forall j = 1, \ldots, q$$

imposed by the NMP zeros and unstable poles.

The next theorem specifies the additional SNR required when $S_{\mathrm{ext}}(s)$ is the sensitivity function of the output feedback control loop. The result is in terms of a lower bound for the SNR and is expressed by two terms. The first term accounts for the stabilisability of the feedback control loop, while the second term accounts for having $S_{\mathrm{ext}}(s)$ instead of $\hat{S}_o(s)$.

Theorem 2 (extra SNR requirement): If the choice of the closed-loop stabilising controller in Fig. 1 is such that the closed-loop sensitivity function is given by $S_{\mathrm{ext}}(s)$ instead of $\hat{S}_o(s)$, then the channel SNR satisfies

$$\frac{\mathcal{P}}{\Phi} > \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{r_i\bar{r}_j}{(p_i+\bar{p}_j)}\mathrm{e}^{(p_i+\bar{p}_j)\tau_o} + \left\|(S_{\mathrm{ext}} - \hat{S}_o)F_o^{-1}H_o\right\|^2_{H_2} \tag{14}$$

in which $\sum_{i=1}^{m}\sum_{j=1}^{m} r_i\bar{r}_j/p_i + \bar{p}_j\mathrm{e}^{(p_i+\bar{p}_j)\tau_o}$ takes into account the stabilisability requirement, while the expression $\left\|\left(S_{\mathrm{ext}} - \hat{S}_o\right)F_o^{-1}H_o\right\|^2_{H_2}$ weights the extra SNR requirement imposed by $S_{\mathrm{ext}}(s)$.

Proof 2: Recall from the proof reported in Appendix 1 that the optimal Youla parameter $\hat{Q}_o(s)$ is given by

$$\hat{Q}_o(s) = -M_o^{-1}(s)N_o^{-1}(s)\Gamma(s)H_o^{-1}(s)$$

Also from the same proof reported in Appendix 1 consider (35) from which we drop the infimal operator and recognise the expression for $Q_o(s)$

$$\|T_{\mathrm{un}}\|^2_{H_2} = \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{r_i\bar{r}_j}{p_i+\bar{p}_j}\mathrm{e}^{\left(p_i+\bar{p}_j\right)\tau_o} + \Bigg\|\underbrace{-M_oN_o\hat{Q}_oH_o}_{\Gamma(s)} + M_oN_oQ_oH_o\Bigg\|^2_{H_2}$$

Reintroduce in the squared H_2 norm term the all-pass Blaschke product factors $B_p(s)$, $B_z(s)$ and the factor $\pm NXF_o^{-1}H_o$

$$\begin{aligned}\|T_{\mathrm{un}}\|^2_{H_2} = {}& \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{r_i\bar{r}_j}{p_i+\bar{p}_j}\mathrm{e}^{\left(p_i+\bar{p}_j\right)\tau_o} \\ &+ \| \pm NXF_o^{-1}H_o + MNQ_oF_o^{-1}H_o \\ &- MN\hat{Q}_oF_o^{-1}H_o\|^2_{H_2}\end{aligned} \tag{15}$$

Finally consider that $Q_o(s) = Q_{\mathrm{ext}}(s)$ such that $Q_{\mathrm{ext}}(s)$ satisfies

$$T_{\mathrm{ext}}(s) = \mathrm{e}^{-s\tau_o}[N(s)X(s) + N(s)M(s)Q_{\mathrm{ext}}(s)] \tag{16}$$

and thus we obtain from (15)

$$\|T_{\mathrm{un}}\|^2_{H_2} = \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{r_i\bar{r}_j}{p_i+\bar{p}_j}\mathrm{e}^{\left(p_i+\bar{p}_j\right)\tau_o} + \left\|\left(T_{\mathrm{ext}} - \hat{T}_o\right)F_o^{-1}H_o\right\|^2_{H_2} \tag{17}$$

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where we used the fact that $e^{-s\tau_o}$ is all-pass. Finally, since $T_{\text{ext}}(s) = 1 - S_{\text{ext}}(s)$ and $\hat{T}_o(s) = 1 - \hat{S}_o(s)$, replacing in (17) gives (14) which ends the proof. □

