ELEC4410

Control System Design

Lecture 2: Mathematical Description of Systems

School of Electrical Engineering and Computer Science
The University of Newcastle

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- A Taxonomy of Systems

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- A Taxonomy of Systems
- Linear Systems

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- Linear Time-Invariant Systems

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- Linearisation

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- Discrete-Time Systems

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- A Few General Facts to Remember

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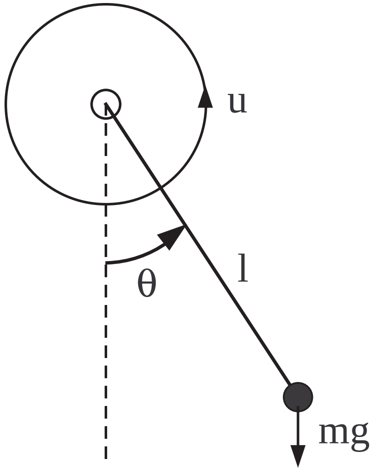
Reference: Linear System Theory and Design, Chen.

A Taxonomy of Systems

- Consider a simple problem in robotics, i.e. control of the position of a robot arm using a motor located at the arm joint.



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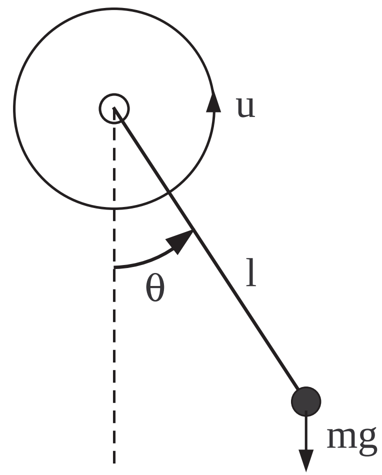


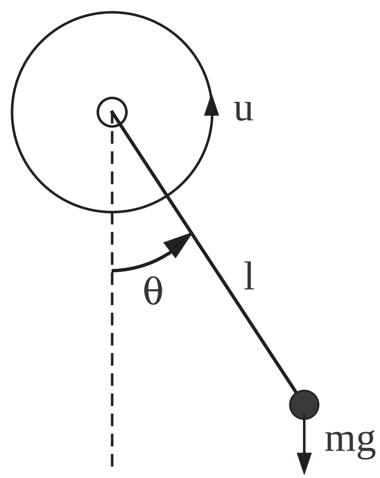
- Mathematically, this system is nothing else than a pendulum controlled by torque.

A Taxonomy of Systems

- Assume:
  - friction at the joint is negligible,
  - the arm is rigid, and
  - all the mass of the arm is concentrated on its free end,

then angle with respect to the vertical θ is given by the differential equation

$$ml^2\ddot{\theta}(t) + mgl\sin\theta(t) = u(t)\,.$$



A Taxonomy of Systems

- The single robot arm model given by the differential equation

$$ml^2\ddot{\theta}(t) + mgl\,\sin\theta(t) = u(t)\,.$$

is an example of a system that is:

- dynamic
- causal
- finite-dimensional
- continuous-time
- nonlinear
- time-invariant

A Taxonomy of Systems

Dynamic /Static?

- *Dynamic* means that the variables θ and $\dot{\theta} \doteq d\theta(t)/dt$, which define the *state* of the arm at a given instant of time t, have a non *instantaneous* dependency on the control torque u. A dynamic system is said to possess memory, i.e. it output depends also on previous inputs.

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- A system that is *not* dynamic is called *static*. In a static system the output has an instantaneous dependency on the evolution of the input. Static systems are also called *memoryless*.

A Taxonomy of Systems

Causal?

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- The current output of a causal dynamic system always depends on past values of the input. But how far back in time do these past values still have an effect on the output?

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- All real physical systems are causal.
- The current output of a causal dynamic system always depends on past values of the input. But how far back in time do these past values still have an effect on the output?
  - Strictly, we would need to go back in time up to $t = -\infty$, which is not very practical. This difficulty is resolved with the concept of *state*.

A Taxonomy of Systems

State?

- The *state* $x(t_0)$ of a system at the time instant t_0 is the information that together with the input $u(t)$ for $t \geq t_0$ univocally determines the output $y(t)$ for all $t \geq t_0$.

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- The state $x(t_0)$ summarises all the system history from $t = -\infty$ to t_0, e.g. with the knowledge of the angle θ and the angular velocity $\dot{\theta}$ at time t_0, we can predict the response of the robot arm to torque inputs u for all time $t \geq t_0$.