Note 1: From (17) in the proof of Theorem 2 we observe that its main result can be equivalently restated as

$$\frac{\mathcal{P}}{\Phi} > \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{r_i \bar{r}_j}{p_i + \bar{p}_j} e^{\left(p_i + \bar{p}_j\right)\tau_o} + \left\| \left(T_{\text{ext}} - \hat{T}_o\right) F_o^{-1} H_o \right\|_{H_2}^2$$

which we shall also use subsequently, where $T_{\text{ext}}(s)$ is as in (16) and $\hat{T}_o(s)$ is given by

$$\hat{T}_o(s) = e^{-s\tau_o} B_z(s) \sum_{i=1}^{m} \left(\frac{r_i e^{p_i \tau_o}}{s + \bar{p}_i} \prod_{\substack{j=1 \\ j\neq i}}^{m} \frac{s - p_j}{s + \bar{p}_j} \right) F_o(s) H_o^{-1}(s) \quad (18)$$

Example 2: We claim that Theorem 2 is tight in the sense that there are controllers that achieve the expressed bounds. As a simple example to illustrate this consider

$$G_o(s) = \frac{1}{s-2}, \quad F_o(s) = 10s + 10, \quad H_o(s) = 1$$

Theorem 1 gives us the sensitivity (and thus the complementary sensitivity) related to the closed-loop infimal SNR required for stabilisability solution for the present example

$$\hat{T}_o(s) = \frac{48}{(s+2)(s+10)}$$

and the infimal SNR for stabilisability satisfies $\inf_{C_o(s)\in\mathcal{K}} \|T_{\text{un}}\|_{H_2}^2 = \|\hat{T}_o F_o^{-1}\|_{H_2}^2 = 5.76$. Consider now that the user is not satisfied with such bandwidth for the closed-loop and decide that it requires the following complementary sensitivity to be in place instead

$$T_{\text{ext}}(s) = \frac{84}{(s+5)(s+10)}$$

Notice that $S_{\text{ext}}(s) = 1 - T_{\text{ext}}(s)$ and that it satisfies $S_{\text{ext}}(2) = 0$. Theorem 2 allows us to quantify the effect of the above choice on the channel SNR through the expression $\|(T_{\text{ext}} - \hat{T}_o)F_o^{-1}\|_{\mathcal{H}_2}^2 = 1.2960$, thus the overall channel SNR now satisfies $\mathcal{P}/\Phi > 5.76 + 1.2960 = 7.0560$. For the present example both lower bounds are achievable and therefore tight. For the infimal SNR lower bound of 5.76 the optimal controller is given by

$$\hat{C}_o(s) = \frac{4.8(s+10)}{(s+14)}$$

while for the case of $T_{\text{ext}}(s)$ the controller achieving the lower bound of 7.0560 is given by

$$C_{\text{ext}}(s) = \frac{8.4(s+10)}{(s+17)}$$

We follow on the result of Theorem 2 by studying the two possible reasons outlined earlier for its use, namely performance and robustness.

3.1 SNR and performance

Consider the performance requirement of having one closed-loop pole located at $-\beta$ (with $\beta \in \mathbb{R}^+$)

$$S_{\text{ext}}(s) = \frac{s-p}{s+\beta} \quad (19)$$

The plant is given by $G_o(s) = 1/(s-p)$ (with $p > 0$) and the channel is a memoryless AWGN channel [i.e. $F_o(s) = 1$ and $H_o(s) = 1$]. The resulting SNR from Theorem 2 satisfies

$$\frac{\mathcal{P}}{\Phi} > 2p + \frac{(p-\beta)^2}{2\beta}$$

If $\beta = p$ we regain the minimum value of $2p$, recovering the result presented in (7). Notice, although, that the present discussion has been developed around the idea of $S_{\text{ext}}(s)$ in Theorem 2 to be known. In particular for the present case we have $S_{\text{ext}}(s)$ as in (19). The drawback of such approach is that it lacks generality. To clarify this statement consider $p = 2$ and $\beta = 3$, then the SNR requirement of having $S_{\text{ext}}(s)$ as in (19) instead of $\hat{S}_o(s)$ is 4.1667 (using Theorem 2). On the other hand if we now choose $S_{\text{ext}}(s)$ to be

$$S_{\text{ext}}(s) = \frac{(s-p)(s+2p)}{(s+\beta)^2} \quad (20)$$

we have that the SNR is then lower bounded by

$$\frac{\mathcal{P}}{\Phi} > 2p + \frac{4p^4 + 5p^2\beta^2 + 5\beta^4 - 12p\beta^3}{4\beta^3}$$

Notice that if $\beta = p$ we obtain an extra SNR of $p/2$ due to the different roll-off of $S_{\text{ext}}(s)$ in (20) and $\hat{S}_o(s) = (s-p)/(s+p)$ when approaching 0 [dB] (alternatively we can observe that $S_{\text{ext}}(s) \neq \hat{S}_o(s)$ when $\beta = p$). For a similar choice of $p = 2$ and $\beta = 3$ (which also locates the closed-loop poles at -3, but with multiplicity 2), the SNR requirement is now 4.0093, more than 4, but less than the previous value of 4.1667.

Thus, another approach to quantify the SNR requirement for performance is desirable. To achieve this consider defining frequency bounds, for example on the required attenuation for the sensitivity function. In that case Theorem 2 can still be of use if we focus on obtaining meaningful lower bounds for the extra SNR term $\left\| \left(S_{\text{ext}} - \hat{S}_o\right) F_o^{-1} H_o \right\|_{H_2}^2$.

Theorem 3 (performance SNR requirement): Assume that the performance requirement of sensitivity reduction over a non-trivial bandwidth is defined by a function $S_{\max}(s)$, and that for any $S_{\text{ext}}(s)$ we have $|S_{\text{ext}}| \leq |S_{\max}|$.

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Assume also that the complementary sensitivities in both cases are strictly proper, and therefore at high frequencies both magnitudes, $|S_{\max}|$ and $|S_{\text{ext}}|$, will tend to one. Then

$$\left\| \left(S_{\text{ext}} - \hat{S}_o\right) F_o^{-1} H_o \right\|_{H_2}^2 \geq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\left(|S_{\max}(j\omega)| - \left|\hat{S}_o(j\omega)\right| \right)^2 \left|F_o^{-1}(j\omega) H_o(j\omega)\right|^2 \right] \mathrm{d}\omega \tag{21}$$

Proof 3: Take the extra term as defined in (14)

$$\begin{aligned} \left\| \left(S_{\text{ext}} - \hat{S}_o\right) F_o^{-1} H_o \right\|_{H_2}^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|S_{\text{ext}}(j\omega) - \hat{S}_o(j\omega)\right|^2 \\ &\quad \times \left|F_o^{-1}(j\omega) H_o(j\omega)\right|^2 \mathrm{d}\omega \\ &\geq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(|S_{\text{ext}}(j\omega)| - \left|\hat{S}_o(j\omega)\right| \right)^2 \\ &\quad \times \left|F_o^{-1}(j\omega) H_o(j\omega)\right|^2 \mathrm{d}\omega \end{aligned} \tag{22}$$

From the condition of $|S_{\text{ext}}| \leq |S_{\max}|$ we have

$$|S_{\max}| \geq |S_{\text{ext}}| \Rightarrow \left(|S_{\max}| - \left|\hat{S}_o\right| \right)^2 \leq \left(|S_{\text{ext}}| - \left|\hat{S}_o\right| \right)^2$$

Replacing this inequality in (22) we obtain (21). Note that the strictly proper condition for the complementary sensitivities is needed to guarantee the convergence of (21). □

In order to investigate the tightness of this lower bound we consider the following example.

Example 3: Consider a plant with m distinct poles in $\mathbb{C}^+$, with all its zeros in $\mathbb{C}^-$ and $\tau_o = 0$. Assume the communication channel model to be a memoryless AWGN channel (i.e. $F_o(s) = 1$ and $H_o(s) = 1$). The sensitivity function obtained by solving the related continuous-time SNR constrained output feedback stabilisation problem is given in (13). Take also into account the case of a performance requirement defined through $|S_{\max}|$ as

$$|S_{\max}| = \begin{cases} \omega/\omega_o, & 0 \leq \omega \leq \omega_o \\ 1, & \omega_o < \omega \end{cases} \tag{23}$$