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- The *input* at $t \geq t_0$ and the *initial conditions* $x(t_0)$ determine the evolution of the system for $t \geq t_0$, which we could represent as

$$y(t),\ t \geq t_0 \Lleftarrow \begin{cases} x(t_0) \\ u(t),\ t \geq t_0 \end{cases}$$

A Taxonomy of Systems

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- In the case of the robot arm, two parameters: angle θ and angular velocity $\dot{\theta}$.

A Taxonomy of Systems

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- Means that the independent variable, time t, takes values in a *continuum*, the set of real numbers $\mathbb{R}$.

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- In contrast, a system defined by a *difference equation*, like

$$x[k+1] = Ax[k] + Bu[x],$$

the independent variable k can, for example, take values only in the set of integers $\mathbb{N}$, $k = \cdots - 1, 0, 1, 2 \ldots.$

Linear Systems

- A system is said to be linear if it satisfies the *superposition principle*, that is, if given two pairs of initial conditions and inputs,

$$y_i(t),\ t \geq t_0 \Leftarrow \begin{cases} x_i(t_0) \\ u_i(t),\ t \geq t_0 \end{cases} \qquad \text{for } i = 1, 2\,,$$

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then we have that

$$y_1(t) + y_2(t),\ t \geq t_0 \Lleftarrow \begin{cases} x_1(t_0) + x_2(t_0) \\ u_1(t) + u_2(t),\ t \geq t_0 \end{cases} \qquad \text{(additivity)}$$

$$\alpha y_i(t),\ t \geq t_0 \Lleftarrow \begin{cases} \alpha x_i(t_0) \\ \alpha u_i(t),\ t \geq t_0 \end{cases} \quad \alpha \in \mathbb{R} \qquad \text{(homogeneity)}$$

Linear Systems

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- A system that does not satisfy the property of superposition is nonlinear.
- By the property of additivity we can consider the response of the system to initial conditions independently from that due to inputs.

$$y(t) = y_l(t) + y_f(t),\ t \geq t_0 \Leftarrow \begin{cases} y_l(t),\ t \geq t_0 & \Leftarrow \begin{cases} x(t_0) \\ u(t) = 0,\ t \geq t_0 \end{cases} \\ y_f(t),\ t \geq t_0 & \Leftarrow \begin{cases} x(t_0) = 0 \\ u(t),\ t \geq t_0 \end{cases} \end{cases}$$

Linear Systems

The response of a linear system is the superposition of its *free* response (that to initial conditions only, without external input) and its *forced* response (that to an external input, with zero initial conditions).

Linear Time-Invariant Systems

- A system is *time-invariant* if for each pair of initial conditions and inputs

$$y(t),\ t \geq t_0 \Lleftarrow \begin{cases} x(t_0) \\ u(t),\ t \geq t_0 \end{cases}$$

and each $T \in \mathbb{R}$, we have that

$$y(t-T),\ t \geq t_0 + T \Lleftarrow \begin{cases} x(t_0 + T) \\ u(t-T),\ t \geq t_0 + T\,. \end{cases}$$

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- In other words, the system gives the same response, but shifted in time, that if we apply to it the same input shifted in time, while keeping the same initial conditions.

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- A system without this property is called *time-varying*.

Linear Time-Invariant Systems

Input-Output Representation

- From the superposition principle, we can obtain the representation of a linear system by the *convolution integral*

$$y(t) = \int_{-\infty}^{\infty} g(t, \tau) u(\tau)\, d\tau\,, \tag{1}$$

where $g(t, \tau)$ is the *impulse response* of the system, that is, the output produced by a unitary impulse $\delta(t)$ applied at the input at the time instant τ.

Linear Time-Invariant Systems

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  where $g(t,\tau)$ is the *impulse response* of the system, that is, the output produced by a unitary impulse $\delta(t)$ applied at the input at the time instant τ.

- Causality implies that

$$\text{causality} \Leftrightarrow g(t,\tau) = 0 \text{ for } t < \tau,$$

  and on assuming zero initial conditions, Equation (1) then yields

$$y(t) = \int_{t_0}^{t} g(t,\tau)u(\tau)\, d\tau\,.$$

Linear Time-Invariant Systems

Input-Output Representation

- When the system has p inputs and q outputs, then we use the *impulse response matrix* $G(t,\tau) \in \mathbb{R}^{q \times p}$.