By this choice, the lower bound in (21) can be obtained as

$$\left\| S_{\text{ext}} - \hat{S}_o \right\|_{H_2}^2 \geq \frac{\omega_o}{3\pi}$$

To investigate how tight this bound is take the case of a choice of $S_{\text{ext}}(s)$ as

$$S_{\text{ext}}(s) = \frac{s}{s + \omega_o} B_p(s)$$

The magnitude of this selection for $S_{\text{ext}}(s)$ is given by

$$|S_{\text{ext}}| = \frac{\omega}{\sqrt{\omega^2 + \omega_o^2}} \leq \min\left\{1, \frac{\omega}{\omega_o}\right\}$$

Since the magnitude of $S_{\text{ext}}(s)$ is below the magnitude of $S_{\max}(s)$, the bound is valid, but in this case we can also obtain the exact value of $\left\| S_{\text{ext}} - \hat{S}_o \right\|_{H_2}^2$

$$\left\| S_{\text{ext}} - \hat{S}_o \right\|_{H_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| S_{\text{ext}} - \hat{S}_o \right|^2 \mathrm{d}\omega = \frac{\omega_o}{2} \tag{24}$$

The result in (24) tells us that for the present choice of $S_{\text{ext}}(s)$ the bound is off by 78% on the real extra value, but if we compare it with

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(|S_{\text{ext}}| - \left|\hat{S}_o\right| \right)^2 \mathrm{d}\omega = 0.1366\omega_o$$

the proposed lower bound differs only by 22%. This suggests that for the present choice of $S_{\text{ext}}(s)$ and $S_{\max}(s)$, the first approximation of $\left| S_{\text{ext}} - \hat{S}_o \right|^2$ by $(|S_{\text{ext}}| - |\hat{S}_o|)^2$ is the weakest, while the second approximation performed by replacing $(|S_{\text{ext}}| - |\hat{S}_o|)^2$ by $(|S_{\max}| - |\hat{S}_o|)^2$ is less coarse. Nonetheless the lower bound obtained through $|S_{\max}|$ is a more general result than Theorem 2 since it concludes that for any choice of $S_{\text{ext}}(s)$ that satisfies the frequency bounds imposed by $|S_{\max}|$, the extra SNR requirement will be at least of an amount equal to $\omega_o/3\pi$.

Finally, notice that the choice of the magnitude bounds defined through $S_{\max}(s)$ can be different from the one presented in (23). Another possibility is for $|S_{\max}|$ to be given by

$$|S_{\max}| = \begin{cases} \varepsilon, & 0 \leq \omega \leq \omega_o \\ 1, & \omega_o < \omega \end{cases}$$

The resulting lower bound for the above selection and $\hat{S}_o(s)$ as in (13) is obtained as $(1 - \varepsilon)^2/\pi\omega_o$.

3.2 SNR and robustness

In the present subsection we consider the case treated by robust control theory, see for example [22] and [23], when the plant is subject to multiplicative uncertainty

$$G(s) = G_o(s)(1 + G_\Delta(s))$$

where $G_o(s)$ is the nominal plant model and $G_\Delta(s)$ accounts for multiplicative uncertainty in the plant. More specifically, citing [24, pp. 42–44], we are dealing with a nominal plant model, $G_o(s)$, for control-system design purposes. We also consider a calibration model, $G(s)$, which is a more realistic representation of the plant with other features not used for control-system design but having a direct bearing on the achieved performance. Finally, the multiplicative uncertainty in the plant model, $G_\Delta(s)$,

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accounts for the difference between the calibration and nominal plant model. The details of the multiplicative uncertainty in the plant are not necessarily known, but if frequency bounds are available for it they can be used as $G_\Delta(s)$ (see for example [25, eq. 7]), in a worst-case scenario.

For the sake of simplicity, we exclude from the analysis the case of additive uncertainty $G_\epsilon(s)$ in the plant

$$G(s) = G_o(s) + G_\epsilon(s),$$

since under the proper assumptions $G_\epsilon(s)$ can be equivalently represented as a multiplicative uncertainty in the plant model.