Linear Time-Invariant Systems

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- If the system is time-invariant, then for any T we have that

$$g(t,\tau) = g(t+T, \tau+T) = g(t-\tau, 0),$$

and we can redefine $g(t-\tau, 0)$ simply as $g(t-\tau)$. Thus the input-output representation of the system reduces to

$$y(t) = \int_0^t g(t-\tau)u(\tau)\, d\tau = \int_0^t g(\tau)u(t-\tau)\, d\tau\,.$$

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- The condition of causality for a linear time-invariant system can be alternatively stated as $g(t) = 0$ for $t < 0$.

Linear Time-Invariant Systems

State Space Representation

- Every linear finite-dimensional system can be described by *state space equations*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t). \end{aligned} \tag{3}$$

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- For a system with *order* n, the *state vector* is a vector of dimensions $n \times 1$, that is, it stacks n state variables, $x(t) \in \mathbb{R}^n$, for every t. If the system has p inputs and q outputs, then $u(t) \in \mathbb{R}^p$ and $y(t) \in \mathbb{R}^q$.

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- The matrices A, B, C, D are usually called

$$\begin{aligned}A &\in \mathbb{R}^{n\times n} : \textit{evolution matrix}\\ B &\in \mathbb{R}^{n\times p} : \textit{input matrix}\\ C &\in \mathbb{R}^{q\times n} : \textit{output matrix}\\ D &\in \mathbb{R}^{q\times p} : \textit{direct feedthrough matrix}\end{aligned}$$

Linear Time-Invariant Systems

State Space Representation

- When, in addition, the system is time-invariant, then the state space representation (3) reduces to

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\,. \end{aligned} \tag{6}$$

Linear Time-Invariant Systems

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- By applying the Laplace transform to (6) we obtain

$$\begin{aligned} s\hat{x}(s) - x(0) &= A\hat{x}(s) + B\hat{u}(s) \\ \hat{y}(s) &= C\hat{x}(s) + D\hat{u}(s)\,, \end{aligned}$$

from which follow

$$\begin{aligned} \hat{x}(s) &= (sI - A)^{-1}x(0) + (sI - A)^{-1}B\hat{u}(s) \\ \hat{y}(s) &= C(sI - A)^{-1}x(0) + [C(sI - A)^{-1}B + D]\hat{u}(s)\,. \end{aligned} \tag{9}$$

Linear Time-Invariant Systems

State Space Representation

- The algebraic equations (7) allow us to compute $\hat{x}(s)$ and $\hat{y}(s)$ from $x(0)$ and $\hat{u}(s)$. Then the inverse Laplace transform will give $x(t)$ and $y(t)$. By letting $x(0) = 0$ we see that the transfer function of the system is

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- See also the functions `ss`,`tf`,`ssdata` and `tfdata`, for system representations in MATLAB.

Linearisation

- Most physical systems are nonlinear. An important class of them can be represented by state space equations in the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), x(t_0), t)\,, \quad x(t_0) = x_0 \\ y(t) &= h(x(t), u(t), x(t_0), t)\,, \end{aligned} \qquad (10)$$

where f and h are nonlinear vector fields, that is, in scalar terms, the i-component of $\dot{x}(t)$ in (10) is written as

$$\dot{x}_i(t) = f_i(x_1(t), \dots, x_n(t); u_1(t), \dots, u_m(t); x_1(t_0), \dots, x_n(t_0); t) \quad x_i(t_0) = x_{i0}\,.$$

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- A linear state space equation is a useful tool to describe systems like (10) in an *approximate* way.
- The process of obtaining a linear model from a nonlinear one is called *linearisation*.

Linearisation

- The linearisation is performed around a nominal *point* or *trajectory*, defined by *nominal* values $\tilde{x}(t)$, $\tilde{x}_0$ and $\tilde{u}(t)$ that satisfy (10),

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- We are interested in the behaviour of the nonlinear differential equation (10) for an input and initial state which are “close” to the nominal values, that is, $u(t) = \tilde{u}(t) + u_\delta(t)$ and $x_0 = \tilde{x}_0 + x_{0\delta}$ for $u_\delta(t)$ and $x_{0\delta}$ sufficiently small for all $t \geq t_0$.

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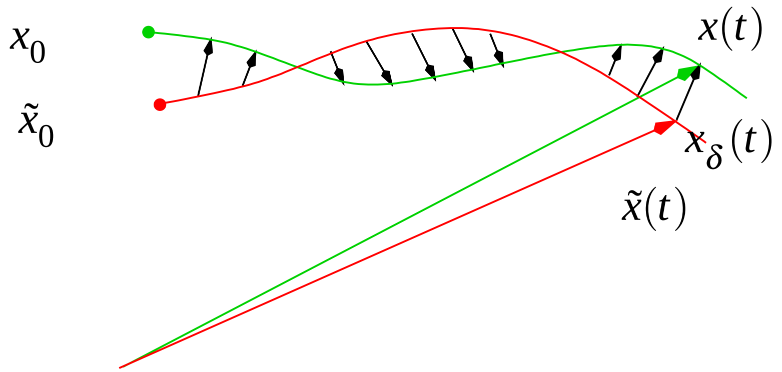

Linearisation

- Suppose that the solution stays close to the nominal trajectory, and write $x(t) = \tilde{x}(t) + x_\delta(t)$ for each $t \geq t_0$.

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- In terms of the nonlinear state space equation (10) we have

$$\dot{\tilde{x}}(t) + \dot{\tilde{x}}_\delta(t) = f(\tilde{x}(t) + x_\delta(t), \tilde{u}(t) + u_\delta(t), t), \quad \tilde{x}(t_0) + x_\delta(t_0) = \tilde{x}_0 + x_{0\delta} \quad (15)$$

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$$\dot{\tilde{x}}(t)+\dot{\tilde{x}}_\delta(t) = f(\tilde{x}(t)+x_\delta(t), \tilde{u}(t)+u_\delta(t), t), \quad \tilde{x}(t_0)+x_\delta(t_0) = \tilde{x}_0+x_{0\delta} \quad (17)$$

- Assuming differentiability, we can expand the right hand side of (13) in Taylor series around $\tilde{x}(t)$ and $\tilde{u}(t)$, keeping only the first order terms. *Note* that the expansion is performed in terms of x and u, and not for the independent variable t.

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- We make the operation more explicit for the i-component, which yields

$$f_i(\tilde{x} + x_\delta, \tilde{u} + u_\delta, t) \approx f_i(\tilde{x}, \tilde{u}, t) + \frac{\partial f_i}{\partial x_1}(\tilde{x}, \tilde{u}, t)x_{\delta 1} + \cdots + \frac{\partial f_i}{\partial x_n}(\tilde{x}, \tilde{u}, t)x_{\delta n}$$
$$+ \frac{\partial f_i}{\partial u_1}(\tilde{x}, \tilde{u}, t)u_{\delta 1} + \cdots + \frac{\partial f_i}{\partial u_m}(\tilde{x}, \tilde{u}, t)u_{\delta m} \quad (20)$$

Linearisation

- By repeating this operation for each $i = 1, \ldots, n$, and returning to the vectorial notation, we have

$$\dot{\tilde{x}}(t) + \dot{\tilde{x}}_\delta(t) \approx f(\tilde{x}(t), \tilde{u}(t)) + \frac{\partial f}{\partial x}(\tilde{x}, \tilde{u}, t)x_\delta + \frac{\partial f}{\partial u}(\tilde{x}, \tilde{u}, t)u_\delta$$

where $\frac{\partial f}{\partial x}$ represents the *Jacobian*, or *Jacobian Matrix*, of the vector field f with respect to x,

$$\frac{\partial f}{\partial x} \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \ldots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ldots & \frac{\partial f_2}{\partial x_n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \ldots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Linearisation

- Since $\dot{\tilde{x}}(t) = f(\tilde{x}(t), \tilde{u}(t), t)$, $\tilde{x}(t_0) = \tilde{x}_0$, the relation between $x_\delta(t)$ and $u_\delta(t)$ (the *incremental model*) is approximately described by a linear, *time-varying* state equation of the form

$$\dot{x}_\delta(t) = A(t)x_\delta(t) + B(t)u_\delta(t), \quad x_\delta(t_0) = x_0 - \tilde{x}_0$$

where

$$A(t) = \frac{\partial f}{\partial x}(\tilde{x}(t), \tilde{u}(t), t), \quad B(t) = \frac{\partial f}{\partial u}(\tilde{x}(t), \tilde{u}(t), t).$$

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- In the same way we can expand the output equation $y(t) = h(x(t), u(t), t)$, from which we obtain the linear approximation

$$y_\delta(t) = C(t)x_\delta(t) + D(t)u_\delta(t),$$

  where $y_\delta(t) = y(t) - \tilde{y}(t)$, with $\tilde{y}(t) = h(\tilde{x}(t), \tilde{u}(t), t)$ and

$$C(t) = \frac{\partial h}{\partial x}(\tilde{x}(t), \tilde{u}(t), t)\,, \quad D(t) = \frac{\partial h}{\partial u}(\tilde{x}(t), \tilde{u}(t), t)\,.$$

Linearisation

Note that the state equations obtained by linearisation will in general be *time-varying*, even when the original vector fields f and h were time-invariant, because the Jacobian matrices are evaluated along trajectories, and not stationary points.