Corollary 4 (robustness SNR requirement): Consider that the multiplicative uncertainty in the plant model lay in RH_∞ ([26, §9.3.2]). Consider also that it does not introduce (nor eliminate) any unstable pole, NMP zero and furthermore do not modify the existing *nominal* plant time delay τ_o. Then the SNR due to the presence of multiplicative uncertainty in the plant and/or channel model satisfies

$$\frac{\mathcal{P}}{\Phi} > \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{r_i\bar{r}_j}{p_i+\bar{p}_j}\mathrm{e}^{(p_i+\bar{p}_j)\tau_o} + \left\| \frac{\hat{S}_o\hat{T}_oG_\Delta}{1+\hat{T}_oG_\Delta}F_o^{-1}H_o \right\|_{H_2}^2 \qquad (25)$$

with r_i as in (8).

Proof 4: From [24, p. 145] we have that the real sensitivity function is given by

$$S_{\text{ext}}(s) = \frac{\hat{S}_o(s)}{1+\hat{T}_o(s)G_\Delta(s)}$$

Direct application of Theorem 2 gives (25) where it is implicitly assumed that the infimal stabilisation result is obtained for the nominal plant $G_o(s)$ and channel model $F_o(s)$. The condition for the real plant to preserve the nominal interpolation conditions and nominal time delay is required in order to be able to claim that (16) holds. □

As an example consider the following case.

Example 4: Assume that the nominal plant model is given by

$$G_o(s) = \frac{2-s\tau_o}{(2+s\tau_o)(s-p)}$$

where we are introducing a first-order Padé approximation with $\tau_o \in \mathbb{R}_o^+$ for the plant time delay $\mathrm{e}^{-s\tau_o}$ and $p \in \mathbb{R}^+$. The real plant model is given by

$$G(s) = \frac{2-s\tau_o}{(2+s\tau_o)(s-p)}\frac{a}{(s+a)} \qquad (26)$$

with $a \in \mathbb{R}^-$, thus $G_\Delta(s) = -s/(s+a)$. Furthermore assume for the sake of simplicity that the channel is a memoryless AWGN channel (that is $F_o(s) = 1$ and $H_o(s) = 1$), thus $\Delta(s) = G_\Delta(s)$. The infimal SNR controller taking into account the nominal plant is given by

$$\hat{C}_o(s) = \frac{2p(2+p\tau_o)(2+s\tau_o)}{(2-p\tau_o)\tau_os + (6p\tau_o+4)}$$

and the nominal complementary sensitivity is

$$\hat{T}_o(s) = \frac{G_o(s)\hat{C}_o(s)}{1+G_o(s)\hat{C}_o(s)} = \frac{2p(2+p\tau_o)(2-s\tau_o)}{(2-p\tau_o)(2+s\tau_o)(s+p)} \qquad (27)$$

On the other hand, the complementary sensitivity function considering the real plant model in (26) is given by

$$\begin{aligned}\hat{T}(s) &= \frac{G(s)\hat{C}_o(s)}{1+G(s)\hat{C}_o(s)} \\ &= \frac{2p(2+p\tau_o)(2-s\tau_o)}{(2-p\tau_o)(2+s\tau_o)(s+p) + (s[(2-p\tau_o)\tau_os^2 + (p\tau_o+2)^2s - p(6p\tau_o+4)]/a}\end{aligned} \qquad (28)$$

Notice that as $1/a \to 0$ we regain the nominal complementary sensitivity function in (27). Recall that the infimal SNR for stabilisability is the squared H_2 norm of $\hat{T}_o(s)$ and that the infimal SNR due to the presence of multiplicative uncertainty in the plant is given by the squared H_2 norm of $\hat{T}(s)$. In order to evaluate the squared H_2 norm of $\hat{T}_o(s)$ in (27) and $\hat{T}(s)$ (28) we can use the command `norm` in Matlab®, version 7.3.0.267 (R2006b). Notice although that robust stability of $\hat{C}_o(s)$ [the infimal SNR controller that achieves $\hat{T}_o(s)$] is not guaranteed a priori. To address the issue of robust stability we present the following argument: for $\tau_o = 0$ the closed-loop characteristic polynomial is given by $s^2 + (a-p)s + ap$, furthermore if we consider $a = (6p + 4p\sqrt{2})/2$ some algebra will confirm that we are locating the closed-loop

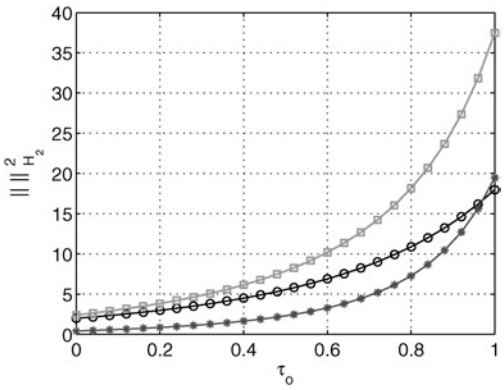

Figure 3 *H_2 norm comparison: $\hat{T}_o(s)$ (solid-circle line), $\hat{T}(s)$ (solid-square line) and $\hat{S}_o\hat{T}_o\Delta/1+\hat{T}_o\Delta$ (solid-dot line)*

The parameters defining the nominal and real plant are: $p = 1$, $\tau_o \in [0, 1]$ and $a = 5.8284$

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poles, for the real plant model, at $s = -p(1 + \sqrt{2})$ with multiplicity 2. Finally, by a continuity argument, the squared H_2 norm of $\hat{T}(s)$ will grow if the value of τ_o increases, and it will become infinite if $\hat{T}(s)$ becomes unstable. In Fig. 3, for $p = 1$ and $\tau_o \in [0, 1]$, we can observe that the value of the squared H_2 norm of $\hat{T}(s)$ is also given by to the sum of the squared H_2 norm of $\hat{T}_o(s)$ and the squared H_2 norm of $\hat{S}_o\hat{T}_o\Delta/1 + \hat{T}_o\Delta$, which agrees with Corollary 4. Not shown in Fig. 3 is the fact that closed-loop stability is lost for values of $\tau_o \geq 1.3588$.

4 Conclusion

We reviewed the infimal solution to the problem of SNR constrained stabilisability of a (non) minimum phase continuous-time LTI unstable plant with time delay over an ACGN channel model with bandwidth limitation. The solution to such a problem is expressed as a tight lower bound on the channel SNR, below which stability is not achievable with an LTI controller. We presented a closed-form expression for the output feedback sensitivity function resulting from the infimal solution for stabilisability.

We then extended the analysis to the case in which performance and robustness are required in addition to closed-loop stability. We showed that the channel SNR, in both cases, will be greater than the stabilisability SNR requirement. The output feedback sensitivity function for the infimal solution for stabilisability is a key element in quantifying the extra SNR requirement.

Most of the ideas presented here have corresponding discrete-time counterpart results.

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6 Appendix 1: Proof of the infimal SNR stabilisability result reported in [19]

Consider a coprime factorisation for $F_o(s)G_o(s)$ as

$$F_o(s)G_o(s) = \frac{e^{-s\tau_o}N(s)}{M(s)} \tag{29}$$

where $N(s), M(s) \in RH_\infty$. Further, without loss of generality, consider

$$N(s) = B_z(s)N_o(s)F_o(s), \quad M(s) = B_p(s)M_o(s)$$

where $N_o(s)$, $M_o(s) \in RH_\infty$, $N_o(s)$ and $M_o(s)$ are stable and MP transfer functions $B_p(s)$, $B_z(s)$ are as defined in (5).

Following [15, Lemma 3.1], a Youla parameterisation of all controllers that stabilise $G_o(s)$ is given by

$$C_o(s) = \frac{X(s) + M(s)Q_o(s)}{Y(s) - e^{-s\tau_o}N(s)Q_o(s)} \tag{30}$$

where $X(s)$ is in RH_∞, $Q_o(s)$, $Y(s)$ are in H_∞ and $X(s)$ and $Y(s)$ satisfy the Bezout identity

$$e^{-s\tau_o}N(s)X(s) + M(s)Y(s) = 1 \tag{31}$$

A demonstration of the Bezout identity (31) can be found for example in [27, Lemma 3.2]. Replacing these factorisations for $F_o(s)G_o(s)$ and $C_o(s)$ into (3) gives

$$\begin{aligned} T_{\mathrm{un}}(s) = -(&e^{-s\tau_o}B_z(s)N_o(s)F_o(s)X(s) \\ &+ e^{-s\tau_o}B_p(s)B_z(s)M_o(s)N_o(s) \\ &\times F_o(s)Q_o(s))F_o^{-1}(s)H_o(s) \end{aligned}$$