Discrete-Time Systems

- Most of the state space concepts for linear continuous-time systems can be directly translated to discrete-time systems, described by *linear difference equations*. In this case the time variable t only takes values on a denumerable set, like the integers.

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- When the discrete-time system is obtained from sampling a continuous-time system, we will only consider *regular* sampling, where $t = kT$, $k = 0, 1, 2, \dots$, and T is the *sampling period*. In this case we denote the discrete-time variables (sequences) as $u[k] \triangleq u(kT)$, etc.

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- The concepts of finite dimensionality, causality, linearity and the superposition principle for responses to initial conditions and inputs are exactly the same as those in the continuous-time case.
- One difference though: pure delays in discrete-time do not give rise to an infinite-dimensional system, as is the case of continuous-time systems, if the delay is a multiple of the sampling period T.

Discrete-Time Systems

Input-Output Representation

- We define the *impulse sequence* $\delta[k]$ as

$$\delta[k-m] = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases}$$

where k and m are integers.

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- Note how in the discrete-time case impulses are easy to implement physically, in contrast to the continuous-time case.

- In a discrete-time linear system every input sequence $u[k]$ can be represented by means of the series

$$u[k] = \sum_{m=-\infty}^{\infty} u[m]\delta[k-m].$$

Discrete-Time Systems

Input-Output Representation

- If $g[k, m]$ denotes the output of a discrete time system to an impulse sequence applied at the instant m, then we have that

$$\delta[k - m] \rightarrow g[k, m]$$

$$\delta[k, m]u[m] \rightarrow g[k, m]u[m] \qquad \text{(by homogeneity)}$$

$$\sum_m \delta[k, m]u[m] \rightarrow \sum_m g[k, m]u[m] \qquad \text{(by additivity)}.$$

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- Thus the output $y[k]$ obtained from the input $u[k]$ can be written by means of the series

$$y[k] = \sum_{m=-\infty}^{\infty} g[k,m]u[m]. \tag{22}$$

Discrete-Time Systems

Input-Output Representation

- If the system is *causal* there wouldn't be output signal before the input is applied, hence

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- For *causal* discrete-time systems the representation (21) reduces to

$$y[k] = \sum_{m=k_0}^{k} g[k, m]u[m]\,,$$

and, if in addition we have *time-invariance*, the property of invariance with respect to shifts in time holds, and thus we arrive to the system representation by the *discrete convolution*

$$y[k] = \sum_{m=0}^{k} g[k - m]u[m] = \sum_{m=0}^{k} g[m]u[k - m]\,.$$

Discrete-Time Systems

State Space Representation

- Every discrete-time, finite dimensional, linear system can be represented by state space difference equations, as in

$$x[k+1] = A[k]x[k] + B[k]u[k]$$
$$y[k] = C[k]x[k] + D[k]u[k],$$

and in the time-invariant case

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k].$$

Discrete-Time Systems

State Space Representation

- In this case, it corresponds to talk about *discrete* transfer functions, $\hat{G}(z) = \mathcal{Z}[g[k]]$. The relation between discrete transfer function representation and state space representation is identical to the continuous-time case,

$$\hat{G}(z) = C(zI - A)^{-1}B + D\,,$$

and the same MATLAB functions can be used.

A Few General Facts to Remember

- A transfer matrix is *rational* if and only if the corresponding system is linear, time-invariant and finite-dimensional.

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- Linearisation is performed around a known *nominal trajectory*, and the incremental models obtained will, in general, be *time-varying*.
- Discrete-time systems have representations equivalent to those of continuous-time systems by *convolution series*, transfer functions in the discrete $\mathcal{Z}$ transform, and state space difference equations.
- In contrast to the continuous time case, pure delays do not necessarily give rise to an infinite-dimensional discrete-time system.

A Few General Facts to Remember

Type of system	Internal representation	External representation
infinite dim. linear		$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$
finite dim., linear	$\dot{x} = A(t)x + B(t)u$ $y = C(t)x + D(t)u$	$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$
infinite dim. LTI		$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$ $\hat{y}(s) = \hat{G}(s)\hat{u}(s)$
finite dim., LTI	$\dot{x} = Ax + Bu$ $y = Cx + Du$	$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$ $\hat{y}(s) = \hat{G}(s)\hat{u}(s)$