Since $B_p(s)$ and $B_z(s)$ are all pass they have norm one, we have

$$\begin{aligned} \inf_{Q_o(s)\in H_\infty} \|T_{\mathrm{un}}\|_{H_2}^2 = \inf_{Q_o(s)\in H_\infty} &\|e^{-s\tau_o}B_p^{-1}N_oXH_o \\ &+ e^{-s\tau_o}M_oN_oQ_oH_o\|_{L_2}^2 \end{aligned} \tag{32}$$

Since $e^{-s\tau_o}$ has magnitude one at all frequencies, the norm expression on the RHS of equation (32) is not affected by it

$$\inf_{Q_o(s)\in H_\infty} \left\|T_{\mathrm{un}}\right\|_{H_2}^2 = \inf_{Q_o(s)\in H_\infty} \left\|B_p^{-1}N_oXH_o + M_oN_oQ_oH_o\right\|_{L_2}^2 \tag{33}$$

Recall next the definitions for H_2 and $H_2^\perp$

$$H_2 = L_2 \cap \left\{G(s) : \text{analytic in } \overline{\mathbb{C}}^+\right\},$$
$$H_2^\perp = L_2 \cap \left\{G(s) : \text{analytic in } \overline{\mathbb{C}}^-\right\}$$

and notice that the second term inside the norm expression in (33) belongs to H_2, while the first term is a mixed term that can be decomposed as

$$B_p^{-1}(s)N_o(s)X(s)H_o(s) = \Gamma^\perp(s) + \Gamma(s)$$

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where $\Gamma(s)$ is in H_2, while $\Gamma^{\perp}(s)$ is in $H_2^{\perp}$ and therefore by Lemma 3 in [28, p. 196]

$$\inf_{Q_o(s)\in H_\infty} \|T_{\mathrm{un}}\|_{H_2}^2 = \|\Gamma^{\perp}\|_{H_2^{\perp}}^2 + \inf_{Q_o(s)\in H_\infty} \|\Gamma + M_o N_o Q_o H_o\|_{H_2}^2 \quad (34)$$

By means of a partial fraction expansion and the Bezout identity in (31), it is possible to quantify $\Gamma^{\perp}(s)$ as $\sum_{i=1}^{m}(r_i e^{p_i\tau_o}/s - p_i)$, where

$$r_i = \mathrm{Res}_{s=p_i} B_p^{-1}(s) B_z^{-1}(p_i) F_o^{-1}(p_i) H_o(p_i)$$

Note that from (31) we have $N_o(p_i)X(p_i) = F_o^{-1}(p_i)B_z^{-1}(p_i)e^{p_i\tau_o}$ at any p_i, $\forall i = 1, \ldots, m$ unstable poles of $G_o(s)$. The result for the first norm term on the RHS of (34), by use of the Residue theorem (see for example [29, pp. 169–172]), is

$$\|\Gamma^{\perp}\|_{H_2^{\perp}}^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{r_i \bar{r}_j}{p_i + \bar{p}_j} e^{(p_i+\bar{p}_j)\tau_o}$$

Replacing in (34) will give

$$\inf_{Q_o(s)\in H_\infty} \|T_{\mathrm{un}}\|_{H_2}^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{r_i \bar{r}_j}{p_i + \bar{p}_j} e^{(p_i+\bar{p}_j)\tau_o} + \inf_{Q_o(s)\in H_\infty} \|\Gamma + M_o N_o Q_o H_o\|_{H_2}^2 \quad (35)$$

We can make the expression $\Gamma(s) + M_o(s)N_o(s)Q_o(s)H_o(s)$ arbitrarily small in H_2 by choosing $Q_o(s)$ as

$$\hat{Q}_o(s) = -M_o^{-1}(s)N_o^{-1}(s)\Gamma(s)H_o^{-1}(s)$$

obtaining the infimal norm that can be achieved in (35) as

$$\inf_{Q_o(s)\in H_\infty} \|T_{\mathrm{un}}\|_{H_2}^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{r_i \bar{r}_j}{p_i + \bar{p}_j} e^{(p_i+\bar{p}_j)\tau_o}$$

which completes the proof. □

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